High performance simplex solvers for linear programming problems

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High performance simplex solvers: Overview

Talk

- A little mathematics
- Some algorithms
- Mainly numerical linear algebra

Content

- Background
- Exploiting hyper-sparsity
- Exploiting parallelism
- Conclusions

Linear programming (LP)

$$\begin{array}{ll} \text{minimize} & f = \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \quad \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

Background

- Fundamental model in optimal decision-making
- Solution techniques
  - Simplex method (1947)
  - Interior point methods (1984)
- Large problems have
  - 10^3–$10^{\cancel{7}8}$ variables
  - 10^3–$10^{\cancel{7}8}$ constraints
- Matrix A is (usually) sparse

Example

STAIR: 356 rows, 467 columns and 3856 nonzeros

Solving LP problems

$$\begin{array}{ll} \text{minimize} & f = \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \quad \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

Partitioned LP

- Let $\mathcal{B} \cup \mathcal{N}$ be a **partition** of the variable set
  - Let A be partitioned as $\begin{bmatrix} B & N \end{bmatrix}$ with nonsingular **basis matrix** B
  - Let $\boldsymbol{c}$ be partitioned as $\begin{bmatrix} \boldsymbol{c}_B \\ \boldsymbol{c}_N \end{bmatrix}$
- Partitioned LP is

$$\begin{array}{ll} \text{minimize} & f = \boldsymbol{c}_B^T\boldsymbol{x}_B + \boldsymbol{c}_N^T\boldsymbol{x}_N \\ \text{subject to} & B\boldsymbol{x}_B + N\boldsymbol{x}_N = \boldsymbol{b} \quad \boldsymbol{x}_B \geq \boldsymbol{0} \quad \boldsymbol{x}_N \geq \boldsymbol{0} \end{array}$$

Solving LP problems

$$\begin{array}{ll} \text{minimize} & f = \boldsymbol{c}_B^T \boldsymbol{x}_B + \boldsymbol{c}_N^T \boldsymbol{x}_N \\ \text{subject to} & B\boldsymbol{x}_B + N\boldsymbol{x}_N = \boldsymbol{b} \quad \boldsymbol{x}_B \geq \boldsymbol{0} \quad \boldsymbol{x}_N \geq \boldsymbol{0} \end{array}$$

Reduced LP

- Equations yield

$$\boldsymbol{x}_B = \widehat{\boldsymbol{b}} - \widehat{N}\boldsymbol{x}_N \quad \text{where} \quad \widehat{N} = B^{-1}N \quad \text{and} \quad \widehat{\boldsymbol{b}} = B^{-1}\boldsymbol{b}$$

- Eliminate $\boldsymbol{x}_B$ from the objective to yield the **reduced LP**

$$\begin{array}{ll} \text{minimize} & f = \widehat{f} + \widehat{\boldsymbol{c}}^T \boldsymbol{x}_N \\ \text{subject to} & \boldsymbol{x}_B + \widehat{N}\boldsymbol{x}_N = \widehat{\boldsymbol{b}} \quad \boldsymbol{x}_B \geq \boldsymbol{0} \quad \boldsymbol{x}_N \geq \boldsymbol{0} \end{array}$$

where

$$\widehat{f} = \boldsymbol{c}_B^T \widehat{\boldsymbol{b}} \quad \text{and} \quad \widehat{\boldsymbol{c}}^T = \boldsymbol{c}_N^T - \boldsymbol{c}_B^T B^{-1} N$$

Solving LP problems

$$\begin{array}{ll} \text{minimize} & f = \widehat{f} + \widehat{\boldsymbol{c}}^T \boldsymbol{x}_N \\ \text{subject to} & \boldsymbol{x}_B + \widehat{N}\boldsymbol{x}_N = \widehat{\boldsymbol{b}} \quad \boldsymbol{x}_B \geq \boldsymbol{0} \quad \boldsymbol{x}_N \geq \boldsymbol{0} \end{array}$$

Sufficient optimality conditions

For $\boldsymbol{x}_N = \boldsymbol{0}$ sufficient optimality conditions are

- Primal feasibility $\widehat{\boldsymbol{b}} \geq \boldsymbol{0}$
- Dual feasibility $\widehat{\boldsymbol{c}} \geq \boldsymbol{0}$
  f cannot be reduced by increasing any component of $\boldsymbol{x}_N$ from zero

Simplex algorithm: concept

- Represent the reduced LP in a tableau
- Find a primal and dual feasible partition $\mathcal{B} \cup \mathcal{N}$

	$\mathcal{N}$	RHS
$\mathcal{B}$	$\widehat{N}$	$\widehat{\boldsymbol{b}}$
	$\widehat{\boldsymbol{c}}^T$	

Simplex algorithm: Primal or dual?

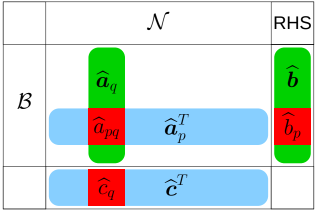
Primal simplex algorithm

- Traditional variant
  - Assume primal feasibility $\widehat{\boldsymbol{b}} \geq \mathbf{0}$
  - Seek dual feasibility $\widehat{\boldsymbol{c}} \geq \mathbf{0}$
- Solution generally not primal feasible when (primal) LP is tightened

Dual simplex algorithm

- Preferred variant
  - Assume dual feasibility $\widehat{\boldsymbol{c}} \geq \mathbf{0}$
  - Seek primal feasibility $\widehat{\boldsymbol{b}} \geq \mathbf{0}$
- Easier to get dual feasibility
- More progress in many iterations
- Solution dual feasible when LP is tightened

Simplex algorithm: Each iteration



Dual algorithm: Assume $\widehat{\boldsymbol{c}} \geq \mathbf{0}$ Seek $\widehat{\boldsymbol{b}} \geq \mathbf{0}$

Scan $\widehat{b}_i$, $i \in \mathcal{B}$, for a good candidate p to leave $\mathcal{B}$ CHUZR

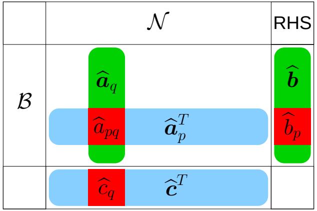
Scan $\widehat{c}_j/\widehat{a}_{pj}$, $j \in \mathcal{N}$, for a good candidate q to leave $\mathcal{N}$ CHUZC

Update: Exchange p and q between $\mathcal{B}$ and $\mathcal{N}$

Update $\widehat{\boldsymbol{b}} := \widehat{\boldsymbol{b}} - \theta_p \widehat{\boldsymbol{a}}_q$ $\theta_p = \widehat{b}_p/\widehat{a}_{pq}$ UPDATE-PRIMAL

Update $\widehat{\boldsymbol{c}}_N^T := \widehat{\boldsymbol{c}}_N^T - \theta_d \widehat{\boldsymbol{a}}_p^T$ $\theta_d = \widehat{c}_q/\widehat{a}_{pq}$ UPDATE-DUAL

Standard simplex method (SSM): Computation



Major computational component

Update of tableau:

$$\widehat{N} := \widehat{N} - \frac{1}{\widehat{a}_{pq}} \widehat{\boldsymbol{a}}_q \widehat{\boldsymbol{a}}_p^T$$

where $\widehat{N} = B^{-1}N$

- Hopelessly inefficient for sparse LP problems
- Prohibitively expensive for large LP problems

Revised simplex method (RSM): Computation

	$\mathcal{N}$	RHS
$\mathcal{B}$	$\widehat{\boldsymbol{a}}_q$ $\widehat{a}_{pq}$ $\widehat{\boldsymbol{a}}_p^T$	$\widehat{\boldsymbol{b}}$ $\widehat{b}_p$
	$\widehat{c}_q$ $\widehat{\boldsymbol{c}}^T$	

Major computational components

$\boldsymbol{\pi}_p^T = \boldsymbol{e}_p^T B^{-1}$ BTRAN $\qquad$ $\widehat{\boldsymbol{a}}_p^T = \boldsymbol{\pi}_p^T N$ PRICE

$\widehat{\boldsymbol{a}}_q = B^{-1}\boldsymbol{a}_q$ FTRAN $\qquad$ Invert B INVERT

Don't form B^{-1}!

- If B is sparse then B^{-1} is generally dense
- INVERT: form sparsity-preserving decomposition $B = LU$ to operate with B^{-1}

Exploiting hyper-sparsity

Exploiting hyper-sparsity in the revised simplex method

Recall: major computational components

- BTRAN: Solve $B^T\boldsymbol{\pi}_p = \boldsymbol{e}_p$
- PRICE: Form $\widehat{\boldsymbol{a}}_p^T = \boldsymbol{\pi}_p^T N$
- FTRAN: Solve $B\,\widehat{\boldsymbol{a}}_q = \boldsymbol{a}_q$

Phenomenon of hyper-sparsity

- Vectors $\boldsymbol{\pi}_p$, $\widehat{\boldsymbol{a}}_p^T$ and $\widehat{\boldsymbol{a}}_q$ may be sparse
- Why?

Because B^{-1} is sparse

- So?

Exploiting hyper-sparsity: Representing B^{-1}

- **Recall:** `INVERT` forms sparsity-preserving decomposition $B = LU$
  - Can use this to solve $B\boldsymbol{x} = \boldsymbol{r}$ using column-wise foward/backward substitution
  - Many columns are trivial
- Remove the trivial columns to represent B^{-1} by the eta file $\{p_k, \mu_k, \boldsymbol{\eta}_k\}_{k=1}^{K}$
- Derived directly from the results of Gaussian elimination
  - The pivots μ_k are in rows p_k
  - $\boldsymbol{\eta}_k$ are the eta vectors
  - $K \ll 2m$ is common
- Operating with the eta file $\equiv$ Column-wise foward/backward substitution

Exploiting hyper-sparsity: When solving $Bx = \boldsymbol{r}$

Traditional technique transforms $\boldsymbol{r}$ into $\boldsymbol{x}$

```
do k = 1, K
```

$$
\begin{aligned}
r_{p_k} &:= r_{p_k}/\mu_k \\
\boldsymbol{r} &:= \boldsymbol{r} - r_{p_k}\boldsymbol{\eta}_k
\end{aligned}
$$

```
end do
```

Exploiting hyper-sparsity: When solving $Bx = \boldsymbol{r}$

When $\boldsymbol{r}$ is sparse skip $\boldsymbol{\eta}_k$ if r_{p_k} is zero

```
do k = 1, K
    if (r_{p_k} .ne. 0) then
        r_{p_k} := r_{p_k}/μ_k
        r := r − r_{p_k} η_k
    end if
end do
```

- When $\boldsymbol{x}$ is sparse, the dominant cost is the test for zero
- Requires efficient identification of vectors $\boldsymbol{\eta}_k$ to be applied

Gilbert and Peierls (1988)
H and McKinnon (1998–2005)

Exploiting hyper-sparsity: When solving $B^T \boldsymbol{x} = \boldsymbol{r}$

Traditional technique transforms $\boldsymbol{r}$ into $\boldsymbol{x}$

```
do k = K, 1
    r_{p_k} := (r_{p_k} - r^T η_k)/μ_k
end do
```

$$
\begin{array}{l}
\texttt{do } k = K, 1 \\
\quad r_{p_k} := (r_{p_k} - \boldsymbol{r}^T \boldsymbol{\eta}_k)/\mu_k \\
\texttt{end do}
\end{array}
$$

- When $\boldsymbol{x}$ is sparse most $\boldsymbol{r}^T \boldsymbol{\eta}_k$ are zero
- No way to exploit hyper-sparsity properly with “column-wise” eta file
- After `INVERT`: Form a “row-wise” copy of the eta file
- Pass row-wise eta file to hyper-sparse forward solution code

H and McKinnon (1998–2005)

Speedup in total solution time and computational components

Problem	Dimension	Solution	$B^{-1}\boldsymbol{r}_F$	$\boldsymbol{r}_B^T B^{-1}$	$\boldsymbol{r}_\pi^T N$
80bau3b	2262	3.34	5.13	3.51	6.06
fit2p	3000	1.75	1.30	12.22	13.47
stocfor3	16675	1.85	1.14	7.26	7.61
dcp2	32388	5.32	8.24	6.21	6.20
ken-11	14694	22.84	98.04	27.22	66.36
ken-13	28632	12.12	104.09	12.87	17.60
ken-18	105127	15.27	263.94	13.91	19.92
pds-06	9881	17.48	24.07	21.58	28.18
pds-10	16558	10.36	11.24	16.60	17.55
pds-20	33874	10.35	5.96	14.33	15.40

H and McKinnon (1998–2005)
[Won COAP best paper prize for 2005]

Exploiting parallelism

Parallelising the simplex method: Background

Data parallel standard simplex method

- Good parallel efficiency *was* achieved
- Only relevant for dense LP problems

Data parallel revised simplex method

- Only immediate parallelism is in forming $\pi_p^T N$
- When $n \gg m$ significant speed-up *was* achieved — Bixby and Martin (2000)

Task parallel revised simplex method

- Overlap computational components for different iterations
  Wunderling (1996), H and McKinnon (1995-2005)
- Modest speed-up *was* achieved on general sparse LP problems

Parallelising the dual revised simplex method: Overview

Single iteration parallelism for general LP

- Pure dual revised simplex
- **Data parallelism:** Form $\boldsymbol{\pi}_p^T N$
- **Task parallelism:** Identify serial computation which can be overlapped

Multiple iteration parallelism for general LP

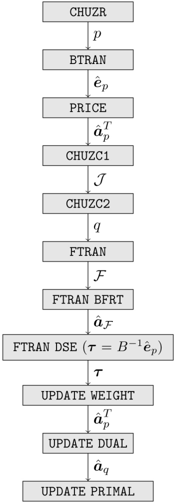
- Dual revised simplex with minor iterations of dual standard simplex
- **Data parallelism:** Form $\boldsymbol{\pi}_p^T N$ and update (slice of) dual standard simplex tableau
- **Task parallelism:** Identify serial computation which can be overlapped

Data parallelism for stochastic LP

- Pure dual revised simplex for column-linked block angular LP problems
- **Data parallelism:** Solve $B^T\boldsymbol{\pi} = \boldsymbol{e}_p$, $B\widehat{\boldsymbol{a}}_q = \boldsymbol{a}_q$ and form $\boldsymbol{\pi}_p^T N$

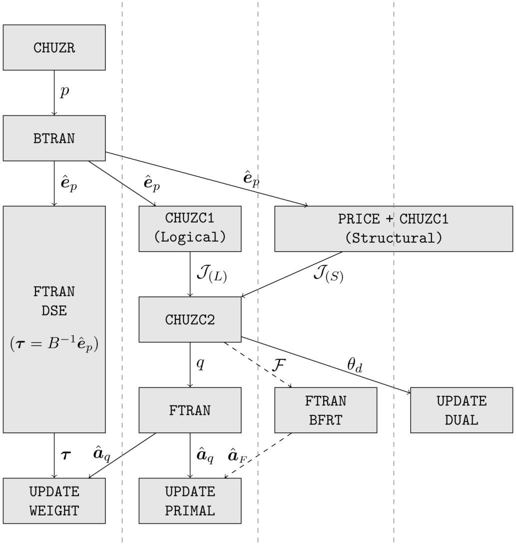
Single iteration parallelism

Single iteration parallelism: Dual revised simplex method



- Computational components appear sequential
- Each has highly-tuned sparsity-exploiting serial implementation
- Exploit “slack” in data dependencies

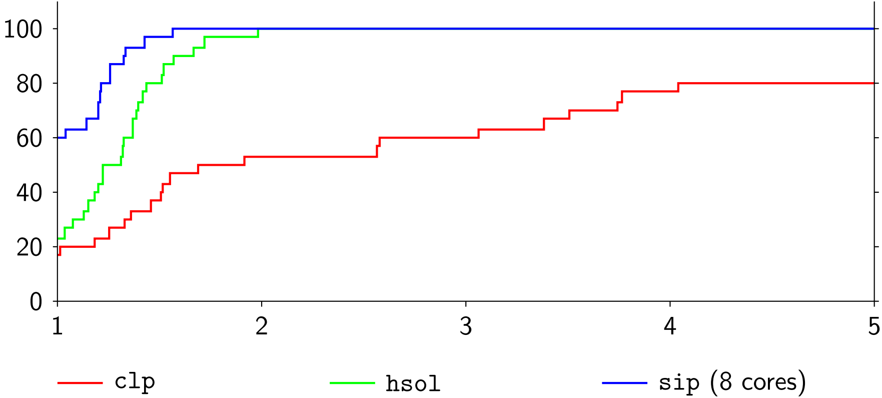
Single iteration parallelism: Computational scheme



- Parallel PRICE to form $\hat{\boldsymbol{a}}_p^T = \boldsymbol{\pi}_p^T N$
- Other computational components serial
- Overlap any independent calculations
- Only four worthwhile threads unless $n \gg m$ so PRICE dominates
- More than Bixby and Martin (2000)
- Better than Forrest (2012)

Huangfu and H (2014)

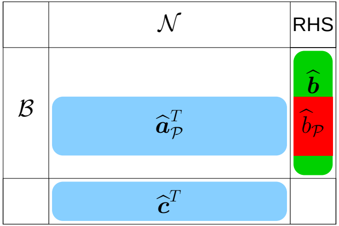
Single iteration parallelism: clp vs hsol vs sip



Multiple iteration parallelism

Multiple iteration parallelism

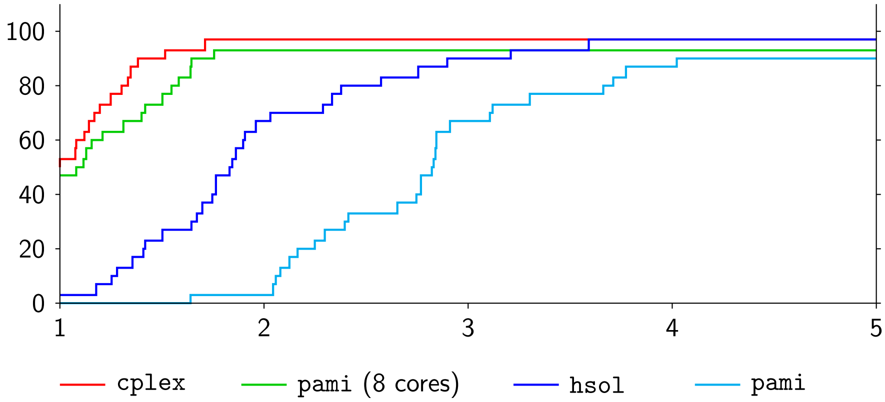
- `sip` has too little work to be performed in parallel to get good speedup
- Perform standard dual simplex minor iterations for rows in set $\mathcal{P}$ ($|\mathcal{P}| \ll m$)
- Suggested by Rosander (1975) but never implemented efficiently *in serial*



- Task-parallel multiple `BTRAN` to form $\boldsymbol{\pi}_{\mathcal{P}} = B^{-1}\boldsymbol{e}_{\mathcal{P}}$
- Data-parallel `PRICE` to form $\widehat{\boldsymbol{a}}_p^T$ (as required)
- Task-parallel multiple `FTRAN` for primal, dual and weight updates

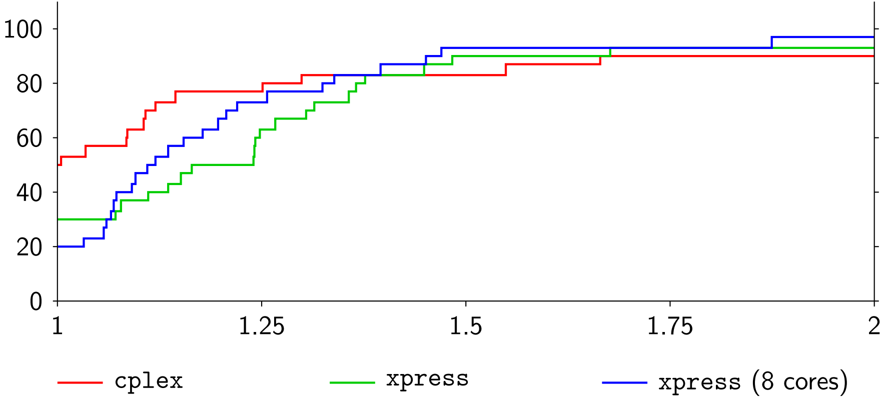
Huangfu and H (2011–2014)

Multiple iteration parallelism: cplex vs pami vs hsol



- pami is less efficient than hsol in serial
- pami speedup more than compensates
- pami performance approaching cplex

Multiple iteration parallelism: cplex vs xpress



- pami ideas incorporated in FICO Xpress (Huangfu 2014)

Data parallelism for stochastic LPs

Stochastic MIP problems: General

Two-stage stochastic LPs have column-linked block angular structure

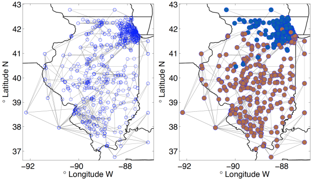
$$
\begin{array}{rcccccccccccc}
\text{minimize} & \boldsymbol{c}_0^T\boldsymbol{x}_0 & + & \boldsymbol{c}_1^T\boldsymbol{x}_1 & + & \boldsymbol{c}_2^T\boldsymbol{x}_2 & + & \dots & + & \boldsymbol{c}_N^T\boldsymbol{x}_N & & \\
\text{subject to} & A\boldsymbol{x}_0 & & & & & & & & & = & \boldsymbol{b}_0 \\
& T_1\boldsymbol{x}_0 & + & W_1\boldsymbol{x}_1 & & & & & & & = & \boldsymbol{b}_1 \\
& T_2\boldsymbol{x}_0 & & & + & W_2\boldsymbol{x}_2 & & & & & = & \boldsymbol{b}_2 \\
& \vdots & & & & & & \ddots & & & & \vdots \\
& T_N\boldsymbol{x}_0 & & & & & & & + & W_N\boldsymbol{x}_N & = & \boldsymbol{b}_N \\
& \boldsymbol{x}_0 \geq \boldsymbol{0} & & \boldsymbol{x}_1 \geq \boldsymbol{0} & & \boldsymbol{x}_2 \geq \boldsymbol{0} & & \dots & & \boldsymbol{x}_N \geq \boldsymbol{0} & &
\end{array}
$$

- Variables $\boldsymbol{x}_0 \in \mathbb{R}^{n_0}$ are **first stage** decisions
- Variables $\boldsymbol{x}_i \in \mathbb{R}^{n_i}$ for $i = 1, \dots, N$ are **second stage** decisions
  Each corresponds to a **scenario** which occurs with modelled probability
- The objective is the expected cost of the decisions
- In stochastic MIP problems, some/all decisions are discrete

Stochastic MIP problems: For Argonne

- Power systems optimization project at Argonne
- Integer second-stage decisions
- Stochasticity from wind generation
- Initial experiments carried out using model problem
- Number of scenarios increases with refinement of probability distribution sampling
- Solution via branch-and-bound
    - Solve root using parallel IPM solver `PIPS`

      Lubin, Petra *et al.* (2011)
    - Solve nodes using parallel dual simplex solver `PIPS-S`



Stochastic MIP problems: General

Convenient to permute the LP thus:

$$
\begin{array}{rcccccccccccc}
\text{minimize} & \boldsymbol{c}_1^T\boldsymbol{x}_1 & + & \boldsymbol{c}_2^T\boldsymbol{x}_2 & + & \dots & + & \boldsymbol{c}_N^T\boldsymbol{x}_N & + & \boldsymbol{c}_0^T\boldsymbol{x}_0 & & \\
\text{subject to} & W_1\boldsymbol{x}_1 & & & & & & & + & T_1\boldsymbol{x}_0 & = & \boldsymbol{b}_1 \\
 & & & W_2\boldsymbol{x}_2 & & & & & + & T_2\boldsymbol{x}_0 & = & \boldsymbol{b}_2 \\
 & & & & & \ddots & & & \vdots & & & \vdots \\
 & & & & & & & W_N\boldsymbol{x}_N & + & T_N\boldsymbol{x}_0 & = & \boldsymbol{b}_N \\
 & & & & & & & & & A\boldsymbol{x}_0 & = & \boldsymbol{b}_0 \\
 & \boldsymbol{x}_1 \geq \boldsymbol{0} & & \boldsymbol{x}_2 \geq \boldsymbol{0} & & \dots & & \boldsymbol{x}_N \geq \boldsymbol{0} & & \boldsymbol{x}_0 \geq \boldsymbol{0} & &
\end{array}
$$

Exploiting problem structure

- Inversion of the basis matrix B is key to revised simplex efficiency
- For column-linked BALP problems

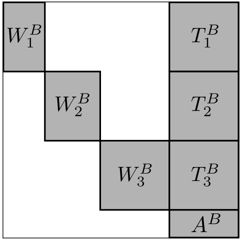
$$B = \begin{bmatrix} W_1^B & & & T_1^B \\ & \ddots & & \vdots \\ & & W_N^B & T_N^B \\ & & & A^B \end{bmatrix}$$

- W_i^B are columns corresponding to n_i^B basic variables in scenario i
- $\begin{bmatrix} T_1^B \\ \vdots \\ T_N^B \\ A^B \end{bmatrix}$ are columns corresponding to n_0^B basic first stage decisions

Exploiting problem structure

- Inversion of the basis matrix B is key to revised simplex efficiency
- For column-linked BALP problems

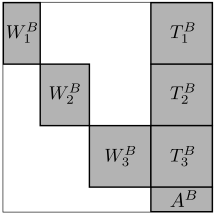
$$B = \begin{bmatrix} W_1^B & & & T_1^B \\ & \ddots & & \vdots \\ & & W_N^B & T_N^B \\ & & & A^B \end{bmatrix}$$



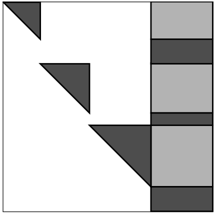
- B is nonsingular so
  - W_i^B are "tall": full column rank
  - $\begin{bmatrix} W_i^B & T_i^B \end{bmatrix}$ are "wide": full row rank
  - A^B is "wide": full row rank

- Scope for parallel inversion is immediate and well known

Exploiting problem structure

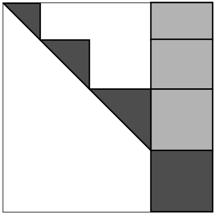


- Eliminate sub-diagonal entries in each W_i^B (independently)



- Apply elimination operations to each T_i^B (independently)

- Accumulate non-pivoted rows from the W_i^B with A^B and complete elimination



Parallel distributed-memory simplex for large-scale stochastic LP problems

Scope for parallelism

- Parallel Gaussian elimination yields **block LU** decomposition of B
- Scope for parallelism in block forward and block backward substitution
- Scope for parallelism in `PRICE`

Implementation

- Distribute problem data over processes
- Perform data-parallel `BTRAN`, `FTRAN` and `PRICE` over processes
- Used MPI

Paper: Lubin, H *et al.* (2013)

- Won COIN-OR INFORMS 2013 Cup
- Won COAP best paper prize for 2013

Results: Stochastic LP test problems

Test Problem	1st Stage n_0	m_0	2nd-Stage Scenario n_i	m_i	Nonzero Elements A	W_i	T_i
`Storm`	121	185	1,259	528	696	3,220	121
`SSN`	89	1	706	175	89	2,284	89
`UC12`	3,132	0	56,532	59,436	0	163,839	3,132
`UC24`	6,264	0	113,064	118,872	0	327,939	6,264

- `Storm` and `SSN` are publicly available
- `UC12` and `UC24` are stochastic unit commitment problems developed at Argonne
  - Aim to choose optimal on/off schedules for generators on the power grid of the state of Illinois over a 12-hour and 24-hour horizon
  - In practice each scenario corresponds to a weather simulation
    Model problem generates scenarios by normal perturbations

Zavala (2011)

Results: Baseline serial performance for large instances

Serial performance of `PIPS-S` and `clp`

Problem	Dimensions	Solver	Iterations	Time (s)	Iter/sec
`Storm`	$n =$10,313,849	`PIPS-S`	6,353,593	385,825	16.5
8,192 scen.	$m = 4,325,561$	`clp`	6,706,401	133,047	50.4
`SSN`	$n = 5,783,651$	`PIPS-S`	1,025,279	58,425	17.5
8,192 scen.	$m = 1,433,601$	`clp`	1,175,282	12,619	93.1
`UC12`	$n = 1,812,156$	`PIPS-S`	1,968,400	236,219	8.3
32 scen.	$m = 1,901,952$	`clp`	2,474,175	39,722	62.3
`UC24`	$n = 1,815,288$	`PIPS-S`	2,142,962	543,272	3.9
16 scen.	$m = 1,901,952$	`clp`	2,441,374	41,708	58.5

Results: On Fusion cluster

Speed-up of PIPS-S relative to 1-core PIPS-S and 1-core clp

Cores	Storm	SSN	UC12	UC24
1	1.0	1.0	1.0	1.0
4	3.6	3.5	2.7	3.0
8	7.3	7.5	6.1	5.3
16	13.6	15.1	8.5	8.9
32	24.6	30.3	14.5	
clp	8.5	6.5	2.4	0.7

Results: On Fusion cluster - larger instances

	Storm	SSN	UC12	UC24
Scenarios	32,768	32,768	512	256
Variables	41,255,033	23,134,297	28,947,516	28,950,648
Constraints	17,301,689	5,734,401	30,431,232	30,431,232

Results: On Fusion cluster - larger instances, from an advanced basis

Speed-up of PIPS-S relative to 1-core PIPS-S and 1-core clp

Cores	Storm	SSN	UC12	UC24
1	1	1	1	1
8	15	19	7	6
16	52	45	14	12
32	117	103	26	22
64	152	181	44	41
128	202	289	60	64
256	285	383	70	80
clp	299	45	67	68

Results: On Blue Gene supercomputer - very large instance

- Instance of `UC12`
  - 8,192 scenarios
  - 463,113,276 variables
  - 486,899,712 constraints
- Requires 1 TB of RAM
  $\geq$ 1024 Blue Gene cores
- Runs from an advanced basis

Cores	Iterations	Time (h)	Iter/sec
1024	Exceeded execution time limit		
2048	82,638	6.14	3.74
4096	75,732	5.03	4.18
8192	86,439	4.67	5.14

High performance simplex solvers: Conclusions

- Use the dual simplex method
- Exploit hyper-sparsity
- Two parallel schemes for general LP problems
  - Meaningful performance improvement
  - Have led to publicised advances in a leading commercial solver
- One parallel scheme for stochastic LP problems
  - Demonstrated scalable parallel performance... for highly specialised problems... on highly specialised machines
  - Solved problems which would be intractable using commercial serial solvers
- Helped develop two really talented young researchers: Qi Huangfu and Miles Lubin

Slides: http://www.maths.ed.ac.uk/hall/Google15/

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