High performance numerical linear algebra for the revised simplex method

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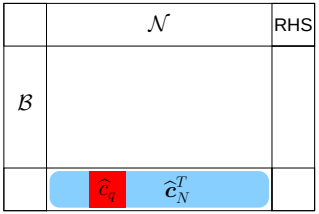
Overview

- Primal revised simplex method
  - Solving linear systems (FTRAN and BTRAN) with focus on exploiting hyper-sparsity
  - Exploiting $\bar{B} = B + (\boldsymbol{a}_q - B\boldsymbol{e}_p)\boldsymbol{e}_p^T$ (UPDATE-BASIS)
    - Traditional techniques
    - Novel techniques
- Exploiting parallelism
  - Background
  - Data parallelism by exploiting LP structure
  - Task parallelism for general LP problems
- Conclusions

The simplex algorithm: Choosing a column

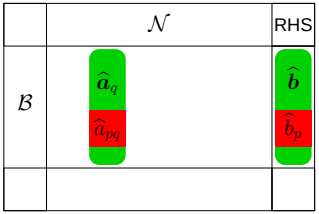


Primal algorithm: Assume $\widehat{\boldsymbol{b}} \geq \boldsymbol{0}$ Seek $\widehat{\boldsymbol{c}}_N \geq \boldsymbol{0}$

Scan $\widehat{c}_j < 0$, $i \in \mathcal{N}$, for a good candidate q to leave $\mathcal{N}$ CHUZC

Scan $\widehat{b}_i / \widehat{a}_{iq} > 0$, $i \in \mathcal{B}$, for a good candidate p to leave $\mathcal{B}$ CHUZR

The simplex algorithm: Choosing a row



Primal algorithm: Assume $\widehat{\boldsymbol{b}} \geq \mathbf{0}$ Seek $\widehat{\boldsymbol{c}}_N \geq \mathbf{0}$

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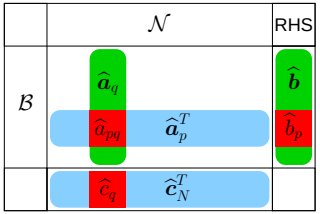
Scan $\widehat{b}_i/\widehat{a}_{iq} > 0$, $i \in \mathcal{B}$, for a good candidate p to leave $\mathcal{B}$ CHUZR

The simplex algorithm: Update



Update: Exchange p and q between $\mathcal{B}$ and $\mathcal{N}$

Update $\widehat{\boldsymbol{b}} := \widehat{\boldsymbol{b}} - \alpha_P \widehat{\boldsymbol{a}}_q$ $\quad \alpha_P = \widehat{b}_p / \widehat{a}_{pq}$ UPDATE-PRIMAL

Update $\widehat{\boldsymbol{c}}_N^T := \widehat{\boldsymbol{c}}_N^T + \alpha_D \widehat{\boldsymbol{a}}_p^T$ $\quad \alpha_D = -\widehat{c}_q / \widehat{a}_{pq}$ UPDATE-DUAL

Data required

- Pivotal row $\widehat{\boldsymbol{a}}_p^T = \boldsymbol{e}_p^T B^{-1} N$
- Pivotal column $\widehat{\boldsymbol{a}}_q = B^{-1} \boldsymbol{a}_q$

Why does it work?

Objective improves by $\left| \dfrac{\widehat{b}_p \times \widehat{c}_q}{\widehat{a}_{pq}} \right|$ each iteration

Simplex method: Computation

Standard simplex method (SSM): Major computational component

	$\mathcal{N}$	RHS
$\mathcal{B}$	$\widehat{N}$	$\widehat{\boldsymbol{b}}$
	$\widehat{\boldsymbol{c}}_N^T$	

Update of tableau: $\widehat{N} := \widehat{N} - \frac{1}{\widehat{a}_{pq}} \widehat{\boldsymbol{a}}_q \widehat{\boldsymbol{a}}_p^T$

where $\widehat{N} = B^{-1}N$

- Hopelessly inefficient for sparse LP problems
- Prohibitively expensive for large LP problems

Revised simplex method (RSM): Major computational components

Pivotal row via $B^T\boldsymbol{\pi}_p = \boldsymbol{e}_p$ BTRAN and $\widehat{\boldsymbol{a}}_p^T = \boldsymbol{\pi}_p^T N$ PRICE

Pivotal column via $B\,\widehat{\boldsymbol{a}}_q = \boldsymbol{a}_q$ FTRAN Represent B^{-1} INVERT

Simplex method: Hyper-sparsity

Recall: major computational components

- BTRAN: Form $\boldsymbol{\pi}_p = B^{-T}\boldsymbol{e}_p$
- PRICE: Form $\widehat{\boldsymbol{a}}_p^T = \boldsymbol{\pi}_p^T N$
- FTRAN: Form $\widehat{\boldsymbol{a}}_q = B^{-1}\boldsymbol{a}_q$

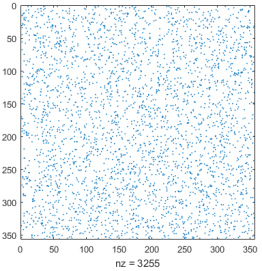
Phenomenon of hyper-sparsity

- Vectors $\boldsymbol{e}_p$ and $\boldsymbol{a}_q$ are sparse
- Results $\boldsymbol{\pi}_p$, $\widehat{\boldsymbol{a}}_p^T$ and $\widehat{\boldsymbol{a}}_q$ may be sparse—because B^{-1} is sparse
  - In BTRAN, $\boldsymbol{\pi}_p$ is a row of B^{-1}
  - In PRICE, $\widehat{\boldsymbol{a}}_p^T$ is a linear combination of a few rows of N
  - In FTRAN, $\widehat{\boldsymbol{a}}_q$ is a linear combination of a few columns of B^{-1}

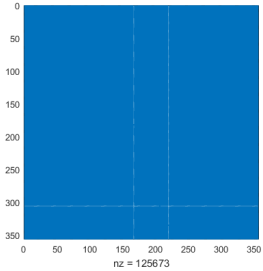
Simplex method: Hyper-sparsity

Simplex method: Inverse of a sparse matrix and solution of $B\boldsymbol{x} = \boldsymbol{b}$

Random sparse matrix B
$m = 356$ and density 2.5%



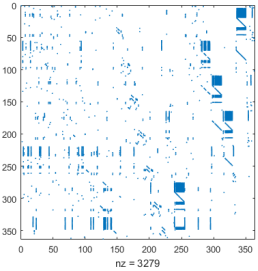
B^{-1} has density of 99%



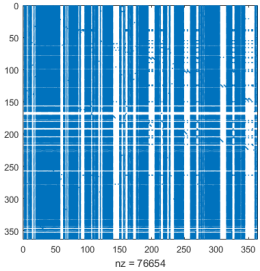
Simplex method: Hyper-sparsity

Simplex method: Inverse of a sparse matrix and solution of $B\boldsymbol{x} = \boldsymbol{b}$

Optimal B for LP problem `stair`
$m = 356$ and density 2.5%



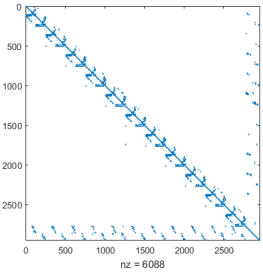
B^{-1} has density of 58%
$B^{-1}\boldsymbol{b}$ for sparse $\boldsymbol{b}$ is typically dense



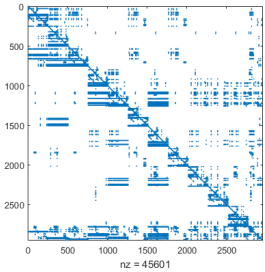
Simplex method: Hyper-sparsity

Simplex method: Inverse of a sparse matrix and solution of $B\boldsymbol{x} = \boldsymbol{b}$

Optimal B for LP problem pds-02
$m = 2953$ and density 0.07%



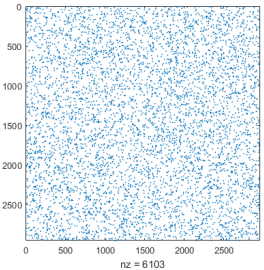
B^{-1} has density of 0.52%
$B^{-1}\boldsymbol{b}$ for sparse $\boldsymbol{b}$ is typically sparse



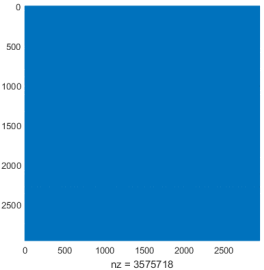
Simplex method: Hyper-sparsity

Simplex method: Inverse of a sparse matrix and solution of $B\boldsymbol{x} = \boldsymbol{b}$

Random matrix
$m = 2953$ and density 0.07%



B^{-1} has density of 100%



Simplex method: Hyper-sparsity

- Use solution of $\boldsymbol{Lx} = \boldsymbol{b}$
  - To illustrate the phenomenon of hyper-sparsity
  - To demonstrate how to exploit hyper-sparsity
- Apply principles to other triangular solves in the simplex method

Simplex method: Hyper-sparsity

Recall: Solve $L\boldsymbol{x} = \boldsymbol{b}$ using

function `ftranL`(L, $\boldsymbol{b}$, $\boldsymbol{x}$)

$\boldsymbol{r} = \boldsymbol{b}$
for all $j \in \{1, \ldots, m\}$ **do**
  for all $i : L_{ij} \neq 0$ **do**
    $r_i = r_i - L_{ij} r_j$
$\boldsymbol{x} = \boldsymbol{r}$

When $\boldsymbol{b}$ is **sparse**

- Inefficient until $\boldsymbol{r}$ fills in

Simplex method: Hyper-sparsity

Better: Check r_j for zero

function `ftranL`(L, $\boldsymbol{b}$, $\boldsymbol{x}$)
  $\boldsymbol{r} = \boldsymbol{b}$
  for all $j \in \{1, \ldots, m\}$ **do**
    if $r_j \neq 0$ **then**
      for all $i : L_{ij} \neq 0$ **do**
        $r_i = r_i - L_{ij} r_j$
  $\boldsymbol{x} = \boldsymbol{r}$

When $\boldsymbol{x}$ is **sparse**

- Few values of r_j are nonzero
- Check for zero dominates
- Requires more efficient identification of set $\mathcal{X}$ of indices j such that $r_j \neq 0$

Gilbert and Peierls (1988)
H and McKinnon (1998–2005)
COAP best paper prize (2005)

Simplex method: UPDATE-BASIS

- **Each iteration:**
  - Solve $B^T \boldsymbol{\pi}_p = \boldsymbol{e}_p$
  - Solve $B \widehat{\boldsymbol{a}}_q = \boldsymbol{a}_q$
  - Column p of B replaced by $\boldsymbol{a}_q$ to give $\bar{B} = B + (\boldsymbol{a}_q - B\boldsymbol{e}_p)\boldsymbol{e}_p^T$
- **Sparsity-exploiting decomposition:** $PBQ = LU$
- **Challenge:** Solve systems involving $\bar{B}$ at minimal cost

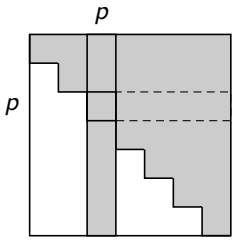
Simplex method: Product form update (PFI) for B

- Given
$$\bar{B} = B + (\boldsymbol{a}_q - B\boldsymbol{e}_p)\boldsymbol{e}_p^T$$
- Take B out as a factor on the left
$$\bar{B} \;=\; B[I + (B^{-1}\boldsymbol{a}_q - \boldsymbol{e}_p)\boldsymbol{e}_p^T] \;=\; BE$$
$$\text{where} \quad E \;=\; I + (\widehat{\boldsymbol{a}}_q - \boldsymbol{e}_p)\boldsymbol{e}_p^T \;=\; \begin{bmatrix} 1 & & \eta_1 & & \\ & \ddots & \vdots & & \\ & & \mu & & \\ & & \vdots & \ddots & \\ & & \eta_m & & 1 \end{bmatrix}$$
$\mu = \widehat{a}_{pq}$ is the **pivot**; remaining entries in $\widehat{\boldsymbol{a}}_q$ form the **eta vector** $\boldsymbol{\eta}$
- Can solve $\bar{B}\boldsymbol{x} = \boldsymbol{r}$ as $B\boldsymbol{x} = \boldsymbol{r}$ then $\boldsymbol{x} := E^{-1}\boldsymbol{x}$ as
$$x_p := x_p/\mu \qquad \text{then} \qquad \boldsymbol{x} := \boldsymbol{x} - x_p\boldsymbol{\eta}$$

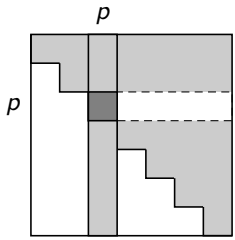
Dantzig and Orchard-Hays (1954)

Simplex method: Forrest-Tomlin update (FT) for B

- Given
$$\bar{B} = B + (\boldsymbol{a}_q - B\boldsymbol{e}_p)\boldsymbol{e}_p^T \quad \text{where (wlog)} \quad B = LU$$
- Multiply $\bar{B}$ by L^{-1} to give
$$L^{-1}\bar{B} \;=\; U + (L^{-1}\boldsymbol{a}_q - U\boldsymbol{e}_p)\boldsymbol{e}_p^T \;=\; U + (\widetilde{\boldsymbol{a}}_q - \boldsymbol{u}_p)\boldsymbol{e}_p^T \;=\; U' \qquad \text{(a)}$$
- Eliminate entries in row p to give $R^{-1}U' = \bar{U}$ (b)



(a) Spiked upper: U'



(b) After elimination: $\bar{U}$

- Yields $\bar{B} = LR\bar{U}$
- Compute $\widetilde{\boldsymbol{a}}_q$ when forming $\widehat{\boldsymbol{a}}_q$
- Represent R like E
- FT more efficient than PFI with respect to sparsity

Forrest and Tomlin (1972)

Simplex method: Multiple updates

- Suppose $B_0 = L_0 U_0$ and k updates are performed to obtain B_k
  - Pivot in rows $\{p_i\}_{i=1}^{k}$
  - Introduce columns $\{\boldsymbol{a}_{q_i}\}_{i=1}^{k}$
- PFI generalizes as

$$B_k = B_0 E_1 E_2 \dots E_k$$

- FT generalizes as

$$B_k = L_0 R_1 R_2 \dots R_k U_k$$

Eventually more computationally efficient or numerically prudent to reinvert some B_k

Novel update techniques for the revised simplex method: 1

Alternative product form update (APF)

- **Recall:** Column p of B is replaced by $\boldsymbol{a}_q$ to give $\bar{B} = B + (\boldsymbol{a}_q - B\boldsymbol{e}_p)\boldsymbol{e}_p^T$
  - Traditional PFI takes B out as a factor on the left so $\bar{B} = BE$
- **Idea:** Why not take it out on the right!

$$\begin{aligned} \bar{B} &= [I + (\boldsymbol{a}_q - B\boldsymbol{e}_p)\boldsymbol{e}_p^T B^{-1}]B &= TB \\ \text{where} \quad T &= I + (\boldsymbol{a}_q - \boldsymbol{a}_{p'})\widehat{\boldsymbol{e}}_p^T \end{aligned}$$

- T is formed of known data and readily invertible (like E for PFI)
  Naturally compute $\widehat{\boldsymbol{e}}_p$ when solving $B^T\boldsymbol{\pi}_p = \boldsymbol{e}_p$
- **But:** Is this useful?

Novel update techniques for the revised simplex method: 2

Middle product form update (MPF)

- **Recall:** Column p of B is replaced by $\boldsymbol{a}_q$ to give $\bar{B} = B + (\boldsymbol{a}_q - B\boldsymbol{e}_p)\boldsymbol{e}_p^T$
- **Idea:** Substitute $B = LU$ and take factors L on the left and U on the right!

$$\begin{aligned}
\bar{B} &= LU + (\boldsymbol{a}_q - B\boldsymbol{e}_p)\boldsymbol{e}_p^T \\
&= LU + LL^{-1}(\boldsymbol{a}_q - B\boldsymbol{e}_p)\boldsymbol{e}_p^T U^{-1}U \\
&= L[I + (\widetilde{\boldsymbol{a}}_q - U\boldsymbol{e}_p)\widetilde{\boldsymbol{e}}_p^T]U \\
&= LMU \qquad \text{where} \quad M = I + (\widetilde{\boldsymbol{a}}_q - \boldsymbol{u}_p)\widetilde{\boldsymbol{e}}_p^T
\end{aligned}$$

- M is formed of known data and readily invertible (like E for PFI)
  Naturally compute $\widetilde{\boldsymbol{a}}_q$ when solving $B\,\widehat{\boldsymbol{a}}_q = \boldsymbol{a}_q$ and $\widetilde{\boldsymbol{e}}_p$ when solving $B^T\boldsymbol{\pi}_p = \boldsymbol{e}_p$
- **But:** Is this useful?

Novel update techniques for the revised simplex method: 3

Forrest-Tomlin update

- **Recall:** Column p of B is replaced by $\boldsymbol{a}_q$ to give $\bar{B} = B + (\boldsymbol{a}_q - B\boldsymbol{e}_p)\boldsymbol{e}_p^T$
- **Idea:** Substitute $B = LU$ and take factor L on the left

$$\bar{B} \quad = \quad L[U + (L^{-1}\boldsymbol{a}_q - U\boldsymbol{e}_p)\boldsymbol{e}_p^T] \quad = \quad LU'$$

where $U' = U + (\widetilde{\boldsymbol{a}}_q - \boldsymbol{u}_p)\boldsymbol{e}_p^T$ is "spiked" upper triangular and then eliminate

Collective Forrest-Tomlin (CFT) update

- Update Forrest-Tomlin representation of B after multiple basis changes
- Don't have data to perform a sequence of standard FT updates
- Have to perform elimination corresponding to multiple spikes
- **But:** Is this useful?

Huangfu and H (2013)

Novel update techniques for the revised simplex method: Multiple updates

- Suppose $B_0 = L_0 U_0$ and k updates are performed to obtain B_k
- **Recall:** PFI generalizes as $B_k = B_0 E_1 E_2 \dots E_k$
- **Recall:** FT generalizes as $B_k = L_0 R_1 R_2 \dots R_k U_k$
- APF generalizes as

$$B_k = T_k \dots T_2 T_1 B$$

- MPF generalizes as

$$B_k = L_0 M_1 M_2 \dots M_k U_0$$

Novel update techniques for the revised simplex method: Results

- Test environment
  - 30 representative LP problems from standard test sets
  - Same sequence of basis changes for all update techniques
- Geometric mean time for operations to form and use update data

Update	PFI	FT	APF	MPF	CFT
Mean	1.00	0.30	0.95	0.45	0.29

Conclusions:

- FT much better than PFI
- APF little better than PFI
- MPF closer to FT than PFI
- CFT as good as FT

Parallelising the revised simplex method

Parallelising the simplex method

History

- SSM: Good parallel efficiency of $\widehat{N} := \widehat{N} - \frac{1}{\widehat{a}_{pq}} \widehat{\boldsymbol{a}}_q \widehat{\boldsymbol{a}}_p^T$ *was* achieved
  Many! (1988–date)
- Only parallel *revised* simplex is worthwhile: goal of H and McKinnon in 1992!

Parallel primal revised simplex method

- Overlap computational components for different iterations
  Wunderling (1996), H and McKinnon (1995-2005)
- Modest speed-up *was* achieved on general sparse LP problems

Parallel dual revised simplex method

- Only immediate parallelism is in forming $\boldsymbol{\pi}_p^T N$
- When $n \gg m$ significant speed-up *was* achieved Bixby and Martin (2000)

Parallelising the simplex method: Matrix-vector product $\boldsymbol{\pi}_p^T N$ (PRICE)

- **In theory:**
  - Partition $N = \begin{bmatrix} N_1 & N_2 & \dots & N_P \end{bmatrix}$, where P is the number of processes
  - Compute $\boldsymbol{\pi}_p^T N_j$ on process j for $j = 1, 2, \dots P$
  - Execution time is reduced by a factor P: **linear speed-up**
- **In practice**
  - On a distributed memory machine with N_j on processor j
    Linear speed-up achieved
  - On a shared memory machine
    - If N_j fits into cache for process j
      Linear speed-up achieved
    - Otherwise, computation is limited by speed of reading data from memory
      Speed-up limited to number of memory channels

Data parallelism by exploiting LP structure

PIPS-S (2011–date)

Overview

- Written in C++ to solve stochastic MIP relaxations in parallel
- Dual simplex
- Based on NLA routines in `clp`
- Product form update

Concept

- Exploit data parallelism due to block structure of LPs
- Distribute problem over processes

Paper: Lubin, H, Petra and Anitescu (2013)

- COIN-OR INFORMS 2013 Cup
- COAP best paper prize (2013)

PIPS-S: Stochastic MIP problems

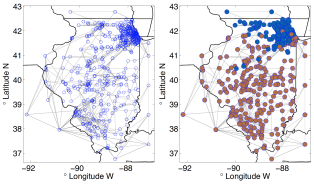
Two-stage stochastic LPs have column-linked block angular (BALP) structure

$$
\begin{array}{lccccccccc}
\text{minimize} & \boldsymbol{c}_0^T\boldsymbol{x}_0 & + & \boldsymbol{c}_1^T\boldsymbol{x}_1 & + & \boldsymbol{c}_2^T\boldsymbol{x}_2 & + & \dots & + & \boldsymbol{c}_N^T\boldsymbol{x}_N \\
\text{subject to} & A\boldsymbol{x}_0 & & & & & & & & = \boldsymbol{b}_0 \\
& T_1\boldsymbol{x}_0 & + & W_1\boldsymbol{x}_1 & & & & & & = \boldsymbol{b}_1 \\
& T_2\boldsymbol{x}_0 & & & + & W_2\boldsymbol{x}_2 & & & & = \boldsymbol{b}_2 \\
& \vdots & & & & & \ddots & & & \vdots \\
& T_N\boldsymbol{x}_0 & & & & & & + & W_N\boldsymbol{x}_N & = \boldsymbol{b}_N \\
& \boldsymbol{x}_0 \geq \boldsymbol{0} & & \boldsymbol{x}_1 \geq \boldsymbol{0} & & \boldsymbol{x}_2 \geq \boldsymbol{0} & & \dots & & \boldsymbol{x}_N \geq \boldsymbol{0}
\end{array}
$$

- Variables $\boldsymbol{x}_0 \in \mathbb{R}^{n_0}$ are **first stage** decisions
- Variables $\boldsymbol{x}_i \in \mathbb{R}^{n_i}$ for $i = 1, \dots, N$ are **second stage** decisions
  Each corresponds to a **scenario** which occurs with modelled probability
- The objective is the expected cost of the decisions
- In stochastic MIP problems, some/all decisions are discrete

PIPS-S: Stochastic MIP problems

- Power systems optimization project at Argonne
- Integer second-stage decisions
- Stochasticity from wind generation
- Solution via branch-and-bound
  - Solve root using parallel IPM solver PIPS
    Lubin, Petra *et al.* (2011)
  - Solve nodes using parallel dual simplex solver PIPS-S



PIPS-S: Stochastic MIP problems

Convenient to permute the LP thus:

$$
\begin{array}{lccccccccccc}
\text{minimize} & \boldsymbol{c}_1^T\boldsymbol{x}_1 & + & \boldsymbol{c}_2^T\boldsymbol{x}_2 & + & \ldots & + & \boldsymbol{c}_N^T\boldsymbol{x}_N & + & \boldsymbol{c}_0^T\boldsymbol{x}_0 & & \\
\text{subject to} & W_1\boldsymbol{x}_1 & & & & & & & + & T_1\boldsymbol{x}_0 & = & \boldsymbol{b}_1 \\
 & & & W_2\boldsymbol{x}_2 & & & & & + & T_2\boldsymbol{x}_0 & = & \boldsymbol{b}_2 \\
 & & & & & \ddots & & & \vdots & & & \vdots \\
 & & & & & & & W_N\boldsymbol{x}_N & + & T_N\boldsymbol{x}_0 & = & \boldsymbol{b}_N \\
 & & & & & & & & & A\boldsymbol{x}_0 & = & \boldsymbol{b}_0 \\
 & \boldsymbol{x}_1 \geq \boldsymbol{0} & & \boldsymbol{x}_2 \geq \boldsymbol{0} & & \ldots & & \boldsymbol{x}_N \geq \boldsymbol{0} & & \boldsymbol{x}_0 \geq \boldsymbol{0} & &
\end{array}
$$

PIPS-S: Exploiting problem structure

- Inversion of the basis matrix B is key to revised simplex efficiency
- For column-linked BALP problems

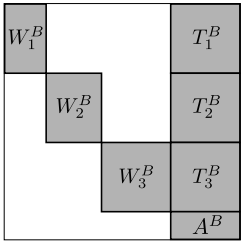
$$B = \begin{bmatrix} W_1^B & & & T_1^B \\ & \ddots & & \vdots \\ & & W_N^B & T_N^B \\ & & & A^B \end{bmatrix}$$

- W_i^B are columns corresponding to n_i^B basic variables in scenario i
- $\begin{bmatrix} T_1^B \\ \vdots \\ T_N^B \\ A^B \end{bmatrix}$ are columns corresponding to n_0^B basic first stage decisions

PIPS-S: Exploiting problem structure

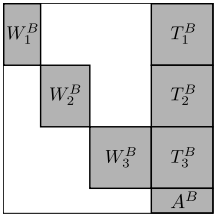
- Inversion of the basis matrix B is key to revised simplex efficiency
- For column-linked BALP problems

$$B = \begin{bmatrix} W_1^B & & & T_1^B \\ & \ddots & & \vdots \\ & & W_N^B & T_N^B \\ & & & A^B \end{bmatrix}$$

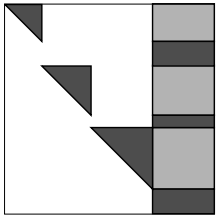


- B is nonsingular so
  - W_i^B are "tall": full column rank
  - $\begin{bmatrix} W_i^B & T_i^B \end{bmatrix}$ are "wide": full row rank
  - A^B is "wide": full row rank
- Scope for parallel inversion is immediate and well known

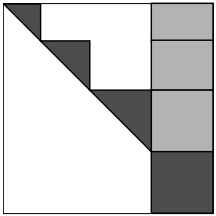
PIPS-S: Exploiting problem structure



- Eliminate sub-diagonal entries in each W_i^B (independently)



- Apply elimination operations to each T_i^B (independently)



- Accumulate non-pivoted rows from the W_i^B with A^B and complete elimination

PIPS-S: Overview

Scope for parallelism

- Parallel Gaussian elimination yields **block LU** decomposition of B
- Scope for parallelism in block forward and block backward substitution
- Scope for parallelism in `PRICE`

Implementation

- Distribute problem data over processes
- Perform data-parallel `BTRAN`, `FTRAN` and `PRICE` over processes
- Used MPI

PIPS-S: Results

On Fusion cluster: Performance relative to clp

Dimension	Cores	Storm	SSN	UC12	UC24
$m + n = O(10^6)$	1	0.34	0.22	0.17	0.08
	32	8.5	6.5	2.4	0.7
$m + n = O(10^7)$	256	299	45	67	68

On Blue Gene

- Instance of UC12
- $m + n = O(10^8)$
- Requires 1 TB of RAM
- Runs from an advanced basis

Cores	Iterations	Time (h)	Iter/sec
1024	Exceeded execution time limit		
2048	82,638	6.14	3.74
4096	75,732	5.03	4.18
8192	86,439	4.67	5.14

Task parallelism for general LP problems

$h_2$gmp (2011–date)

Overview

- Written in C++ to study parallel simplex
- Dual simplex with steepest edge and BFRT
- Forrest-Tomlin update
  - complex and inherently serial
  - efficient and numerically stable

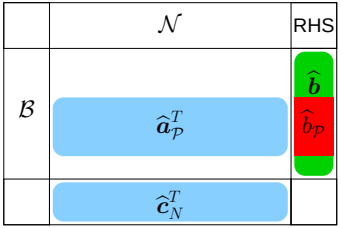
Concept

- Exploit limited task and data parallelism in standard dual RSM iterations (`sip`)
- Exploit greater task and data parallelism via minor iterations of dual SSM (`pami`)
- Test-bed for research
- Work-horse for consultancy

Huangfu, H and Galabova (2011–date)

h2gmp: Multiple iteration parallelism with pami option

- Perform standard dual simplex minor iterations for rows in set $\mathcal{P}$ ($|\mathcal{P}| \ll m$)
- Suggested by Rosander (1975) but never implemented efficiently *in serial*



- Task-parallel multiple BTRAN to form $\boldsymbol{\pi}_{\mathcal{P}} = B^{-1}\boldsymbol{e}_{\mathcal{P}}$
- Data-parallel PRICE to form $\widehat{\boldsymbol{a}}_{p}^{T}$ (as required)
- Task-parallel multiple FTRAN for primal, dual and weight updates

Huangfu and H (2011–2014)
COAP best paper prize (2015)

Novel update procedures: are they useful?

Update 1: Alternative product form update (APF)

- **Recall:** $\bar{B} = TB$ with T easily invertible
- Used to get $\boldsymbol{\pi}_{\mathcal{P}} = \bar{B}^{-T}\boldsymbol{e}_{\mathcal{P}}$ from $B^{-T}\boldsymbol{e}_{\mathcal{P}}$ in `pami`
- Used to compute multiple $B^{-1}\boldsymbol{a}_F$ efficiently after multiple BFRT in `pami`

Update 2: Middle product form update (MPF)

- **Recall:** $\bar{B} = LMU$ with M easily invertible
- Not used by `pami`!
- Used by Google in `glop`

Update 3: Multiple Forrest-Tomlin update

Used to perform multiple Forrest-Tomlin updates after minor iterations in `pami`

h2gmp: Performance and reliability

Extended testing using 159 test problems

- 98 Netlib
- 16 Kennington
- 4 Industrial
- 41 Mittelmann

Exclude 7 which are "hard"

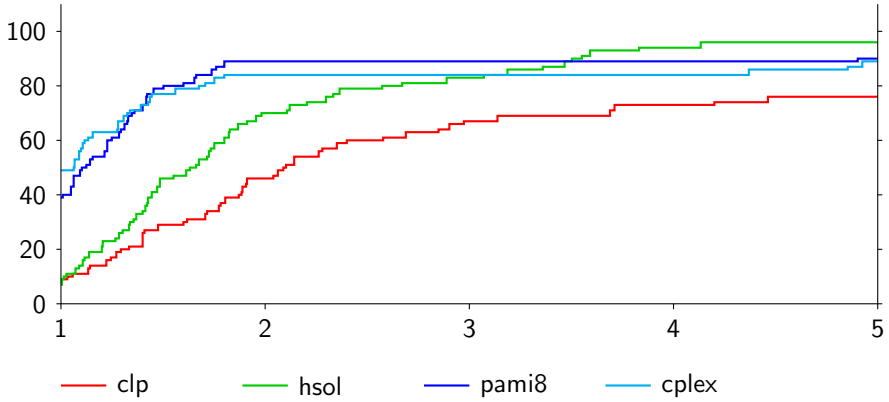
Performance

Benchmark against `clp` (v1.16) and `cplex` (v12.5)

- Dual simplex
- No presolve
- No crash

Ignore results for 82 LPs with minimum solution time below 0.1s

h2gmp: Performance



Conclusions

High performance NLA is essential for the revised simplex method

- Hyper-sparsity: advance in serial simplex
- Stochastic LP: large scale data-parallelism for special problems
- Novel updates: underpin open-source parallel simplex solver

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Slides: http://www.maths.ed.ac.uk/hall/NLAPDEO17/