Linear Programming and Numerical Analysis
MATH08037
Lecture 3
Ratio constraints, multidimensional LP problems and vertex enumeration

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Ratio constraints

- Original blending problem had fixed blending proportions
- In general, suppose product P is a blend of material X and material Y
  - Let x be the amount of material X used
  - Let y be the amount of material Y used
  - Amount of P made is $x + y$
  - Proportion of X in the amount of P made is

$$\frac{x}{x+y}$$

- In practice, there would be tolerances on this proportion
  - A lower bound l
  - An upper bound u
- Hence x and y must satisfy the **nonlinear constraints**

$$l \leq \frac{x}{x+y} \leq u$$

Transforming the nonlinear constraints

- x and y must satisfy

$$l \leq \frac{x}{x+y} \qquad \text{and} \qquad \frac{x}{x+y} \leq u$$

- Multiply both inequalities by (the positive quantity) $x + y$ to give

$$l(x+y) \leq x \qquad \text{and} \qquad x \leq u(x+y)$$

- yielding the **linear constraints**

$$(l-1)x + ly \leq 0 \qquad \text{and} \qquad (1-u)x - uy \leq 0$$

Adding ratio constraints to the blending problem

- A petroleum company blends two oil fractions F_1 and F_2 to produce two products P_1 and P_2
  - P_1 is a blend of F_1 and F_2 and sells for £8/tonne
  - P_2 is a blend of F_1 and F_2 and sells for £10/tonne
- The company has 120 tonnes of F_1 and 210 tonnes of F_2
  - The proportion of F_1 in P_1 must be between 0.25 and 0.35
  - The proportion of F_2 in P_1 must be between 0.60 and 0.80
  - The proportion of F_1 in P_2 must be between 0.45 and 0.55
  - The proportion of F_2 in P_2 must be between 0.40 and 0.60
- How much of F_1 and F_2 should be used to make P_1 and P_2 in order to maximize revenue?

Let x_1 be the amount of F_1 used to make P_1 (tonnes)

Let x_2 be the amount of F_2 used to make P_1 (tonnes)

Let x_3 be the amount of F_1 used to make P_2 (tonnes)

Let x_4 be the amount of F_2 used to make P_2 (tonnes)

Blending problem with ratio constraints: objective and availability constraints

- Total amount of P_1 made is $x_1 + x_2$
- Total amount of P_2 made is $x_3 + x_4$
- Objective is to maximize total revenue

$$f = 8(x_1 + x_2) + 10(x_3 + x_4)$$

- Total amount of F_1 used is $x_1 + x_3$ and must not exceed availability

$$x_1 + x_3 \leq 120$$

- Total amount of F_2 used is $x_2 + x_4$ and must not exceed availability

$$x_2 + x_4 \leq 210$$

Blending problem: ratio constraints

- The proportion of F_1 in P_1 must be between 0.25 and 0.35
- **Recall:**
  - The amount of F_1 used to make P_1 is x_1
  - The amount of P_1 made is $x_1 + x_2$
- Hence

$$0.25 \leq \frac{x_1}{x_1 + x_2} \leq 0.35$$

  so

$$0.25(x_1 + x_2) \leq x_1 \qquad \Rightarrow \qquad -0.75x_1 + 0.25x_2 \leq 0$$

  and

$$x_1 \leq 0.35(x_1 + x_2) \qquad \Rightarrow \qquad 0.65x_1 - 0.35x_2 \leq 0$$

- Six more constraints derived similarly

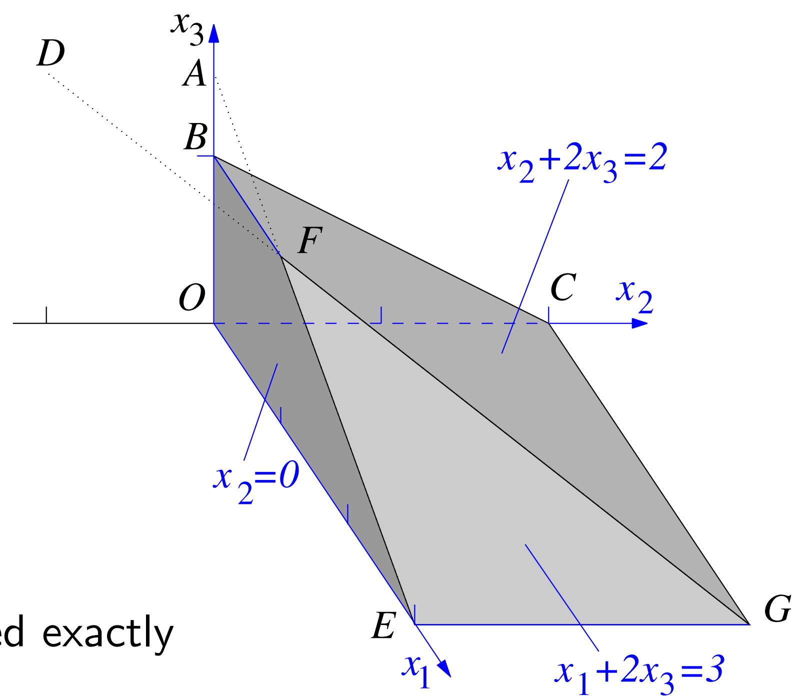
Blending problem with ratio constraints: the complete LP

$$
\begin{array}{llrcrcrcrcr}
\text{maximize} & f = & 8x_1 & + & 8x_2 & + & 10x_3 & + & 10x_4 & & \\
\text{subject to} & & x_1 & & & + & x_3 & & & \leq & 120 \\
& & & & x_2 & & & + & x_4 & \leq & 210 \\
& & -0.75x_1 & + & 0.25x_2 & & & & & \leq & 0 \\
& & 0.65x_1 & - & 0.35x_2 & & & & & \leq & 0 \\
& & 0.6x_1 & - & 0.4x_2 & & & & & \leq & 0 \\
& & -0.8x_1 & + & 0.2x_2 & & & & & \leq & 0 \\
& & & & & - & 0.55x_3 & + & 0.45x_4 & \leq & 0 \\
& & & & & & 0.45x_3 & - & 0.55x_4 & \leq & 0 \\
& & & & & & 0.4x_3 & - & 0.6x_4 & \leq & 0 \\
& & & & & - & 0.6x_3 & + & 0.4x_4 & \leq & 0 \\
& & & & & & \multicolumn{5}{r}{x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \quad x_4 \geq 0}
\end{array}
$$

- **4** variables and **10** constraints
- Problem is larger and has **structure**

A 3-dimensional LP problem

$$
\begin{array}{llcrcrcrcc}
\text{maximize} & f & = & x_1 & + & 2x_2 & + & 3x_3 & & \\
\text{subject to} & & & x_1 & & & + & 2x_3 & \leq & 3 \\
& & & & & x_2 & + & 2x_3 & \leq & 2 \\
& & & \multicolumn{5}{c}{x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0.} & &
\end{array}
$$



- Feasible region is the **polyhedron** $OBCEFG$
- At each vertex there are 3 constraints/bounds satisfied exactly

General LP problems

A general LP problem is in the form

$$
\begin{array}{lccccccccccc}
\text{maximize} & f & = & c_1x_1 & + & c_2x_2 & + & \dots & + & c_nx_n & & \\
\text{subject to} & & & a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & \le & b_1 \\
 & & & a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & \le & b_2 \\
 & & & & \vdots & & \vdots & & \vdots & & \vdots & \\
 & & & a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & \le & b_m \\
 & & & \multicolumn{9}{l}{x_1 \ge 0, \quad x_2 \ge 0, \quad \dots \quad x_n \ge 0}
\end{array}
$$

- There are n variables and m constraints
- The feasible region is a convex **polyhedron**
- At each vertex there are n constraints/bounds satisfied exactly

Solving general LP problems

- The graphical method of solution cannot be applied to problems with more than 3 variables
- For all the problems solved graphically there is an optimal solution at a **vertex**
  - For an LP in 2 variables, a vertex is the intersection of two constraint/bound lines
  - For a general LP (with n variables) a vertex is the intersection of n constraint/bound (hyper-)planes
- **Observation 1**

  If an LP problem has an optimal solution then there is an optimal solution at a vertex

- This can be proved... but we won't
- **Observation 2**

  If all the vertices are known then the one with the best objective value is optimal

- This yields the solution method known as **vertex enumeration**

Slack variables

- Vertices are identified systematically by introducing **slack variables**
- A slack variable transforms an inequality into
  - an equation
  - a bound on the slack variable
- For example

$$x_1 + 2x_2 \leq 3$$

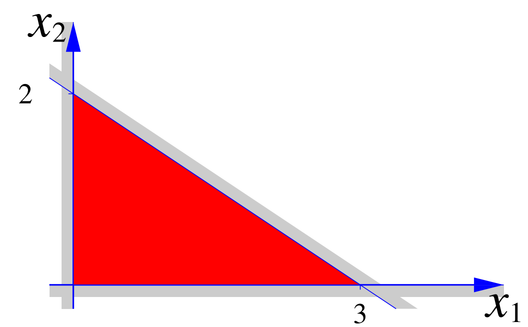
  is equivalent to

$$x_1 + 2x_2 + s = 3 \quad \text{and} \quad s \geq 0$$

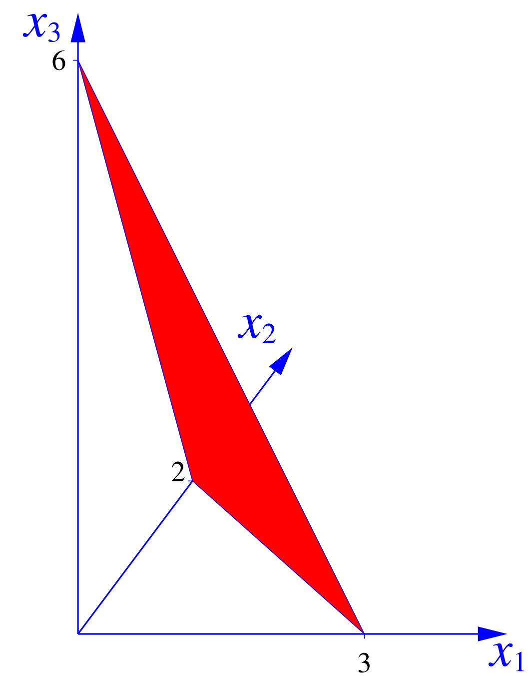
- For an LP problem with n (original) variables, the slack variables are x_{n+1}, x_{n+2}, $\ldots$

Geometric view of slack variables

For the simple LP problem



$$\begin{array}{lcccccc} \text{maximize} & f & = & 3x_1 & + & 5x_2 & \\ \text{subject to} & & & 2x_1 & + & 3x_2 & \leq & 6 \\ & & & x_1 \geq 0 & \text{and} & x_2 \geq 0 & \end{array}$$



After adding slack variable x_3, the problem in standard form is

$$\begin{array}{lcccccccc} \text{maximize} & f & = & 3x_1 & + & 5x_2 & & & \\ \text{subject to} & & & 2x_1 & + & 3x_2 & + & x_3 & = & 6 \\ & & & x_1 \geq 0, & & x_2 \geq 0 & \text{and} & x_3 \geq 0 & \end{array}$$

Slack variables for the blending problem

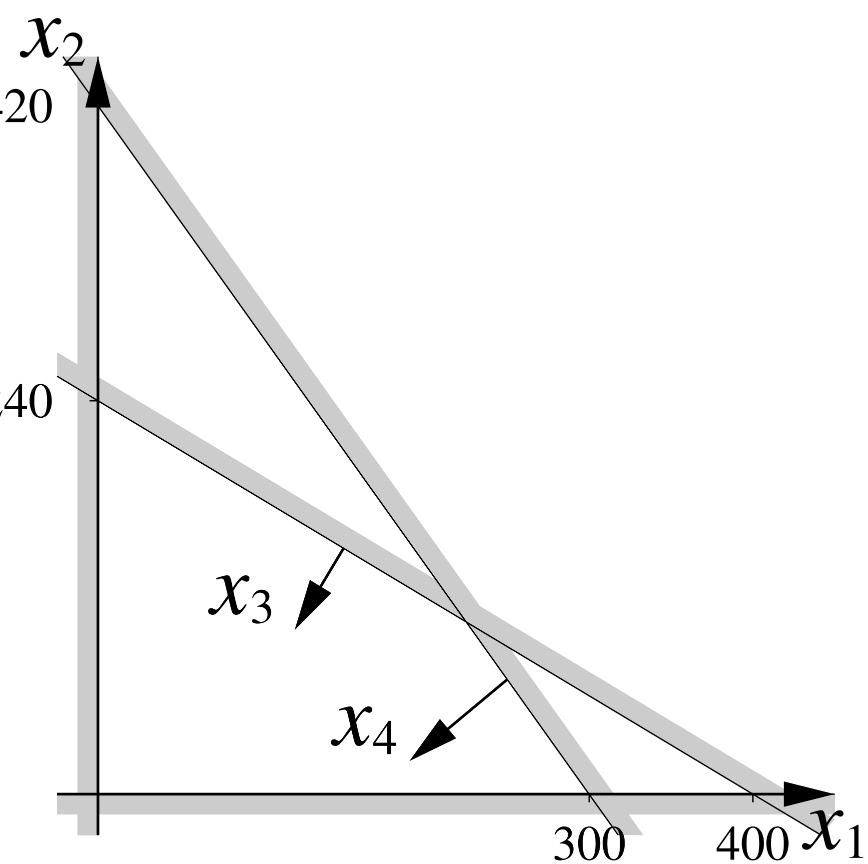
- For the blending problem

$$
\begin{array}{llcrcrcr}
\text{maximize} & f & = & 8x_1 & + & 10x_2 & & \\
\text{subject to} & & & 0.3x_1 & + & 0.5x_2 & \leq & 120 \\
 & & & 0.7x_1 & + & 0.5x_2 & \leq & 210 \\
 & & & & & x_1 \geq 0 & & x_2 \geq 0
\end{array}
$$

- Introduce slack variables x_3 and x_4 to transform the problem into **standard form**

$$
\begin{array}{llcrcrcrcrcr}
\text{maximize} & f & = & 8x_1 & + & 10x_2 & & & & & & \\
\text{subject to} & & & 0.3x_1 & + & 0.5x_2 & + & x_3 & & & = & 120 \\
 & & & 0.7x_1 & + & 0.5x_2 & & & + & x_4 & = & 210 \\
 & & & & & x_1 \geq 0 & & x_2 \geq 0 & & x_3 \geq 0 & & x_4 \geq 0
\end{array}
$$

Graphical interpretation of slack variables



Slack variables measure how far a point is from the constraint line