The Mathematical legacy of ECOSSE

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University of Edinburgh

With help from Sven Leyffer

April 17th 2009

Mathematics within ECOSSE

- Centred on Roger Fletcher's group in Dundee

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- Engineers were also using Maths!

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- Otherwise, we took the money and ran...
- and went hill-walking...
- and did maths...
- and paved the way for the German invasion

The Dundee group

- Major focus was the solution of mixed-integer nonlinear programming **MINLP** problems

$$\begin{array}{ll} \text{minimize} & f(\boldsymbol{x}, \boldsymbol{y}) \\ \text{subject to} & \boldsymbol{c}(\boldsymbol{x}, \boldsymbol{y}) \geq \boldsymbol{0} \\ & y_i \in \{0,\ 1\}\ \forall i \end{array}$$

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- **Why?**
  Retrofit design problems required the solution of (convex) MINLP problems
- What is required in order to solve MINLP problems?

Outer-approximation: Duran and Grossmann (1986)

Choose $\boldsymbol{y}^{(0)}$; $k = 0$

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Solve **NLP**$^{(k)}$

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s.t. $\boldsymbol{c}(\boldsymbol{x},\ \boldsymbol{y}^{(k)}) \geq \boldsymbol{0}$

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Linearise $f(\boldsymbol{x},\ \boldsymbol{y})$ and $\boldsymbol{c}(\boldsymbol{x},\ \boldsymbol{y})$ about $(\boldsymbol{x}^{(k)},\ \boldsymbol{y}^{(k)})$

to form $A^{(k)}\boldsymbol{x} + B^{(k)}\boldsymbol{y} + \boldsymbol{g}^{(k)}z \geq \boldsymbol{b}^{(k)}$

from $A^{(k-1)}$, $B^{(k-1)}$, $\boldsymbol{g}^{(k-1)}$ and $\boldsymbol{b}^{(k-1)}$

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- *Towards reliable linear programming*
  Fletcher and Hall (1990)
  Pitman Research Notes in Mathematics Series 228 89–104
- *Ordering algorithms for irreducible sparse linear systems*
  Fletcher and Hall (1993)
  Annals of Operations Research **43** 15–32

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What is hyper-sparsity

Consider solving $A\boldsymbol{x} = \boldsymbol{b}$ when A is **sparse**

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Consider solving $\boldsymbol{Ax} = \boldsymbol{b}$ when A is **sparse**

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Given sparse $LU = A$, solving $A\boldsymbol{x} = \boldsymbol{b}$ by traditional substitution can be very inefficient

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When does hyper-sparsity occur?

- During left-looking implementations of Gaussian elimination
- When applying the revised simplex method to important classes of LP problems

Exploiting hyper-sparsity

- Traditional technique for solving $L\boldsymbol{x} = \boldsymbol{b}$ by transforming $\boldsymbol{b}$ into $\boldsymbol{x}$

```
do k = 1, n
    b := b − b_k l_k
end do
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do $k = 1$, n

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- When $\boldsymbol{x}$ is sparse, the dominant cost is the test for zero
- Efficient identification of vectors $\boldsymbol{l}_k$ to be applied
  - Gilbert and Peierls (1988)
  - Hall and McKinnon (1998–2005)

The Leyffer years (1989–2002)

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- MIQP

  Numerical Experience with lower bounds for MIQP branch–and–bound
  Fletcher and Leyffer (1998)
  SIAM J. Optimization **8(2)** 604–616

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    Solving mixed integer nonlinear programs by outer approximation
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    Mathematical Programming **66** 327–349

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  - Integrated NLP within branch-and-bound

    Integrating SQP and branch-and-bound for mixed integer nonlinear programming
    Leyffer (2001)
    Computational Optimization & Applications **18** 295–309

The Leyffer years (1989–2002)

- Eventually did some chemical engineering!

  Comparison of certain MINLP algorithms when applied to a model structure determination and parameter estimation problem
  Skrifvars, Leyffer and Westerlund (1998)
  Computers and Chemical Engineering **22(12)** 1829–1835

The “Filter” method

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For NLP

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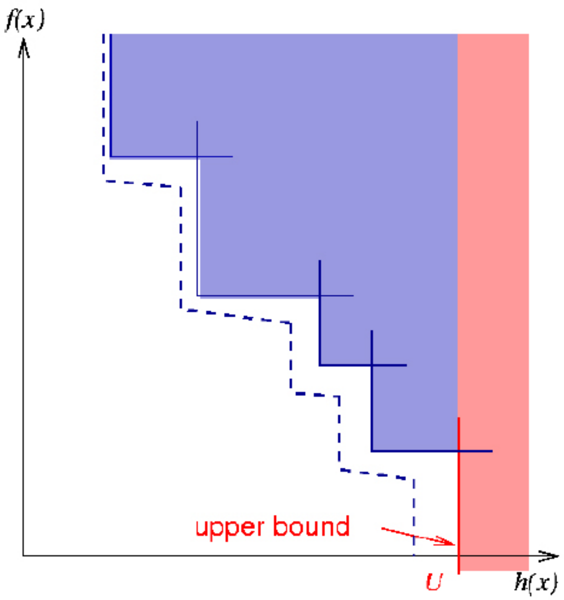

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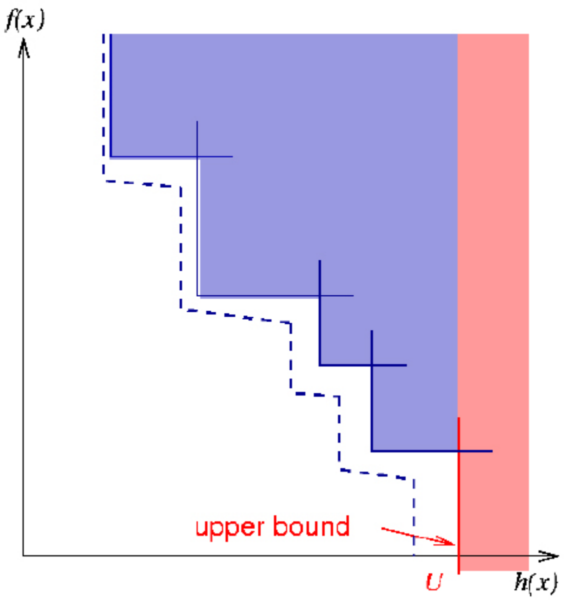

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Where did the idea come from?

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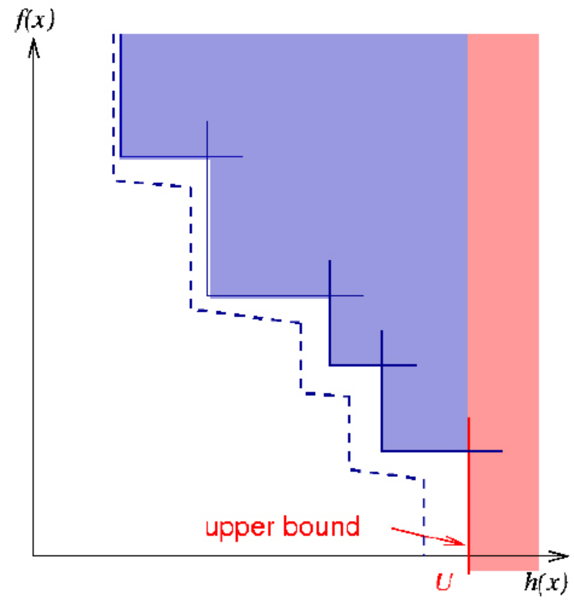
- Want to minimize *both* $f(\boldsymbol{x})$ *and* $h(\boldsymbol{x}) = \|\boldsymbol{c}(\boldsymbol{x})^-\|$
- **Traditionally:** minimize single objective $f(\boldsymbol{x}) + \pi h(\boldsymbol{x})$
- Hard to identify penalty parameter π
- Filter avoided this using a multi-objective approach



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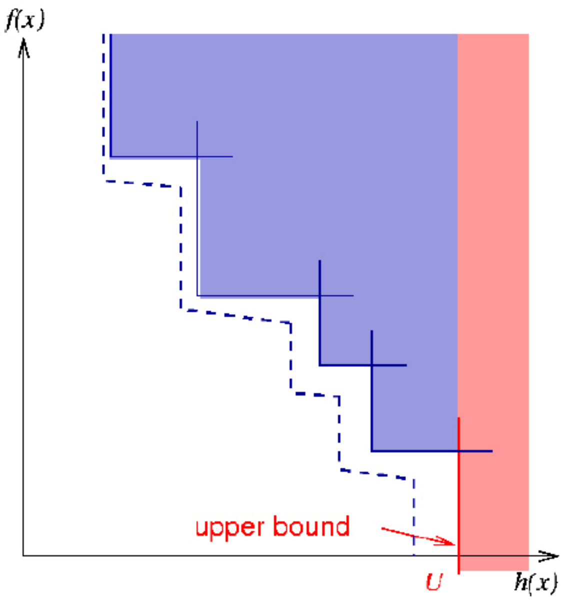

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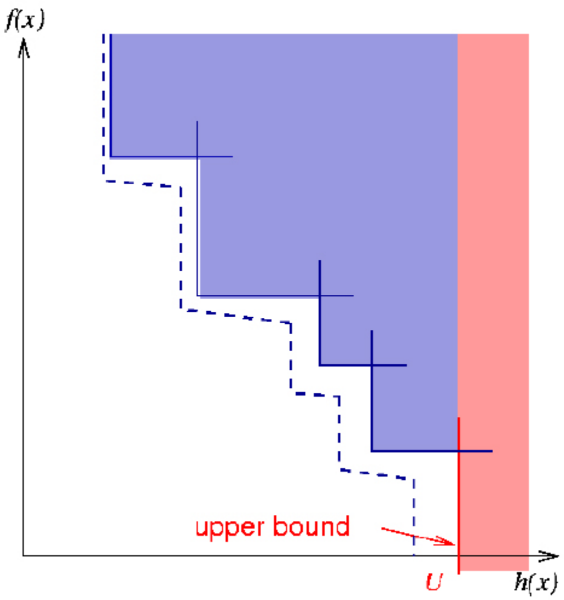

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Legacy

- Convergence proof and extensions generated a sequence of landmark papers
  - *Nonlinear programming without a penalty function*
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    Mathematical Programming **91** 239–270
  - *On the global convergence of a filter-SQP algorithm*
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The Return of the Filter	Dundee/SIAM
How the Grinch Solved MPECs	Dundee
Filter Methods By Dummies (*)	Argonne
MPECs: Much Ado About Nothing?	ISMP-00
The M&M of Optimization: Modeling and Methods	Argonne
QPPAL: A Friendly Solver for Large-Scale QPs	INFORMS

(*) actually a misprint: should have been "For"

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- Where are they now?
  - Marcel Mongeau: maître de conférences at Université Paul Sabatier, Toulouse
  - Colin Millar: DJ?

Future work?

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Thanks Jack!