Xpress Non-Linear Solvers

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FICO

Xpress Solver Development Team

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  Mixed Integer
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    Interior Point Solver
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    Continuous Optimization
  - Pietro Belotti
    Mixed Integer Non-Linear
  - Qi Huangfu
    Dual Simplex
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    Mixed Integer
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FICO Snapshot

Profile	The leader in predictive analytics for decision management Founded: 1956 NYSE: FICO Annual Global Revenues Approximately: $700 million
Products and Services	Scores and related analytic models Analytic Applications for risk management, fraud, marketing, mobility Tools for decision management
Clients and Markets	5,000+ clients in 80 countries Industry focus: Banking, insurance, retail, health care
Recent Rankings	#1 in services operations analytics (IDC) #6 in worldwide analytics software (IDC) #7 in Business Intelligence, CPM and Analytic Applications (Gartner) #26 in the FinTech 100 (*American Banker*)
Offices	20+ offices worldwide, HQ in San Jose, California, USA 2,400+ employees Regional Hubs: San Rafael (CA); San Diego (CA); New York; Roseville, MN; London; Birmingham (UK); Istanbul; Madrid; Munich; Sao Paulo; Bangalore; Beijing; and Singapore.

FICO Product Portfolio

For Specific Decision Processes

	Marketing	Origination	Customer Management	Collections and Recovery	Fraud Management	Mobile
Applications	FICO® Customer Dialogue Manager FICO® Analytic Offer Manager	FICO® Origination Manager	FICO® TRIAD® Customer Manager	FICO® Debt Manager™ FICO® Recovery Management System™	FICO® Falcon® Fraud Manager FICO® Insurance Fraud Manager FICO® Claims Fraud Manager	FICO® Adeptra® Fraud Resolution FICO® Adeptra® Risk Intervention Manager
Custom / Embedded Analytics	Targeting Models Time-to-Event Analytics	Consumer and Small Business Risk Models Economic Impact Models	Behavior Scorecards Transaction Analytics	Collections Scores	Consortium Fraud Models Custom Fraud Models Application Fraud Models	

For Any Decision Process

Scores	B2B: B2C:	FICO® Score • FICO® Credit Capacity Index™ • FICO® Insurance Risk Scores myFICO®
Tools	Business Rules Management: Predictive Analytics: Optimization:	FICO® Blaze Advisor® FICO® Model Builder • FICO® Model Central FICO® Optimization Modeler • FICO® Xpress • FICO® Decision Optimizer
Professional Services	Custom Analytics Operational Best Practices Strategy Design and Optimization	

FICO

A Network of Intelligence

Accelerating the Development of Ideas

CONSUME

CONTRIBUTE

Researchers

Entrepreneurs

Businesses

FICO

ISVs

FICO ANALYTIC CLOUD

Systems Integrators/
Consultants

Governments

Corporate
Developers

Academics

COLLABORATE

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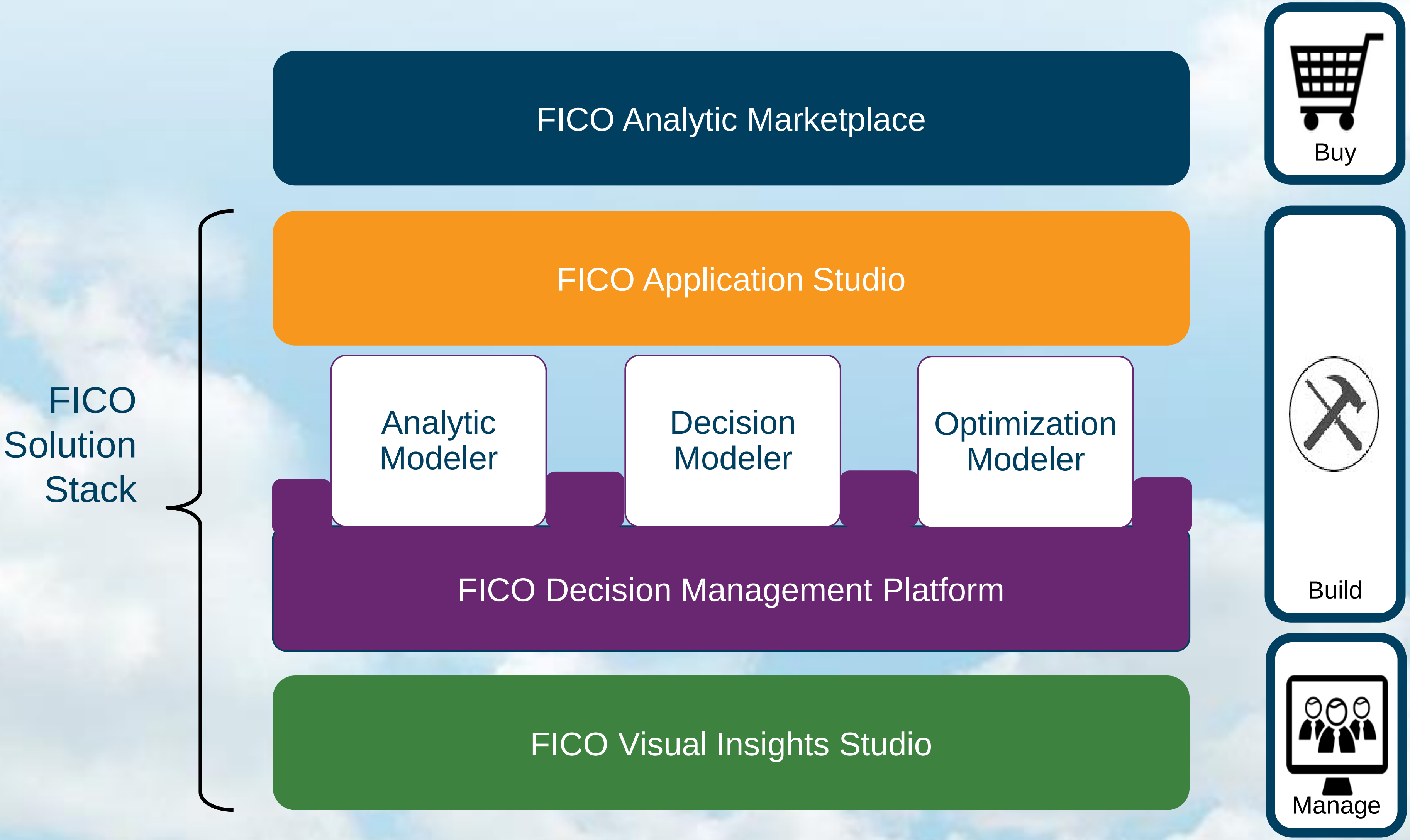
FICO Solutions on FICO Solution Stack



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History of Xpress

1983	Dash Optimization founded by Bob Daniel and Robert Ashford. Xpress-MP LP solver released.
1985	MP-Model – modelling language
1986	Mixed Integer Programming (MIP) solver added
2001	Mosel modelling language
2003	XSLP non-linear solver added
2008	Dash Optimization bought by Fair Isaac (now FICO)
2012	Xpress Insight added (now Xpress Optimization Modeller)

Xpress Optimization Suite

Applications Services **Optimization Modeler**	BENEFITS	▶ Adapt data and parameters to create and compare scenarios ▶ Understand trade-offs and sensitivities ▶ Visualize data and results for analysis ▶ Collaborate in a multi-user environment ▶ Works in a rich client and a web browser — on premise and in the cloud ▶ Fully featured APIs including web
Modeling **Mosel**	FEATURES	▶ Flexible, modular, easy-to-learn and use ▶ Development IDE ▶ Distributed modeling and cloud enablement ▶ Data connections (file, excel, databases, web services) ▶ Precompiled for efficiency and IP protection ▶ Fully featured APIs
Optimization **Optimizer NonLinear Kalis**	FEATURES	▶ High-performance, scalable and robust LP (Simplex|Barrier), MIP, QP, MIQP, QCQP, MIQCQP, SOCP, MISOCP, NLP, MINLP, and CP engines ▶ Great out-of-the-box performance — advanced users have full control over solution process ▶ Utilizes multi-core/CPU machines, automatic tuning ▶ N-best solutions capabilities and advanced infeasibility handling ▶ Fully featured APIs

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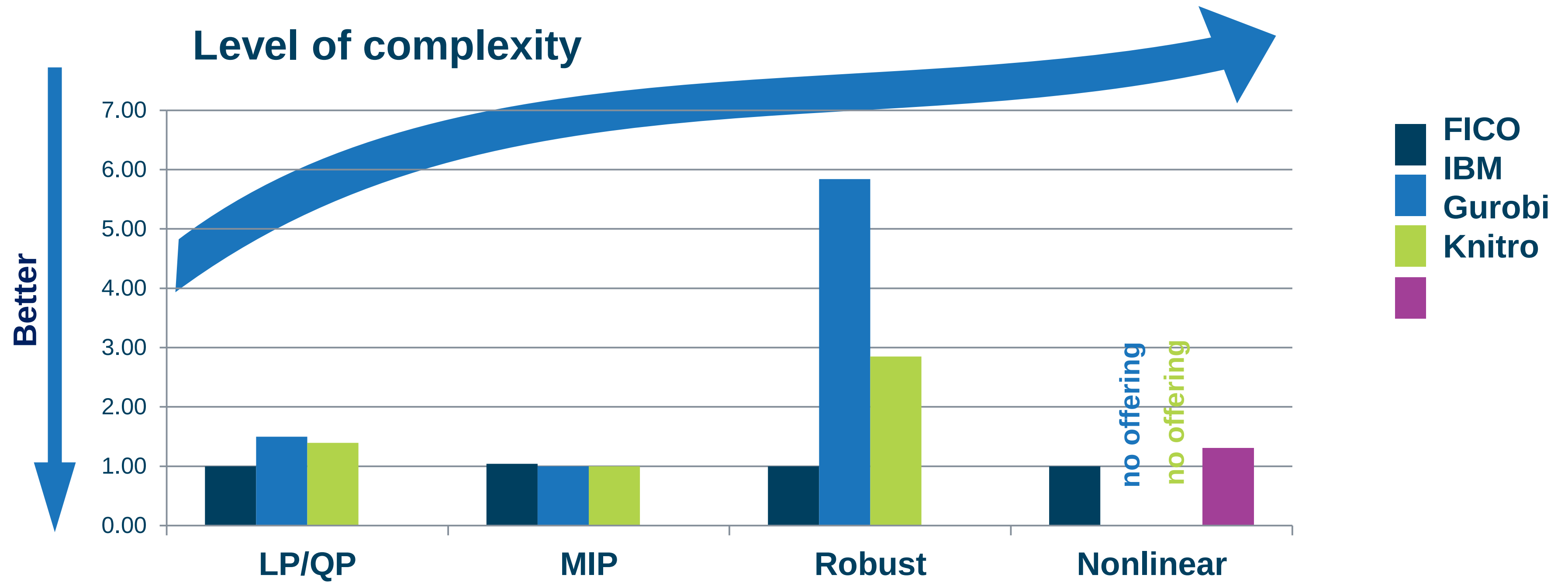
FICO Xpress Solver Performance



- FICO has the most complete optimization offering and all solvers are very competitive
- Robust / (MI)SOCP solver dominates the competition
- FICO has the leading nonlinear offering with applications in particular in finance, insurance, and power/gas/oil industries

Results as of March 30, geometric means of time to optimality, LP/QP geometric mean computed by FICO, MIP 12 threads, nonlinear numbers directly computed from the logs and computed by FICO

FICO Xpress Users

Basic Problem Types

LP

$$\begin{aligned} \min \quad & cx \\ s.t. \quad & a^i x \geq b_i, i \in M \end{aligned}$$

Primal simplex
Dual simplex
Interior point

QP

$$\begin{aligned} \min \quad & cx + xQx \\ s.t. \quad & a^i x \geq b_i, i \in M \end{aligned}$$

Quadratic primal simplex
Quadratic dual simplex
Interior point

QCQP

$$\begin{aligned} \min \quad & cx + xQx \\ s.t. \quad & a^i x + xQ^i x \geq b_i, i \in M \end{aligned}$$

Interior point

SOCP

$$\begin{aligned} \min \quad & cx + xQx \\ s.t. \quad & a^i x + xQ^i x \geq b_i, i \in M \\ & \left| D^k x + f_k \right| \leq g^k x + h_k, k \in C \end{aligned}$$

Interior point

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Basic Problem Types

MILP

$$\begin{aligned} \min \quad & cx \\ s.t. \quad & a^i x \geq b_i, i \in M \\ & x_j \in \mathbb{Z}, j \in I \end{aligned}$$

Branch and bound

MIQP

$$\begin{aligned} \min \quad & cx + xQx \\ s.t. \quad & a^i x \geq b_i, i \in M \\ & x_j \in \mathbb{Z}, j \in I \end{aligned}$$

Branch and bound
+ quadratic dual simplex

MIQCQP

$$\begin{aligned} \min \quad & cx + xQx \\ s.t. \quad & a^i x + xQ^i x \geq b_i, i \in M \\ & x_j \in \mathbb{Z}, j \in I \end{aligned}$$

Branch and bound
+ outer approximation

MISOCP

$$\begin{aligned} \min \quad & cx + xQx \\ s.t. \quad & a^i x + xQ^i x \geq b_i, i \in M \\ & |D^k x + f_k| \leq g^k x + h_k, k \in C \\ & x_j \in \mathbb{Z}, j \in I \end{aligned}$$

Branch and bound
+ outer approximation

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Non-linear Problems

Standard non-linear formulation:

$$\begin{array}{ll} min & f(x) \\ s.t. & g_i(x) \geq 0, i \in M \end{array}$$

Xpress non-linear formulation:

$$\begin{array}{ll} min & cx + xQx + f(x) \\ s.t. & a^i x + xQ^i x + g_i(x) \geq b_i, i \in M \\ & \left|D^k x + f_k\right| \leq g^k x + h_k, k \in C \end{array}$$

- Xpress embeds non-linear formulas in a standard problem.
- Solvers: SLP or Knitro

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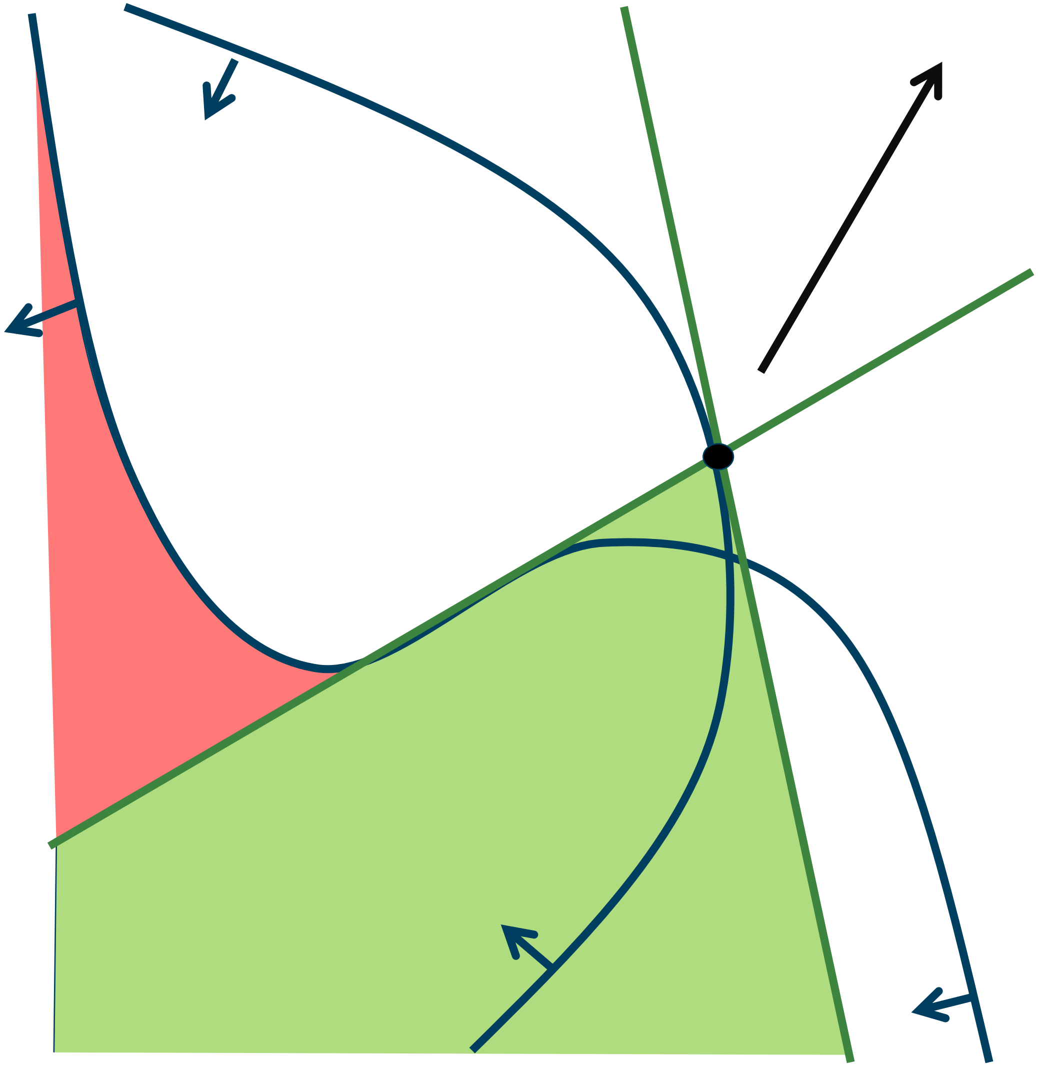
SLP – Sequential Linear Programming



Create linear approximation

Might cut off feasible regions

Solve LP

Repeat

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Sequential Linear Programming

Non-linear problem:
(assume non-linear objective moved into constraints)

$$\begin{aligned} &min \quad cx \\ &s.t. \quad a^i x + g_i(x) \geq b_i, i \in M \end{aligned}$$

Linearize non-linear functions around a solution x^k:

$$g_i(x) \approx g_i\left(x^k\right) + \nabla g_i(x - x^k)$$

to create LP

$$\begin{aligned} &min \quad cx \\ &s.t. \quad a^i x + g_i\left(x^k\right) + \nabla g_i(x - x^k) \geq b_i, i \in M \end{aligned}$$

Solve LP to get next iterate solution: x^{k+1}

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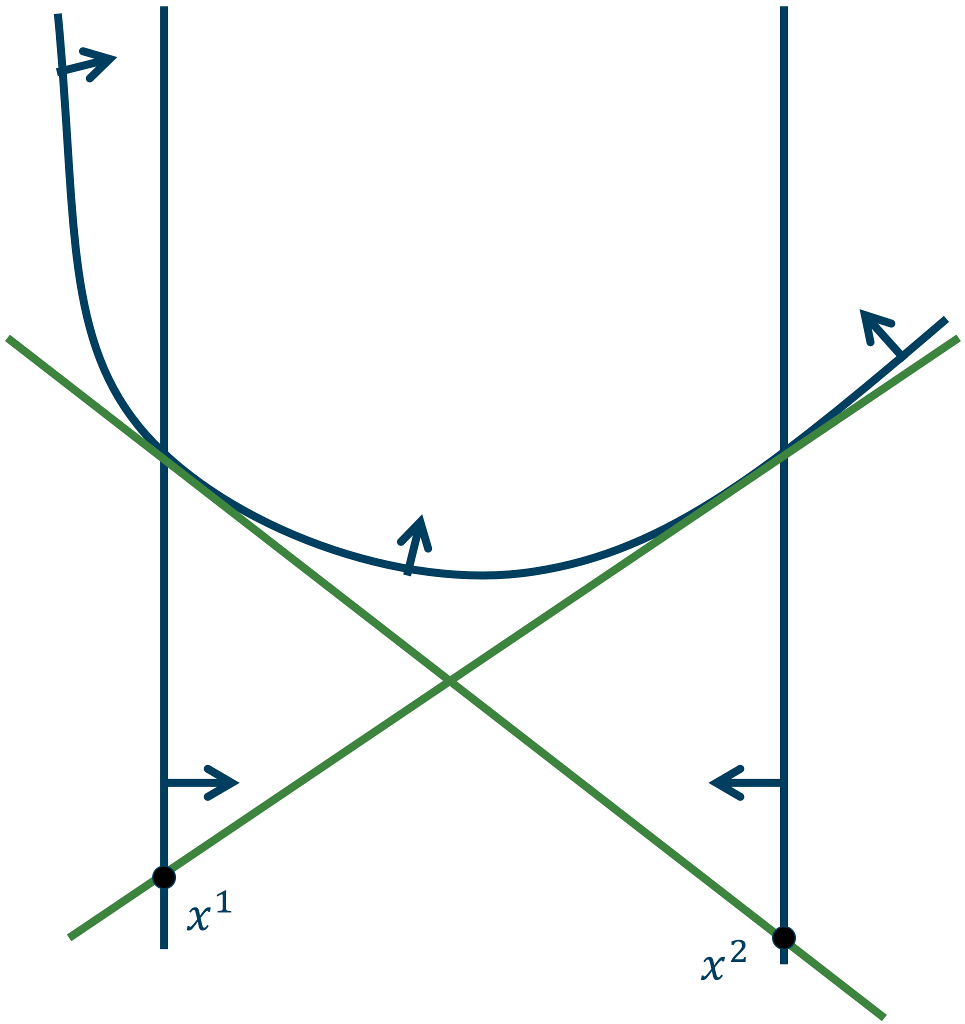
Sequential Linear Programming

Trust regions



Solution will bounce between x^1 and x^2

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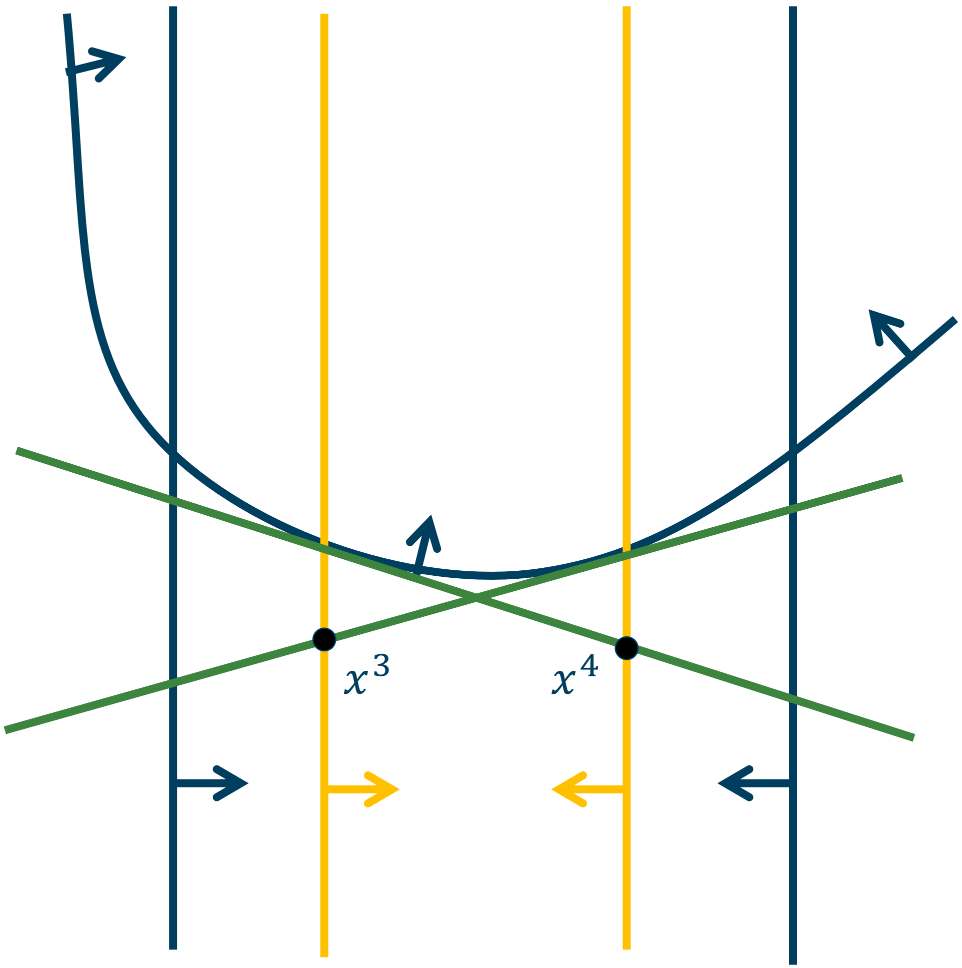
Sequential Linear Programming

Trust regions



Solution will bounce between x^1 and x^2

Use trust region

Smaller movement

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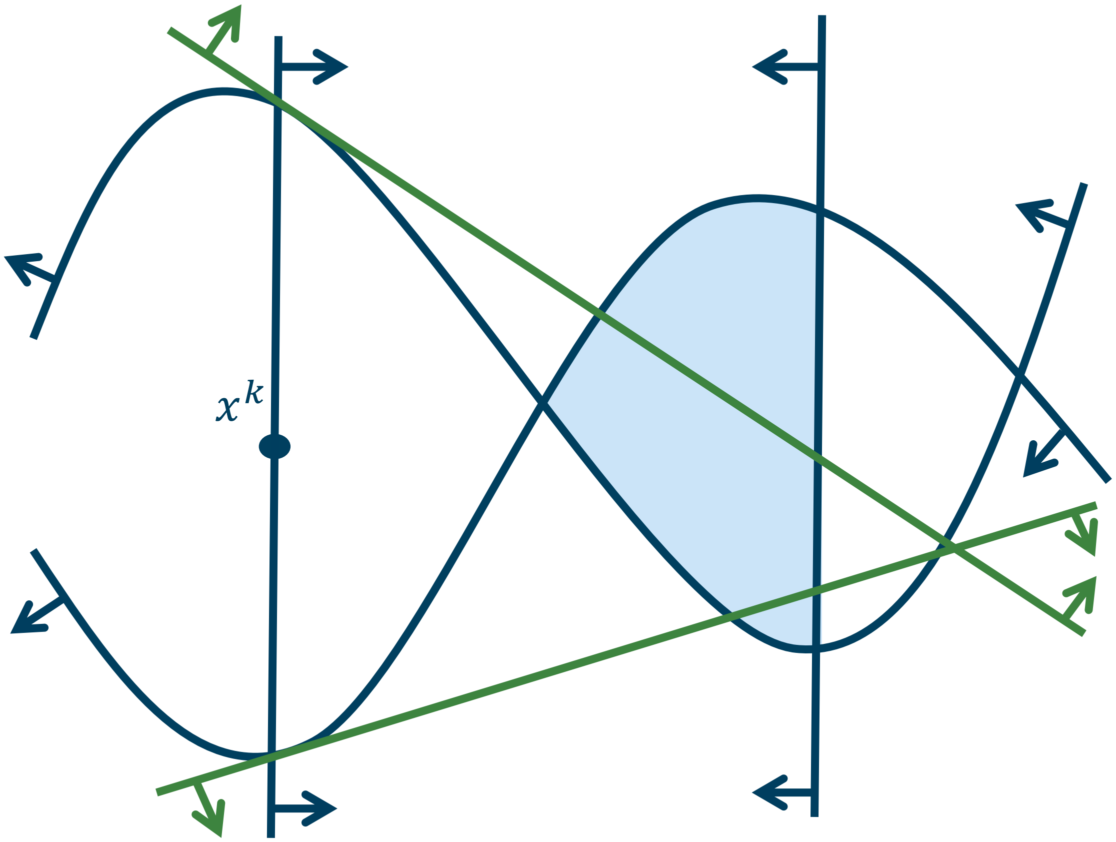
Sequential Linear Programming

Infeasibility



Non-convex non-linear with feasible region.

Iterate solution x^k in infeasible region.

Can result in infeasible LP

Use penalty variables to make LP feasible.

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Sequential Linear Programming

Extended LP

$$\begin{array}{ll} min & cx \\ s.t. & a^i x + g_i(x) \geq b_i, i \in M \end{array}$$

Linearize:

$$\begin{array}{ll} min & cx \\ s.t. & a^i x + g_i\left(x^k\right) + \nabla g_i(x - x^k) \geq b_i, i \in M \end{array}$$

Add trust region:

$$\begin{array}{ll} min & cx \\ s.t. & a^i x + g_i\left(x^k\right) + \nabla g_i \Delta x \geq b_i, i \in M \\ & x = x^k + \Delta x \\ & L \leq \Delta x \leq U \end{array}$$

Add penalty variables:

$$\begin{array}{ll} min & cx + Py \\ s.t. & a^i x + g_i\left(x^k\right) + \nabla g_i \Delta x + y \geq b_i, i \in M \\ & x = x^k + \Delta x \\ & L \leq \Delta x \leq U, y \geq 0 \end{array}$$

Sequential Linear Programming
Optimality

Non-linear problem:

$$\begin{aligned} &\min && cx \\ &\text{s.t.} && a^i x + g_i(x) \geq b_i, i \in M \end{aligned}$$

SLP problem:

$$\begin{aligned} &\min && cx + Py \\ &\text{s.t.} && a^i x + g_i\left(x^k\right) + \nabla g_i \Delta x + y \geq b_i, i \in M \\ & && x = x^k + \Delta x \\ & && L \leq \Delta x \leq U, y \geq 0 \end{aligned}$$

KKT optimality conditions:
(under regularity conditions)

$$\begin{aligned} & a^i x^* + g_i(x^*) \geq b_i, i \in M \\ & c = \sum_{i \in M} \mu_i (a^i + \nabla g_i(x^*)) \\ & \mu_i\left(a^i x^* + g_i(x^*)\right) = 0, i \in M \\ & \mu_i \geq 0, i \in M \end{aligned}$$

- Trust bounds and penalty variables bound primal and dual variables.
- SLP solution feasible if penalties are zero.
- SLP solution optimal if dual multipliers for trust bounds are zero.
- Optimal, feasible LP solution satisfies KKT complementarity.

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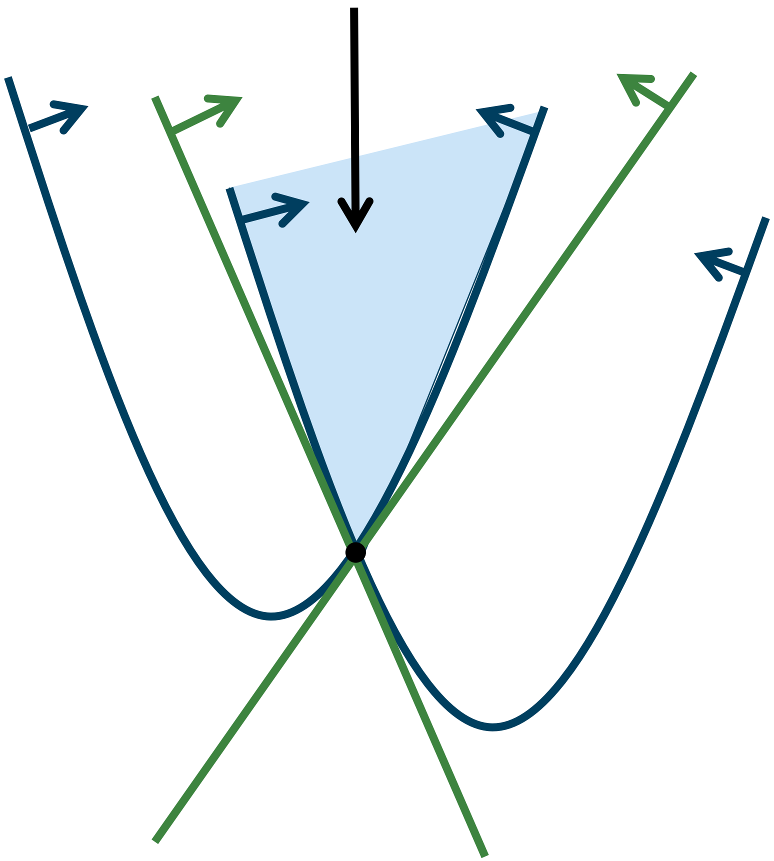
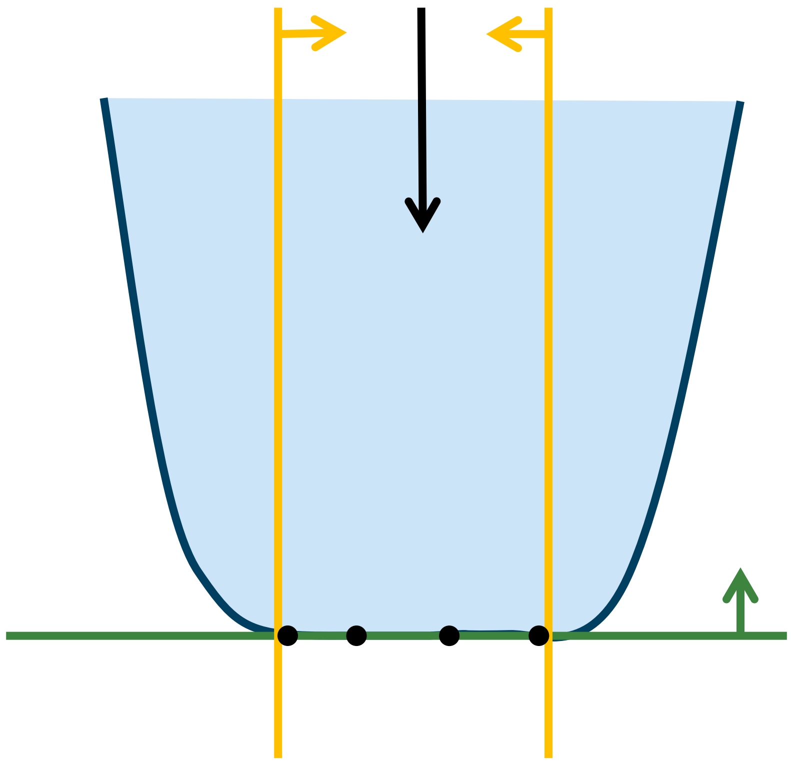
Sequential Linear Programming

Convergence

Strong convergence



Extended convergence



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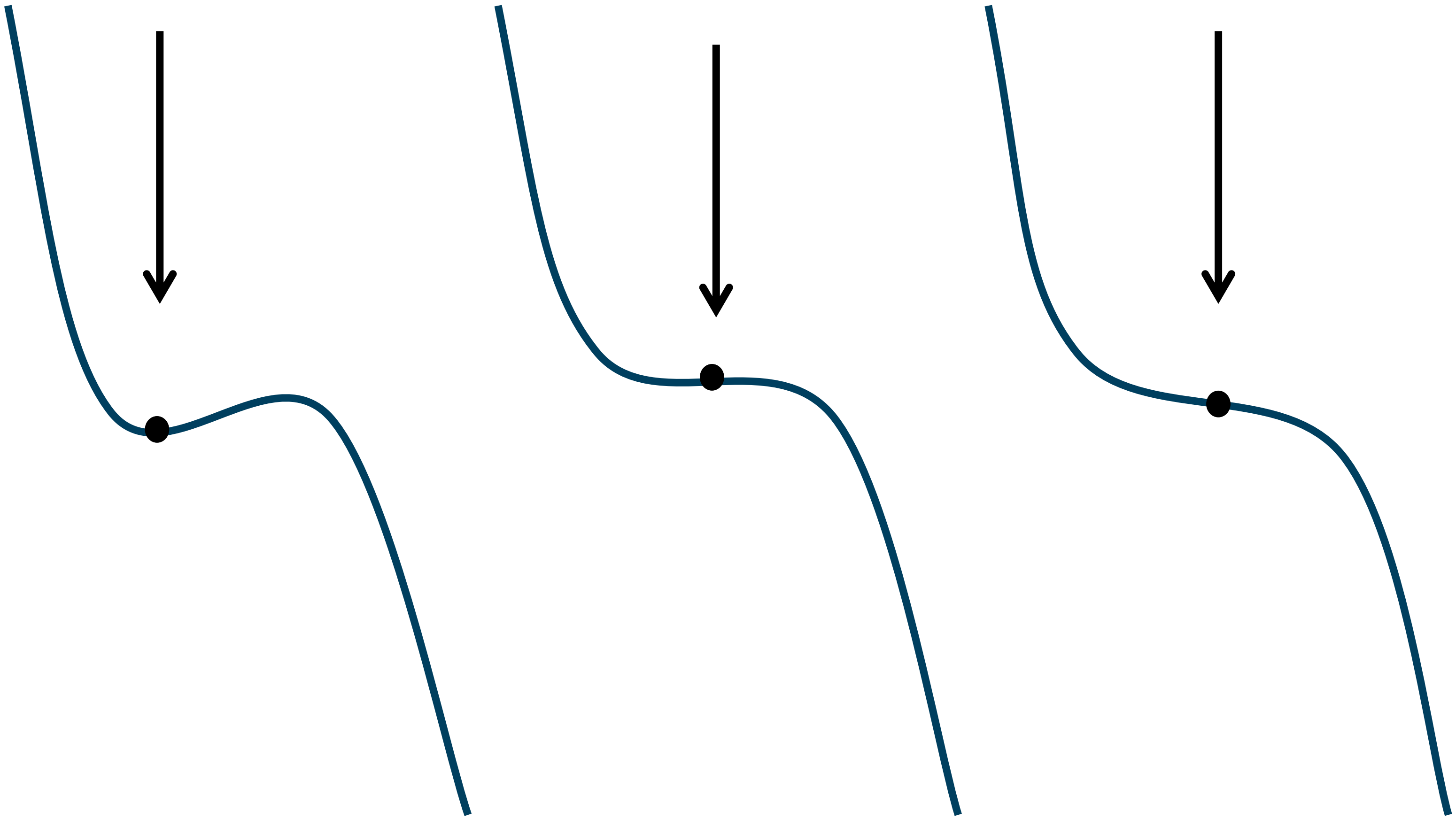
Sequential Linear Programming
Local Optima



Sequential Linear Programming

Blending Example

- Liquids A and B with densities $\delta(A)$ and $\delta(B)$.
- Create blend C with density $L_C \leq \delta(C) \leq U_C$ in amount $V(C)$.
- Select amounts $V(A)$ and $V(B)$ to blend.

$$V(C) = V(A) + V(B)$$
$$\delta(C)V(C) = \delta(A)V(A) + \delta(B)V(B)$$

- Linear problem if A and B given.
- Bi-linear if A and B are also blends.

- Refinery problems can contain 1000s of blends.
- Blends can be returned to earlier stages!

Sequential Linear Programming

Blending Example

$$\delta(C)V(C) = \delta(A)V(A) + \delta(B)V(B)$$

- $V(C) = 0$ results in $\delta(C)$ undefined.
  - Causes “infinite” LP coefficients
  - XSLP detects such edge cases.
- Non-convex problem.
  - Use restarts to find better solutions.
  - XSLP provides parallel multi-start feature with different starting points.
- Other solutions iterate between fixing δ or V.
  - XSLP approximates both at the same time.

Sequential Linear Programming
Overview

- Solves 1st order approximations.
- Builds on top of a strong LP solver.
- Highly efficient for bi-linear or problems with a large amount of linear constraints.
- Local solver: Global optimality guaranteed only for convex problems.
- Applications:
  - Petro-chemical industry.
  - Finance.
  - Price optimization.
  - …
- Comes with a Mixed Integer Programming solver.

Knitro Non-linear Solver

- Knitro licensed from
- 2nd order interior point solver.
- Strong for highly non-linear, medium sized problems.
- Integrated with Xpress non-linear solver
  - Usable with any non-linear model.
  - Xpress non-linear solver decides whether to use Knitro or XSLP.
- Xpress calculates 1st and 2nd order derivatives.
  - Numerical derivatives
  - Symbolic differentiation
  - Automatic differentiation.

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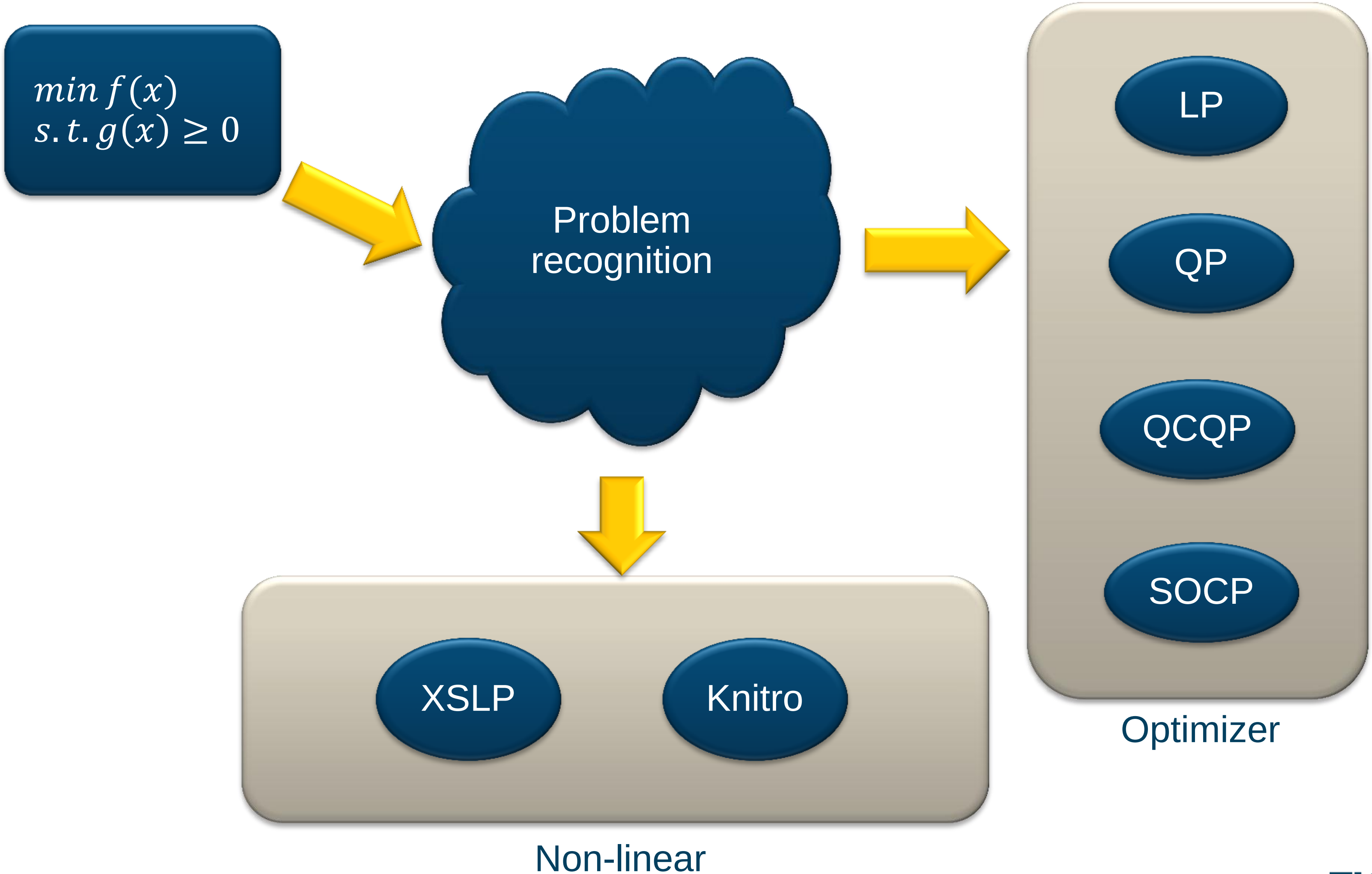
XSLP vs. Knitro

XSLP	Knitro
1^{st} order	2^{nd} order
Bilinear or highly linear problems	Highly non-linear problems
Large sized problems	Medium sized problems
Local solver	Local solver

XNLP Automatic Solver Selection



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Example!

Thank You

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