Maximising the impact of Mathematics at the University of Edinburgh

Rational surfaces and CAD applications

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Project Aim: The first aim of this project is to study the role played by rational surfaces (toric surfaces, del Pezzo surfaces, rational ruled surfaces, Bordiga surfaces, etc) in computer-aided design applications (CAD applications) and to find the possible applications of classical and contemporary results in the two-dimensional birational geometry to two-dimensional non-uniform rational B-splines (NURBS). The second aim of this project is to use two-dimensional NURBS to construct rational surfaces from the given geometrical data.

Background: Current CAD technology is based on NURBS. Two-dimensional NURBS are pieces of real rational surfaces that are patched together by continuous boundary data. The most commonly used rational surfaces in NURBS are toric (projective plane, product of two projective lines, minimal ruled surfaces, etc). Nevertheless, other rational surfaces can be used in NURBS as well (del Pezzo surfaces, projective plane blown up in N different points, Bordiga surfaces). Whenever CAD application generates surface, it is desirable to represent it in NURBS form. This naturally leads to the following parameterization problem: to find an optimal collection of rational surfaces with given boundary curves. This problem is similar to the Cayley-Bacharach problem in Algebraic Geometry (for one-dimensional NURBS, both problems are the same). The difference is that the constraints in the Cayley-Bacharach theorem are points, and the constraints in NURBS are algebraic curves.

Methodology: The basic tools of Algebraic Geometry such as projective geometry of surfaces in the three-dimensional projective space and elementary birational geometry of rational surfaces need to be extended and developed for the real, affine, and bounded cases, having in mind applications to CAD. Particular aspects that we plan to covered are NURBS arising from singular cubic surfaces, transformation of NURBS via birational maps (blowing up points and blowing down curves), topology of NURBS, explicit reconstruction of rational surfaces from one-dimensional geometric data, an analogue of Cayley-Bacharach theory for one-dimensional constraints.

Targeted problems:. Constructing new NURBS using cubic surfaces and other del Pezzo surfaces, explicit description of birational transformations of real rational surfaces (new examples, the structure of the group generated by such transformations, obstructions for existing of such maps), computer aided verification of the old conjecture of Yuri Manin that states that every smooth cubic surface in the three-dimensional projective space can be reconstructed from the set of 27 lines lying on it. The positive answer to the this conjecture means that cubic surfaces are determined up to isomorphism by their generalized genus 0 Gromov–Witten invariants (they are defined over a field of one element).

Expected Results and Impact: We plan to construct new NURBS using low degree hypersurfaces in weighed projective spaces (this includes cubic surfaces) and implement new NURBS into Creo Elements/Pro, formerly known as Pro/ENGINEER, a parametric three-dimensional CAD software package created by Parametric Technology Corporation (PTC). We plan to use NURBS arising from smooth cubic surfaces to verify Manin's conjecture in many cases. PTC is a US based (Needham, MA) company that is a major force in the CAD industry, which was founded in 1985 by a mathematician Samuel Peisakhovich Geisberg. Core Geometry group in PTC hires Mathematics graduates from top international schools who majors in Algebraic or Differential Geometry.