Bridgeland Stability Conditions with a Real Reduction

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$\wedge^2 B_n$: Problem for Fun

Let

$$P_n := \{f(x) \in \mathbb{R}[x] \mid \deg f = n, f(x) = 0 \text{ has } n \text{ distinct real roots.}\};$$
$$B_n := P_n \cup P_{n-1}.$$

For what kind of polynomials $f(x), g(x) \in P_n$, do we have

$$af(x) + bg(x) \in B_n$$

for every $a, b \neq 0$?

Background: slope stability on curves

Let C be a smooth projective curve over $\mathbb{C}$; F be a vector bundle on C. The **slope** of F is:

$$\mu(F) := \frac{\deg(F)}{\mathrm{rank}(F)}.$$

A vector bundle F is slope (semi)**stable** if $\forall 0 \neq E \subsetneq F$, we have

$$\mu(E) < (\leq) \mu(F).$$

- F stable $\iff$ $F \otimes \mathcal{L}$ stable. ($\mathcal{L}$ a line bundle)

Harder–Narasimhan filtration: Every vector bundle F on C admits a unique filtration:

$$0 = F_0 \subset F_1 \subset F_2 \cdots \subset F_m = F$$

such that

- $E_i := F_i / F_{i-1}$ is semistable;
- $\mu(E_1) > \mu(E_2) > \cdots > \mu(E_m)$.

Bogomolov Inequality

Let (S, H) be a polarized smooth surface. The slope of a torsion-free coherent sheaf F is

$$\mu_H(F) := H\mathrm{ch}_1(F)/\mathrm{rank}(F).$$

- (Bogomolov Inequality) For every μ_H-stable F, its discriminant is non-negative.

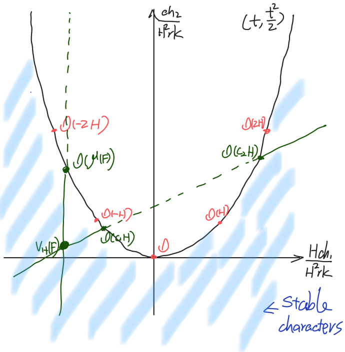
$$\Delta(F) := (\mathrm{ch}_1(F))^2 - 2\mathrm{ch}_2(F)\mathrm{rank}(F) \geq 0$$

- (Bogomolov–Miyaoka–Yau Inequality) The Chern characters of a general type complex surface S satisfies: $(\mathrm{ch}_1(T_S))^2 \geq 6\mathrm{ch}_2(T_S)$.
- (Reider) If there is no curve E on S with ($E^2 = -1$ and $HE = 0$) nor ($E^2 = 0$ and $HE = 1$), then H is base point free.

Bogomolov Inequality (polarized version)

$$\Delta_H(F) := (H\mathrm{ch}_1(F))^2 - 2\mathrm{ch}_2(F)H^2\mathrm{rank}(F) \geq 0$$

We may interpret this as follows:



Let $v_H(F) = (H^2 rank(F), H\mathrm{ch}_1(F), \mathrm{ch}_2(F))$ be a stable character. Then

- $v_H(F) = rv_H(\mathcal{O}(\mu(F)H) - d(0,0,1)$ for some $r, d \geq 0$;
- $v_H(F) = av_H(\mathcal{O}(c_1H)) - bv_H(\mathcal{O}(c_2H))$ for some $a, b \geq 0$.

Bridgeland stability condition

Let $\mathcal{T}$ be a k-linear triangulated category.

Definition (Bridgeland 02, Douglas)

A **stability condition** σ *on* $\mathcal{T}$ *is a pair of datum* $(\mathcal{A}, Z)$, *where*

- $\mathcal{A}$ *is the heart of a bounded t-structure on* $\mathcal{T}$;
- *and* $Z : \mathrm{K}_{\mathrm{num}}(\mathcal{T}) \to \mathbb{C}$ *is a group homomorphism;*

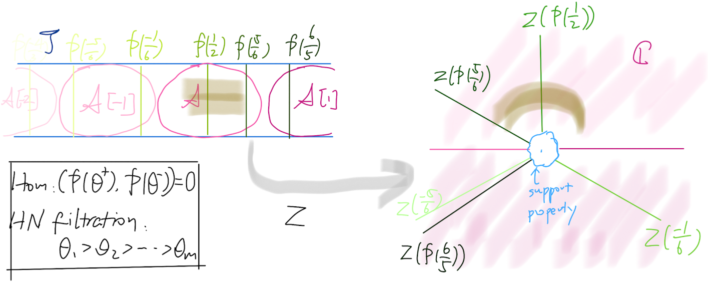
satisfying:

1. $Z(\mathcal{A}) \in \mathbb{R}_+ \cdot e^{\pi i(0,1]}$; *the slope of an object* E *in* $\mathcal{A}$ *is*
$$\mu_\sigma(E) = -\mathrm{Re}Z(E)/\mathrm{Im}Z(E)$$
2. *Harder–Narasimhan filtration property;*
3. *Support property: (a bound for stable characters)* $\exists c > 0$ *such that* $\forall$ σ*-stable* $E \in \mathcal{A}$, *we have* $|Z(E)| \geq c||[E]||$.

A **slicing** $\mathcal{P}$ on $\mathcal{T}$ is a map $\mathcal{P}:\mathbb{R}\to\{$abelian sub-categories of $\mathcal{T}\}$ satisfying:

1. $\mathcal{P}(\theta+1)=\mathcal{P}(\theta)[1]$;
2. $\mathrm{Hom}(F_1,F_2)=0$ for every $F_i\in\mathcal{P}(\theta_i)$, $\theta_1>\theta_2$;
3. For every $0\neq F\in\mathcal{T}$, there is a unique filtration
$$0=F_0\to F_1\to\cdots\to F_m=F,$$
with $0\neq\mathrm{Cone}(F_{i-1}\to F_i)\in\mathcal{P}(\theta_i)$ for some $\theta_1>\cdots>\theta_m$.

Denote by $\phi^+(F)=\theta_1$ and $\phi^-(F)=\theta_m$.



We have the following equivalent definition (also the original definition) for stability conditions:

Definition

A **stability condition** *on* $\mathcal{T}$ *is a pair* $(\mathcal{P}, Z)$ *of slicing and central charge satisfying*

1. $Z(\mathcal{P}(\theta)) \in \mathbb{R}_{>0} \cdot e^{\pi i\theta}$ *for every* $\theta \in \mathbb{R}$;
2. *the support property.*

Denote by $\mathrm{Stab}(\mathcal{T})$ the set of all stability conditions on $\mathcal{T}$.
There is a metric on $\mathrm{Stab}(\mathcal{T})$.

$$d(\mathcal{P}_1, \mathcal{P}_2) := \sup_{0 \neq E \in \mathcal{T}} \{|\phi^+_{\mathcal{P}_1}(E) - \phi^+_{\mathcal{P}_2}(E)|, |\phi^-_{\mathcal{P}_1}(E) - \phi^-_{\mathcal{P}_2}(E)|\};$$

$$d(\sigma_1, \sigma_2) := \max\{d(\mathcal{P}_{\sigma_1}, \mathcal{P}_{\sigma_2}), ||Z_{\sigma_1} - Z_{\sigma_2}||\}.$$

Big Beautiful space

Theorem (Bridgeland 02)

The forgetful map to the central charge

$$\mathrm{Forg}_Z : \mathbf{Stab}(\mathcal{T}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K_{num}}(\mathcal{T}), \mathbb{C})$$
$$(\mathcal{A}, Z) \mapsto Z$$

is local homeomorphic.
The space of all stability conditions on $\mathcal{T}$, whenever non-empty, forms a **complex manifold** *of dimension $\mathrm{rk}(\mathrm{K_{num}}(\mathcal{T}))$.*

Boring case: curve

Example: C a smooth projective curve, $\mathcal{T} = D^b(\mathrm{Coh}(C))$. We may take

$$\sigma_{\mathrm{slope}} = (\mathcal{A} = \mathrm{Coh}(C), Z = -\mathrm{deg} + i\mathrm{rank}).$$

- The $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-action: $\mathrm{Stab}(\mathcal{T}) \times \widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \to \mathrm{Stab}(\mathcal{T})$:

$$((\mathcal{P}, Z), (g, a)) \mapsto (\mathcal{P}_a, g^{-1}Z),$$

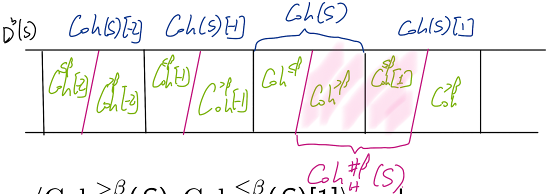
  where $\mathcal{P}_a(\theta) = \mathcal{P}(a(\theta))$. In particular, E is σ-stable if and only if $(\sigma \cdot \tilde{g})$-stable
- The $\mathrm{Aut}(\mathcal{T})$-action: $\mathrm{Aut}(\mathcal{T}) \times \mathrm{Stab}(\mathcal{T}) \to \mathrm{Stab}(\mathcal{T})$: $(\tau, (\mathcal{P}, Z)) \mapsto (\mathcal{P}_\tau, Z(\tau^{-1}(-)))$, where $\mathcal{P}_\tau(\theta) = \tau(\mathcal{P}(\theta))$.
- (Bridgeland, Macrì) When $g(C) \geq 1$, $\mathit{Stab}(C) = \sigma_{\mathrm{slope}} \cdot \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$.
  - E.g. $\sigma_{\mathrm{slope}} \otimes \mathcal{L} = \sigma_{\mathrm{slope}} \cdot \tilde{g}$ for some $\tilde{g} \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$.

For a polarized surface (S, H), the datum $(\mathrm{Coh}(S), -H\mathrm{ch}_1 + i\mathrm{rank})$ fails to become a stability condition as $Z(\mathcal{O}_p) = 0$.

In fact, $\mathrm{Coh}(S)$ can never be the heart structure of a stability condition.

To construct stability conditions on a surface, we consider the tilting pair: for every $\beta \in \mathbb{R}$,

$$\mathrm{Coh}_H^{>\beta}(S) := \{F \in \mathrm{Coh}(S) \mid \forall F \twoheadrightarrow E, \mu_H(E) > \beta\};$$
$$\mathrm{Coh}_H^{\leq\beta}(S) := \{F \in \mathrm{Coh}(S) \mid \forall E \hookrightarrow F, \mu_H(E) \leq \beta\}.$$



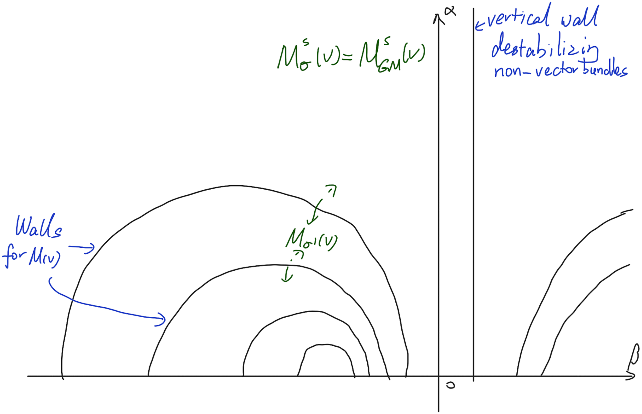
Let $\mathrm{Coh}_H^{\sharp\beta}(S) := \langle \mathrm{Coh}_H^{>\beta}(S), \mathrm{Coh}_H^{\leq\beta}(S)[1] \rangle$ and

$$Z_{\alpha,\beta,H} := -\mathrm{ch}_2^{(\beta+i\alpha)H} = -\mathrm{ch}_2^{\beta H} + \alpha^2 H^2 rank + i(H\mathrm{ch}_1 - \beta H^2 rank).$$

Then for every $\alpha > 0$, the datum $\sigma_{\alpha,\beta} := (\mathrm{Coh}_H^{\sharp\beta}(S), Z_{\alpha,\beta,H})$ is a stability condition on S.

Bertram Nested Walls Theorem

Given a character $v \in K_{num}(S)$, there is a wall and chamber structure for $M_\sigma(v)$ that parametrizes σ-semistable objects in $\mathcal{A}$ with character v. On the (α, β)-upper half plane, all (potential) walls are separated from each other.

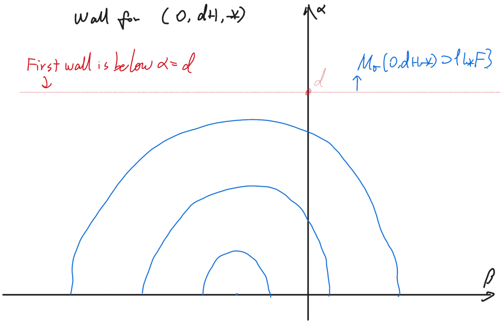


Big Volume Limit

When $\alpha \to +\infty$, the stability condition $\sigma_{\alpha,\beta} \rightsquigarrow$ Gieseker stability:

$$M_{GM}(v) = M_{\alpha \gg 1, \beta < \mu_H(v)}(v).$$

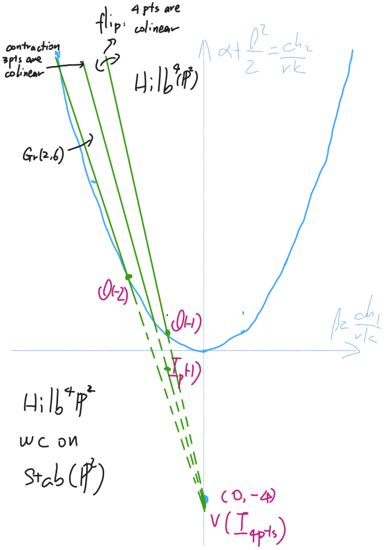
For a smooth curve $C \in |dH|$, $\iota : C \hookrightarrow S$ and slope stable vector bundle F on C, the object $\iota_* F$ is $\sigma_{\alpha,\beta}$-stable when $\alpha > d$.



In other words, when $\alpha > d$, $\sigma_{\alpha,\beta}$ induces the slope stability on $D^b(C)$.

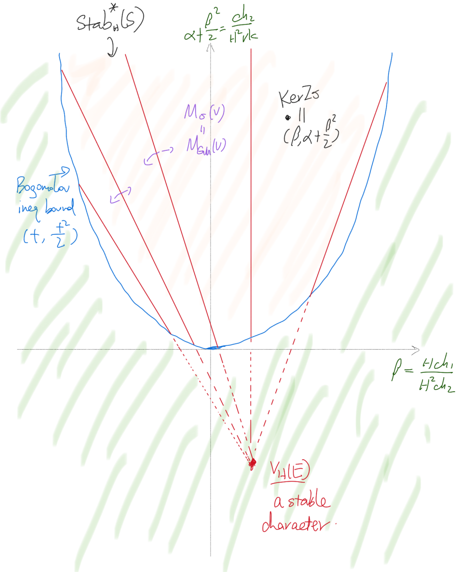
Example: (Arcara–Bertram–Coskun–Huizenga) Wall-crossing for $\mathrm{Hilb}^4\mathbf{P}^2$ on $\mathrm{Stab}(\mathbf{P}^2)$.

Here we adjust the parameter (α, β): representing $\sigma_{\alpha,\beta}$ as $(\alpha + \beta^2/2, \beta)$, which is $\mathrm{Ker}Z_{\alpha,\beta}$.
For every v, the walls of $M(v)$ are line segments through v.



In some cases of S, one can run the minimal model program of the moduli space $M_{GM}(v)$ on $\mathrm{Stab}(S)$.

- Start from the chamber of $M_{GM}(v)$
- $M_\sigma(v) \leftrightarrow M_{\sigma'}(v)$ birational
- (Bayer, Macrì) K3 surfaces
- (Acara, Bertram, Coskun, Huizenga, Woolf, Zhao) $\mathbf{P}^2$
- (Minamide, Nuer, Yanagida, Yoshioka) Abelian surfaces, Enrique surfaces



Bayer Vanishing

There are some technical points in the story above.
In the projective plane case, one of them is about the smoothness of $M_\sigma(v)$. In other words, the vanishing of

$$Ext^2(E,E) = (Hom(E,E(-3))^*.$$

This can be implied by the *Bayer Vanishing Lemma*:

$$Hom(E,F(-dH)) = 0$$

for every $\sigma_{\alpha,\beta}$-stable (non-skyscraper objects) E, F with the same phase and $d > 0$.

BBMST Bogomolov type inequality for 3fold

Let (X, H) be a smooth threefold. The datum $\sigma_{\alpha,\beta}$ is not a stability condition but only a weak stability condition.

Conjecture (Bayer–Bertram–Macrì–Stellari–Toda)

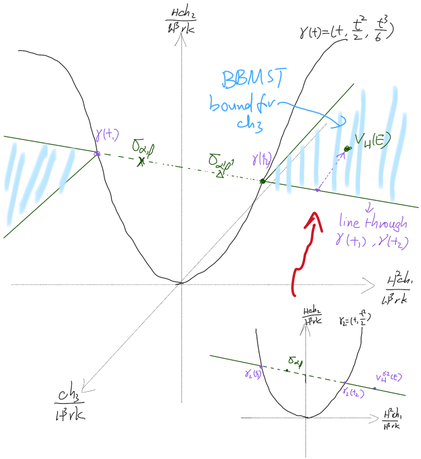
Let E be a $\sigma_{\alpha,\beta}$-stable object, then

$$\alpha\Delta_H(E) + \nabla^\beta_H(E) \geq 0.$$

- $\nabla^\beta_H = 4(H\mathrm{ch}_2^{\beta H})^2 - 6(H^2\mathrm{ch}_1^{\beta H})\mathrm{ch}_3^{\beta H}$.
- BBMST Conjecture $\implies \exists$ a family of stability conditions $\{\sigma^{a,b}_{\alpha,\beta}\}$ with $\alpha > 0$, $a > \frac{1}{6}\alpha^2 + \frac{1}{2}|b|\alpha$.

The space spanned by $v_H(E)$ and $\mathrm{Ker}Z_{\alpha,\beta}$ intersects the twisted curve $\gamma(t) = (1, t, t^2/2, t^3/6)$ at two points $\gamma(t_i)$ with $t_1 < t_2$.

BBMST inequality is to say $\pm v_H(E) = -a_1\gamma(t_1) + a_2\gamma(t_2) - a_3(0, 0, 0, 1)$ for some $a_i \geq 0$.



- (Schmidt) For example, when $E = \mathcal{I}_C$, the ideal sheaf of a curve C, the bound recaptures the Castelnuovo bound for $g(C)$.

The BBMST inequality conjecture is verified or partially verified for

- (Macrì, Schmidt, Bernardara, Zhao) Fano 3folds
- (Bayer, Macrì, Stellari, Maciocia, Piyaratne) Abelian 3folds
- (Koseki, -, S. Liu) some examples of Calabi–Yau 3folds

If not so ambitious for the BBMST bound, but just for the existence of stability conditions, then there are many yoga to do:

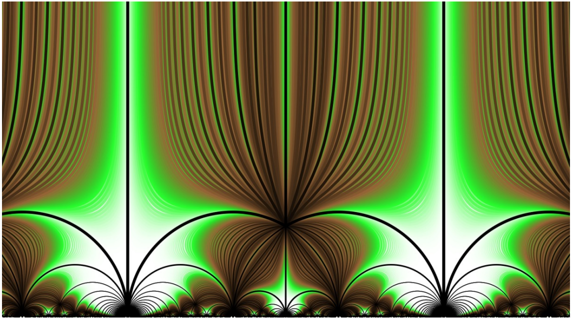
- SOD gluing (Collins, Polishchuk): $\mathbf{P}^n$, quadrics, cubic fourfolds (Bayer, Lahoz, Macrì, Stellari, Zhao), Gushel–Mukai fourfold (Pertusi, Perry, Zhao), cubic fivefolds (P. Liu)
- $X\times$curve: $\mathrm{Stab}(X) \neq \emptyset \implies \mathrm{Stab}(X \times C) \neq \emptyset$ (Yucheng Liu)
    - G-equivariant/BKR: Cynk–Hulek, Borcea–Voisin varieties (Perry, Shah)
    - Fibers of $X \to \mathrm{Alb}(X)$: Kummer varieties (Cheng)

      Based on the work by Fu, Zhao, Dell.
- Mirror: more examples of Calabi–Yau 3folds (Nuer)
- Deformation: (Macrì, Stellari, Perry, Zhao)
- Feyzbakhsh–Thomas Γ-version inequality: c.p.i. CY3 with Picard number one or even more (Feyzbakhsh, Koseki, Z. Liu, Rekuski)

1. How to describe the wall and chamber structure on Stab of a threefold?
    - Bertram Nested Wall Theorem
    - Connection with Gieseker stability/Push forward of stable objects from surface
    - Bayer Vanishing Lemma
2. How to think about stability conditions on higher dimensional varieties?
    - Image of the central charge
    - BBMST inequality

Relevant ideas from Scattering Diagram ....

Stability scattering diagram: Bridgeland (associated to quiver with potentials); Bousseau (projective plane and local P2)...



Cartoon from 'BPS Dendroscopy on Local P2' by Bousseau, Descombes, Le Floch, Pioline

One key construction is by mapping some 'nearby stability conditions' to the imaginary part of the central charge.

Equivalent relation on $Stab(\mathcal{T})$

For $\sigma, \tau \in$Stab$(\mathcal{T})$, we define $\sigma \sim \tau$ if

$$\mathrm{Im} Z_\sigma = \mathrm{Im} Z_\tau \text{ and } d(\mathcal{P}_\sigma, \mathcal{P}_\tau) < 1.$$

This is an equivalence relation and we denote

$$\pi_\sim : Stab(\mathcal{T}) \to Sb(\mathcal{T}) \coloneqq Stab(\mathcal{T})/\sim.$$

- $\sigma \sim \tau$ if and only if $\mathrm{Im} Z_\sigma = \mathrm{Im} Z_\tau$ and they are path connected in the fiber $(Forg_{\mathrm{Im} Z})^{-1}$.
- Each fiber of $\pi_\sim$ is 'convex'.
- If $\sigma \sim \tau$, then $\mathcal{A}_\sigma = \mathcal{A}_\tau$.

We call an element $\tilde{\sigma} \in Sb(\mathcal{T})$ a *reduced stability condition* and denote

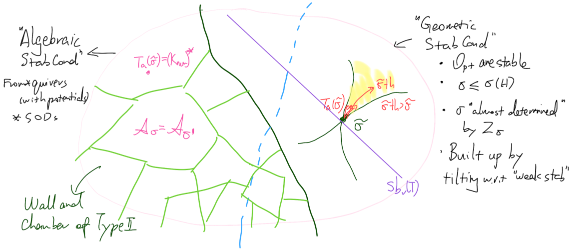
$$B_{\tilde{\sigma}} = \mathrm{Im} Z_\sigma \text{ and } \mathcal{A}_{\tilde{\sigma}} = \mathcal{A}_\sigma.$$

Theorem

The forgetful map

$$Forg : Sb(\mathcal{T}) \to Hom(K_{num}(\mathcal{T}), \mathbb{R})$$
$$\tilde{\sigma} \mapsto B_{\tilde{\sigma}}$$

is a local homeomorphism.



Proposition

The map $\pi_\sim$ preserves all wall and chamber structures on $Stab(\mathcal{T})$.

For every E, when $B_{\tilde{\sigma}}(E) = 0$, the $\tilde{\sigma}$-stability of E is well-defined.
For every $v \in K_{num}(\mathcal{T})$, we denote by

$$Sb_v(\mathcal{T}) = \{\tilde{\sigma} \mid B_{\tilde{\sigma}}(v) = 0\}.$$

The set $M_{\tilde{\sigma}}(v)$ is then well-defined for every $\tilde{\sigma} \in Sb_v(\mathcal{T})$.
The map

$$\pi_\sim : Stab(\mathcal{T})/\mathbb{C} \to Sb_v(\mathcal{T})/\mathbb{R}$$

preserves the wall and chamber structure.

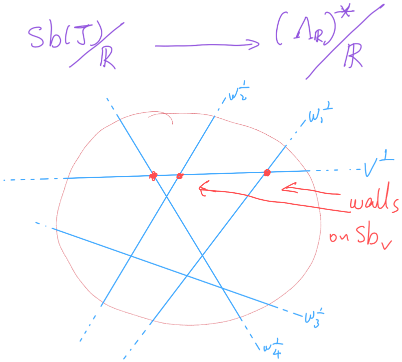
Bertram Nested Wall Theorem

The map

$$Forg : Sb_v(\mathcal{T}) \to v^\perp \subset Hom(K_{num}(\mathcal{T}), \mathbb{R})$$

is a local homeomorphism. Walls on $v^\perp$ are hyperplanes.

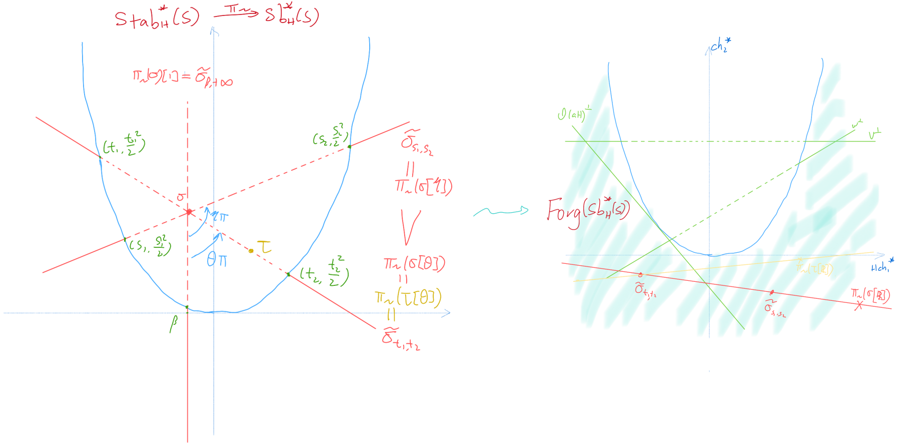
In particular, when $rk(K_{num}(\mathcal{T})) = 3$, this recaptures Bertram nested walls: all walls on $Stab(\mathcal{T})$ are separated from each other.



Surface case

Let (S, H) be a smooth polarized surface, we consider the space of reduced stability conditions in the form of $\pi_\sim(\sigma_{\alpha,\beta}\cdot \widetilde{\mathrm{GL}}^+(2,\mathbb{R}))$.

$$Sb^*_H(S) = \{\tilde{\sigma}_{t_1,t_2} \mid t_1 < t_2 \in \mathbb{R}\cup\{+\infty\}\}$$



We define

$$\tilde{\sigma} < \tilde{\tau} :\iff \mathcal{A}_{\tilde{\sigma}} \subset \mathcal{P}_{\tilde{\tau}}(< 1).$$

The vanishing of $Hom(E, E \otimes \mathcal{O}_S(-H))$ for $\tilde{\sigma}$-stable E can then be implied by

$$\tilde{\sigma} < \tilde{\sigma} \otimes \mathcal{O}_S(H).$$

For $\tilde{\sigma}_{t_1,t_2}$, we have

- $\tilde{\sigma}_{t_1,t_2} \otimes \mathcal{O}_S(H) = \tilde{\sigma}_{t_1+1,t_2+1}$.
- $\tilde{\sigma}_{t_1,t_2} < \tilde{\sigma}_{s_1,s_2}$ when $t_1 < s_1$ and $t_2 < s_2$.
- $\tilde{\sigma}_{t_1,t_2} < \tilde{\sigma}_{s_1,s_2}[1]$ when $t_1 < s_2$.

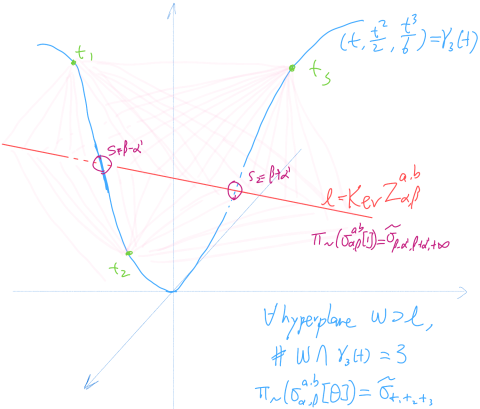
The reduced stability condition $\tilde{\sigma}_{t_1,t_2}$ restricts to $C \in |dH|$ when $t_2 - t_1 > d$.

Threefold case

For every stability condition σ in $\{\sigma^{a,b}_{\alpha,\beta}\} \cdot \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ by the BBMST conjecture, the equation

$$\mathrm{Im} Z_\sigma(\mathcal{O}_X(tH)) = 0$$

has three distinct solutions.



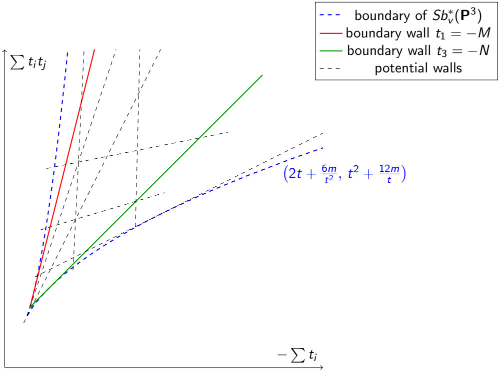
So $\pi_\sim(\{\sigma^{a,b}_{\alpha,\beta}\}\cdot\widetilde{\mathrm{GL}}^+(2,\mathbb{R}))$ form the following space:

$$Sb^*_H(X) = \{\tilde{\sigma}_{\underline{t}} \mid t_1 < t_2 < t_3 \in \mathbb{R}\cup\{+\infty\}\},$$

where $B_{\underline{t}}$ is determined by $B_{\underline{t}}(\mathcal{O}_X(t_iH)) = 0$.

As in the surface case, we have

- $\tilde{\sigma}_{\underline{t}} \otimes \mathcal{O}(H) = \tilde{\sigma}_{\underline{t}+1}$
- $\tilde{\sigma}_{\underline{t}} < \tilde{\sigma}_{\underline{s}}$ when $\underline{t} < \underline{s}$
- $\tilde{\sigma}_{\underline{t}} < \tilde{\sigma}_{\underline{s}}[m]$ when $t_i < s_{i+m}$



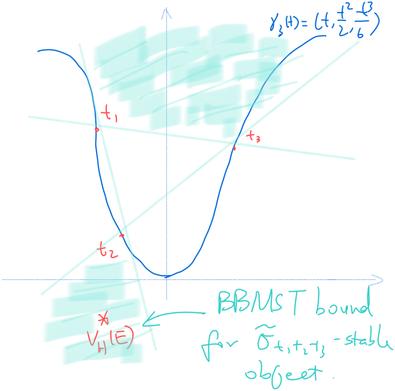
A sketch of walls and chambers for $v = (1, 0, 0, -m)$ on $Sb^*_v(\mathbf{P}^3)$.

BBMST inequality

For a $\tilde{\sigma}_{\underline{t}}$-stable object E, by definition, one has

$$v_H(E) = a_1 v_H(\mathcal{O}_X(t_1 H)) - a_2 v_H(\mathcal{O}_X(t_2 H)) + a_3 v_H(\mathcal{O}_X(t_3 H))$$

the BBMST inequality is to say that all $a_i \geq 0$ or ≤ 0.



Theorem (Polishchuk)

Let $\iota : X \hookrightarrow Y$ be smooth projective varieties and σ be a stability condition $D^b(Y)$. Assume that for every σ-stable object E, one has $\phi^-_\sigma(\iota_*\mathcal{O}_X \otimes E) \geq \phi_\sigma(E)$. Then $\sigma|_{D^b(X)}$ is a stability condition on $D^b(X)$.

- The condition means '$\tilde{\sigma}_{\underline{t}} \leq \iota_*\mathcal{O}_X \otimes \tilde{\sigma}_{\underline{t}}$'.
- Consider the resolution
  $0 \to \mathcal{F} \to \mathcal{O}_Y(-m_{n-1}H)^{\oplus a_{n-1}} \to \cdots \to \mathcal{O}_Y(-m_1H)^{\oplus a_1} \to \mathcal{I}_X \to 0$
  for some n sufficiently large so that '$\tilde{\sigma} \leq Coh(Y)[m] \leq \mathcal{F}[n] \otimes \tilde{\sigma}$'.
  Note that
  $$\iota_*\mathcal{O}_X \in \langle \mathcal{O}_Y, \mathcal{O}_Y(-m_jH)[j], \mathcal{F}[n]\rangle.$$
- When $\tilde{\sigma} \otimes \mathcal{O}_Y(m_jH) < \tilde{\sigma}[j]$ for all j, $\tilde{\sigma}$ restricts to $D^b(X)$.

For $\tilde{\sigma}_{\underline{t}}$, that is $t_{i+j} - t_i > jm_j$ for all i, j.

B_n: Space of reduced central charges

Recall that

$$B_n = P_n \cup P_{n-1} = \{c\prod_{i=1}^{n}(x - t_i) \mid t_1 < \cdots < t_n\}.$$

Let (X, H) be an n-dimensional smooth polarized variety. There is a natural map from:

$$B_n \to Hom(\Lambda_H, \mathbb{R}) : \quad \sum_{j=0}^{n} a_j x^j \mapsto \sum_{j=0}^{n} j! a_j H^{n-j} \mathrm{ch}_j$$

with image $\mathfrak{B}_n = \{cF_{\underline{t}} \mid F_{\underline{t}}(\mathcal{O}(t_i H)) = 0\}$.

Conjecture

There exists a family of reduced stability conditions $Sb_H^*(X)$ on $D^b(X)$ satisfying the following properties:

1. *The forgetful map*

$$Forg : Sb_H^*(X) \to Hom(\Lambda_H, \mathbb{R}) : \tilde{\sigma} \mapsto B_{\tilde{\sigma}}$$

   is a homeomorphism onto $\mathfrak{B}_n$.

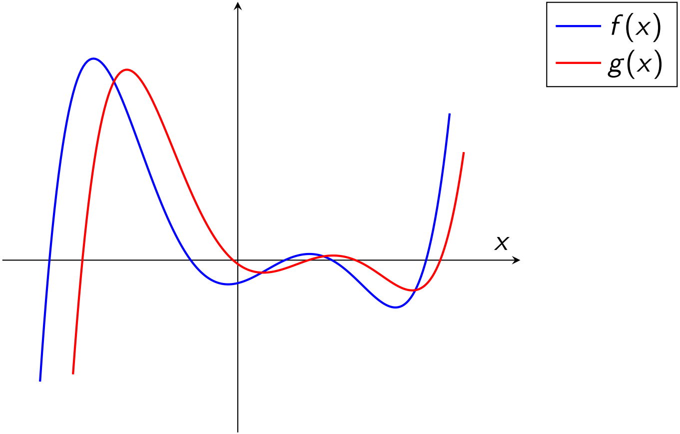
2. $\tilde{\sigma} \in Sb_H^*(X) \implies \tilde{\sigma} \otimes \mathcal{O}_X(H) \in Sb_H^*(X)$.
3. *For any $\tilde{\sigma}_{\underline{s}}, \tilde{\sigma}_{\underline{t}} \in Sb_H^*(X)$ with $\underline{s} < \underline{t}$, the relation $\tilde{\sigma}_{\underline{s}} < \tilde{\sigma}_{\underline{t}}$ holds.*

$\wedge^2 B_n$: Problem at the beginning

For what kind of polynomials $f(x), g(x) \in B_n$, do we have

$$af(x) + bg(x) \in B_n \text{ for every } (a, b) \neq (0, 0)?$$

Interlaced polynomials: $af(x) + bg(x) \in B_n$ for every a, b if and only if their roots alternate.



Denote

$$U_n := \{f_{\underline{t}}(x) + ig_{\underline{s}}(x) \mid \underline{t} < \underline{s} < \underline{t}[1], f, g \text{ monic}\}.$$

Let (X, H) be an n-dimensional smooth polarized variety. There is a natural map

$$U_n \to Hom(\Lambda_H, \mathbb{C}) : \quad \sum a_j x^j \mapsto \sum j! a_j H^{n-j} \mathrm{ch}_j$$

with image $\mathfrak{U}_n$.

Conjecture

There exists a family of stability conditions $Stab^*_H(X)$ *on* $D^b(X)$ *satisfying the following properties:*

1. *The forgetful map*

$$Forg : Stab^*_H(X) \to Hom(\Lambda_H, \mathbb{C}) : \sigma = (\mathcal{P}, Z) \mapsto Z$$

   is a homeomorphism onto $\mathfrak{U}_n$*.*
2. *The space* $Stab^*_H(X)$ *is invariant under the* $\otimes\mathcal{O}_X(H)$*-action.*

Theorem

Conjecture on $Stab^*_H(X)$ *is equivalent to Conjecture* $Sb^*_H(X)$*.*

These conjectures hold for curves and surfaces. In the case of threefolds, it is equivalent to the BBMST conjecture.

Theorem

Assume the Conjecture for X, then:

1. *The family $Stab^*_H(X)$ is unique (up to a homological shift $[2k]$).*
2. *Skyscraper sheaves are σ-stable with respect to all $\sigma \in Stab^*_H(X)$.*
3. *Let Y be a smooth subvariety of X. Then there exists a family of stability conditions on Y.*
4. *For every $\tilde{\sigma}_{\underline{t}}$-stable object E, its character*

$$v_H(E) = \sum (-1)^i a_i v_H(\mathcal{O}_X(t_i H))$$

for some a_i all ≥ 0 or ≤ 0.

Some further questions

- Topology of $Sb(\mathcal{T})$: simply connected?
- Compactification of $Sb(X)$: $\sigma_{\underline{t}}$ with $t_1 \leq t_2 \leq \cdots \leq t_n$.
- $Sb^*(X)$ with respect to the full lattice $K_{num}(X)$.
- About $Stab(X) \neq \emptyset$.

Thank you!