Excellent metrics on triangulated categories

Amnon Neeman

Università degli Studi di Milano

amnon.neeman@unimi.it

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Overview

1. Rickard's 1989 theorem
2. A bunch of definitions
3. The main 2018 theorem
4. Intrinsic equivalence classes of metrics
5. The metrics on $\mathfrak{L}(\mathcal{S})$ and $\mathfrak{S}(\mathcal{S})$

Rickard's 1989 theorem

Theorem

Let R and S be left-coherent rings. Then the following are equivalent:

1. *There exists a triangle equivalence $\mathbf{D}^b(R\text{–proj}) \cong \mathbf{D}^b(S\text{–proj})$.*
2. *There exists a triangle equivalence $\mathbf{D}^b(R\text{–mod}) \cong \mathbf{D}^b(S\text{–mod})$.*

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Questions, Krause 2018:

3. Is it true that $(2) \Longrightarrow (1)$? Challenge: **find a counterexample**.
4. Is there an algorithm to pass directly from the triangulated category $\mathbf{D}^b(R\text{–proj})$ to the triangulated category $\mathbf{D}^b(R\text{–mod})$?

Henning Krause, *Completing perfect complexes*, Math. Z. **296** (2020), no. 3-4, 1387–1427, With appendices by Tobias Barthel, Bernhard Keller and Krause.

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1. For any identity map $\mathrm{id} : X \longrightarrow X$ we have

$$\text{Length}(\mathrm{id}) \quad = \quad 0 \ ,$$

2. and if $x \xrightarrow{f} y \xrightarrow{g} z$ are composable morphisms, then

$$\text{Length}(gf) \quad \leq \quad \text{Length}(f) + \text{Length}(g) \ .$$

The classical literature on the topic

- F. William Lawvere, *Metric spaces, generalized logic, and closed categories*, Rend. Sem. Mat. Fis. Milano **43** (1973), 135–166 (1974).
- Renato Betti and Massimo Galuzzi, *Categorie normate*, Boll. Un. Mat. Ital. (4) **11** (1975), no. 1, 66–75.

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- G. Maxwell Kelly, *Basic concepts of enriched category theory*, London Mathematical Society Lecture Note Series, vol. 64, Cambridge University Press, Cambridge-New York, 1982.
- G. Maxwell Kelly and Vincent Schmitt, *Notes on enriched categories with colimits of some class*, Theory Appl. Categ. **14** (2005), no. 17, 399–423.

Definition (Equivalence of metrics)

We'd like to view two metrics on a category $\mathcal{C}$ as equivalent if the identity functor $\mathrm{id} : \mathcal{C} \longrightarrow \mathcal{C}$ is uniformly continuous in both directions.

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$$\{\text{Length}_1(f) < \delta\} \qquad \Longrightarrow \qquad \{\text{Length}_2(f) < \varepsilon\}$$

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Definition (Cauchy sequences)

Let $\mathcal{C}$ be a category with a metric. A Cauchy sequence in $\mathcal{C}$ is a sequence $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$ of composable morphisms such that, for any $\varepsilon > 0$, there exists an $M > 0$ such that the morphisms $E_i \longrightarrow E_j$ satisfy

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We will assume the category $\mathcal{C}$ is $\mathbb{Z}$–linear. This means that $\mathrm{Hom}(a, b)$ is an abelian group for every pair of objects $a, b \in \mathcal{C}$, and that composition is bilinear.

Definition (The categories $\mathfrak{L}(\mathcal{C})$ and $\mathfrak{S}(\mathcal{C})$)

Let $\mathcal{C}$ be a $\mathbb{Z}$–linear category with a metric. Let $Y : \mathcal{C} \longrightarrow \mathrm{Mod}\text{–}\mathcal{C}$ be the Yoneda map, that is the map sending an object $c \in \mathcal{C}$ to the functor $Y(c) = \mathrm{Hom}(-, c)$, viewed as an additive functor $\mathcal{C}^{\mathrm{op}} \longrightarrow Ab$.

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1. Let $\mathfrak{L}(\mathcal{C})$ be the completion of $\mathcal{C}$, meaning the full subcategory of $\mathrm{Mod}\text{–}\mathcal{C}$ whose objects are the colimits in $\mathrm{Mod}\text{–}\mathcal{C}$ of Cauchy sequences in $\mathcal{C}$.

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   $F : \mathcal{C}^{\mathrm{op}} \longrightarrow Ab$ belongs to $\mathfrak{S}(\mathcal{C})$ if there exists an $\varepsilon > 0$ such that

$$\{\text{Length}(a \to b) < \varepsilon\} \quad \Longrightarrow$$

$$\{F(b) \longrightarrow F(a) \text{ is an isomorphism}\}.$$

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Equivalent metrics lead to identical $\mathfrak{L}(\mathcal{C})$ and $\mathfrak{S}(\mathcal{C})$.

Heuristic

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which means that for any homotopy cartesian square

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ \downarrow & & \downarrow \\ c & \xrightarrow{g} & d \end{array}$$

we must have

$$\text{Length}(f) \quad = \quad \text{Length}(g)$$

Heuristic, continued

Given any $f : a \longrightarrow b$ we may form the homotopy cartesian square

$$\begin{array}{ccc} a & \xrightarrow{\ f\ } & b \\ \big\downarrow & & \big\downarrow \\ 0 & \xrightarrow{\ g\ } & x \end{array}$$

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Hence it suffices to know the lengths of the morphisms

$$0 \longrightarrow x \ .$$

Heuristic, continued

We will soon be assuming that the metric is **non-archimedean**. Replacing the metric by an equivalent (if necessary), we may also assume our metric takes values in the set of rational numbers of the form

$$\{0, \infty\} \cup \{2^n \mid n \in \mathbb{Z}\} .$$

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To know everything about the metric it therefore suffices to specify the balls

$$B_n \quad = \quad \left\{ x \in \mathcal{S} \;\middle|\; \text{the morphism } 0 \longrightarrow x \text{ has length } \leq \frac{1}{2^n} \right\}$$

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If $f:x\longrightarrow y$ is any morphism, to compute its length you complete to a triangle $x\stackrel{f}{\longrightarrow}y\longrightarrow z$, and then

$$\text{Length}(f) \quad = \quad \inf\left\{\frac{1}{2^n}\;\middle|\; z\in B_n\right\}$$

Definition (good metric)

Let $\mathcal{S}$ be a triangulated category. A good metric on $\mathcal{S}$ is a sequence of full subcategories $\{B_n,\ n \in \mathbb{Z}\}$, containing 0 and satisfying

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   This translates to $B_n * B_n = B_n$, which means that if there exists a triangle $b \longrightarrow x \longrightarrow b'$ with $b, b' \in B_n$, then $x \in B_n$.

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Example

Suppose $\mathcal{S}$ has a t-structure. Then $B_n = \mathcal{S}^{\leq -n}$ works.

The main 2018 theorem

Theorem (1)

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Now define the distinguished triangles in $\mathfrak{S}(\mathcal{S})$ to be the colimits in $\mathfrak{S}(\mathcal{S}) \subset \mathrm{Mod}\text{–}\mathcal{S}$ of Cauchy sequences of distinguished triangles in $\mathcal{S}$.

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With this definition of distinguished triangles, the category $\mathfrak{S}(\mathcal{S})$ is triangulated.

Example (the six triangulated categories to keep in mind)

Let R be an associative ring.

1. $\mathbf{D}(R\text{–}\mathrm{Mod})$ has for objects all cochain complexes of R-modules, no conditions.

2. $\mathbf{D}^b(R\text{–}\mathrm{proj})$ is the derived category of bounded complexes of finitely generated, projective R–modules.

3. Suppose the ring R is coherent. Then $\mathbf{D}^b(R\text{–}\mathrm{mod})$ is the bounded derived category of finitely presented R–modules.

Example (the six triangulated categories to keep in mind, continued)

Let X be a quasicompact, quasiseparated scheme, and let $Z \subset X$ be a closed subset with quasicompact complement.

4. $\mathbf{D}_{\mathbf{qc},Z}(X)$ will be our shorthand for $\mathbf{D}_{\mathbf{qc},Z}(\mathcal{O}_X\text{–Mod})$. The objects are the complexes of $\mathcal{O}_X$–modules, and the conditions are that (1) the cohomology must be quasicoherent, and (2) the restriction to $X - Z$ is acyclic.

5. The objects of $\mathbf{D}_Z^{\mathrm{perf}}(X) \subset \mathbf{D}_{\mathbf{qc},Z}(X)$ are the perfect complexes. A complex $F \in \mathbf{D}_{\mathbf{qc}}(X)$ is *perfect* if there exists an open cover $X = \cup_i U_i$ such that, for each U_i, the restriction map $u_i^* : \mathbf{D}_{\mathbf{qc}}(X) \longrightarrow \mathbf{D}_{\mathbf{qc}}(U_i)$ takes F to an object $u_i^*(F)$ isomorphic in $\mathbf{D}_{\mathbf{qc}}(U_i)$ to a bounded complex of vector bundles.

6. Assume X is noetherian. The objects of $\mathbf{D}^b_{\mathbf{coh},Z}(X) \subset \mathbf{D}_{\mathbf{qc},Z}(X)$ are the complexes with coherent cohomology which vanishes in all but finitely many degrees.

Theorem (1, continued)

Now let R be an associative ring. Then the category $\mathbf{D}^b(R\text{–proj})$ admits an **intrinsic metric** *[up to equivalence], so that*

$$\mathfrak{S}\left[\mathbf{D}^b(R\text{–proj})\right] = \mathbf{D}^b(R\text{–mod}).$$

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If we further assume that R is left-coherent then there is on $\left[\mathbf{D}^b(R\text{–mod})\right]^{\mathrm{op}}$ an **intrinsic metric** *[again up to equivalence], such that*

$$\mathfrak{S}\left(\left[\mathbf{D}^b(R\text{–mod})\right]^{\mathrm{op}}\right) = \left[\mathbf{D}^b(R\text{–proj})\right]^{\mathrm{op}} .$$

Theorem (1, continued)

Let X be a quasicompact, quasiseparated scheme, and let $Z \subset X$ be a closed subset with quasicompact complement. There is an **intrinsic equivalence class of metrics** *on* $\mathbf{D}^{\mathrm{perf}}_Z(X)$ *for which*

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Now assume that X is a coherent scheme. Then the category $\left[\mathbf{D}^b_{\mathbf{coh},Z}(X)\right]^{\mathrm{op}}$ can be given an **intrinsic metric** *[up to equivalence], so that*

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Intrinsic equivalence classes of metrics

Recall Rickard's 1989 theorem:

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The theorem makes no mention of metrics.

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The theorem makes no mention of metrics. Until now, what we have honestly explained is that

1. The category $\mathbf{D}^b(R\text{–proj})$ can be given **some metric** $\{B_i,\ i \in \mathbb{N}\}$ for which

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The category $\mathbf{D}^b(R\text{–mod})^{\mathrm{op}}$ can be given **some metric** $\{\widetilde{B}_i,\ i \in \mathbb{N}\}$

$$\mathfrak{S}\Big(\big[\mathbf{D}^b(R\text{–mod})\big]^{\mathrm{op}}\Big) = \big[\mathbf{D}^b(R\text{–proj})\big]^{\mathrm{op}} .$$

The article

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provides recipes, constructing metrics on triangulated categories $\mathcal{S}$.

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Example

Let $\mathcal{S}$ be a triangulated category, and let $G \in \mathcal{S}$ be an object. For any integer $n > 0$, the full subcategory $\langle G \rangle^{(-\infty,-n]}$ is the smallest $\mathcal{L} \subset \mathcal{S}$ subject to

$$G[i] \in \mathcal{L} \quad \forall i \geq n, \qquad \mathcal{L} * \mathcal{L} \subset \mathcal{T}, \qquad \text{add}(\mathcal{L}) \subset \mathcal{L}, \qquad \text{smd}(\mathcal{L}) \subset \mathcal{L}.$$

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$$G[i] \in \mathcal{L} \quad \forall i \geq n, \qquad \mathcal{L} * \mathcal{L} \subset \mathcal{T}, \qquad \text{add}(\mathcal{L}) \subset \mathcal{L}, \qquad \text{smd}(\mathcal{L}) \subset \mathcal{L}.$$

With this notation, the recipe

$$B_n(G) = \langle G \rangle^{(-\infty,-n]}$$

provides a good metric on the category $\mathcal{S}$, for any choice of object $G \in \mathcal{S}$.

The article

Amnon Neeman, *The categories $\mathcal{T}^c$ and $\mathcal{T}^b_c$ determine each other*, https://arxiv.org/abs/1806.06471.

provides recipes, constructing metrics on triangulated categories $\mathcal{S}$.

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With this notation, the recipe

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provides a good metric on the category $\mathcal{S}$, for any choice of object $G \in \mathcal{S}$. And if we stipulate that $G \in \mathcal{S}$ is a classical generator, then the metrics $\{B_n(G),\ n \in \mathbb{N}\}$ **are all equivalent**.

Theorem

The category $\mathbf{D}(R\text{–}\mathrm{proj})$ *has a classical generator. And with the metric being any member of the equivalence class* $\{B_n(G),\ n \in \mathbb{N}\}$ *in the example above, we obtain*

$$\mathfrak{S}\left[\mathbf{D}^b(R\text{–}\mathrm{proj})\right] = \mathbf{D}^b(R\text{–}\mathrm{mod}).$$

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Corollary

Any autoequivalence of the category $\mathbf{D}(R\text{–proj})$ *takes a metric* $\{B_n(G),\ n \in \mathbb{N}\}$ *to an equivalent one* $\{B_n(H),\ n \in \mathbb{N}\}$*, and hence induces an autoequivalence on*

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$$\mathfrak{S}\left[\mathbf{D}^b(R\text{–proj})\right] = \mathbf{D}^b(R\text{–mod}).$$

The category $\mathbf{D}^b(R\text{–mod})$ does not in general have a classical generator. But there is a (more complicated) recipe, providing an equivalence class of metrics that works.

Summarizing: in the article

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And in the recent work which I will discuss today, we reverse this. The question we want to ask is: what hypotheses do we have to impose on the metric, for the passage from $\mathcal{S}$ to $\mathfrak{S}(\mathcal{S})^{\mathrm{op}}$ to be an involution?

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And in the recent work which I will discuss today, we reverse this. The question we want to ask is: what hypotheses do we have to impose on the metric, for the passage from $\mathcal{S}$ to $\mathfrak{S}(\mathcal{S})^{\mathrm{op}}$ to be an involution?

Note that this really is a question about the metric. For any triangulated category $\mathcal{S}$, we can define a good metric $\{B_n,\ n \in \mathbb{N}\}$ by the formula $B_n = \mathcal{S}$. And it is easy to show that, for this metric, $\mathfrak{S}(\mathcal{S}) = \{0\}$. Hence this metric will only be involutive if $\mathcal{S} = \{0\}$.

The metrics on $\mathfrak{L}(\mathcal{S})$ and $\mathfrak{S}(\mathcal{S})$

Definition

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$$\begin{array}{ccc} & & \mathcal{L}_n \\ & & \big\downarrow \\ \mathfrak{S}(\mathcal{S}) & \hookrightarrow & \mathfrak{L}(\mathcal{S}) \end{array}$$

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$$\begin{array}{ccc} \mathcal{N}_n & \hookrightarrow & \mathcal{L}_n \\ \downarrow & & \downarrow \\ \mathfrak{S}(\mathcal{S}) & \hookrightarrow & \mathfrak{L}(\mathcal{S}) \end{array}$$

It can be proved that $\{\mathcal{N}_i,\ i \in \mathbb{N}\}$ **is a good metric on** $\mathfrak{S}(\mathcal{S})$.

The category $\mathfrak{S}(\mathcal{S})^{\mathrm{op}}$ has a good metric $\mathcal{N}_i^{\mathrm{op}}$, and we can perform on $\mathfrak{S}(\mathcal{S})^{\mathrm{op}}$ the constructions

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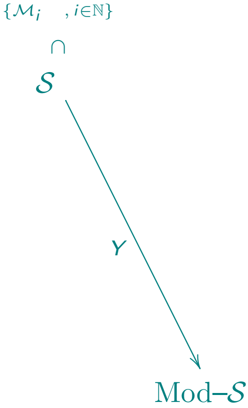
But all of this data came from $\mathcal{S}$ and its metric, and there is a Yoneda map

$$\widehat{Y} \quad : \quad (\mathrm{Mod}\text{–}\mathcal{S})^{\mathrm{op}} \longrightarrow \mathrm{Mod}\text{–}\mathfrak{S}(\mathcal{S})^{\mathrm{op}}$$

$\{\mathcal{M}_i \quad , i \in \mathbb{N}\}$

$\cap$

$\mathcal{S}$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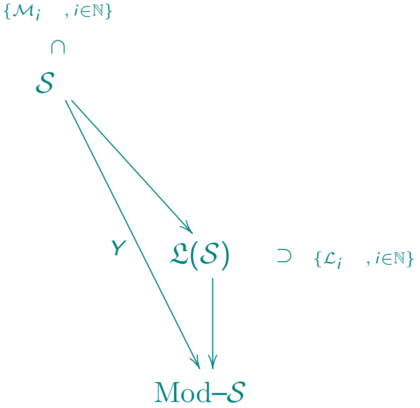
{$\mathcal{M}_i$, $i \in \mathbb{N}$}
$\cap$
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Y
$\mathfrak{L}(\mathcal{S})$
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{$\mathcal{L}_i$, $i \in \mathbb{N}$}
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$$\{\mathcal{M}_i \quad , i\in\mathbb{N}\} \qquad\qquad\qquad \{\mathcal{N}_i \quad , i\in\mathbb{N}\}$$

$$\cap \qquad\qquad\qquad\qquad \cap$$

$$\mathcal{S} \qquad\qquad\qquad\qquad \mathfrak{S}(\mathcal{S})$$

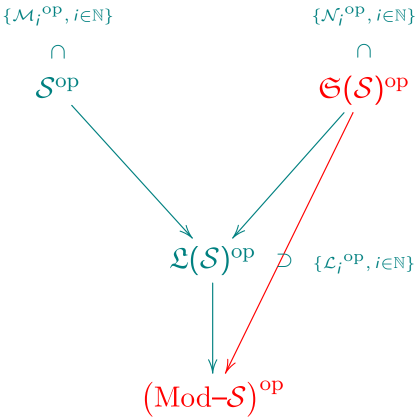
$$\searrow \qquad\qquad \swarrow$$

$$\mathfrak{L}(\mathcal{S}) \quad \supset \quad \{\mathcal{L}_i \quad , i\in\mathbb{N}\}$$

$$\downarrow$$

$$\text{Mod–}\mathcal{S}$$

$$\begin{array}{ccccc} \{\mathcal{M}_i{}^{\mathrm{op}}, i\in\mathbb{N}\} & & & & \{\mathcal{N}_i{}^{\mathrm{op}}, i\in\mathbb{N}\} \\ \cap & & & & \cap \\ \mathcal{S}^{\mathrm{op}} & & & & \mathfrak{S}(\mathcal{S})^{\mathrm{op}} \\ & \searrow & & \swarrow & \\ & & \mathfrak{L}(\mathcal{S})^{\mathrm{op}} \supset \{\mathcal{L}_i{}^{\mathrm{op}}, i\in\mathbb{N}\} & & \\ & & \downarrow & & \\ & & \big(\mathrm{Mod}\text{–}\mathcal{S}\big)^{\mathrm{op}} & & \end{array}$$


{ℳ_i^op, i∈ℕ}
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{𝒩_i^op, i∈ℕ}
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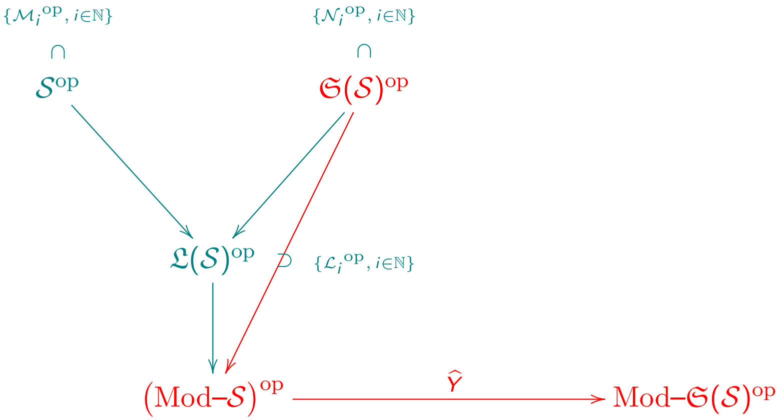
(Mod–𝒮)^op


$\{\mathcal{M}_i{}^{\mathrm{op}}, i\in\mathbb{N}\}$
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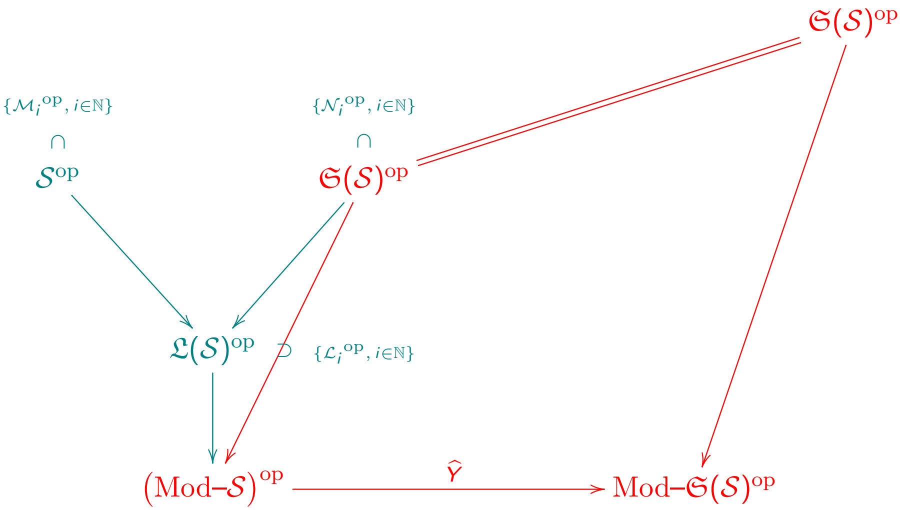
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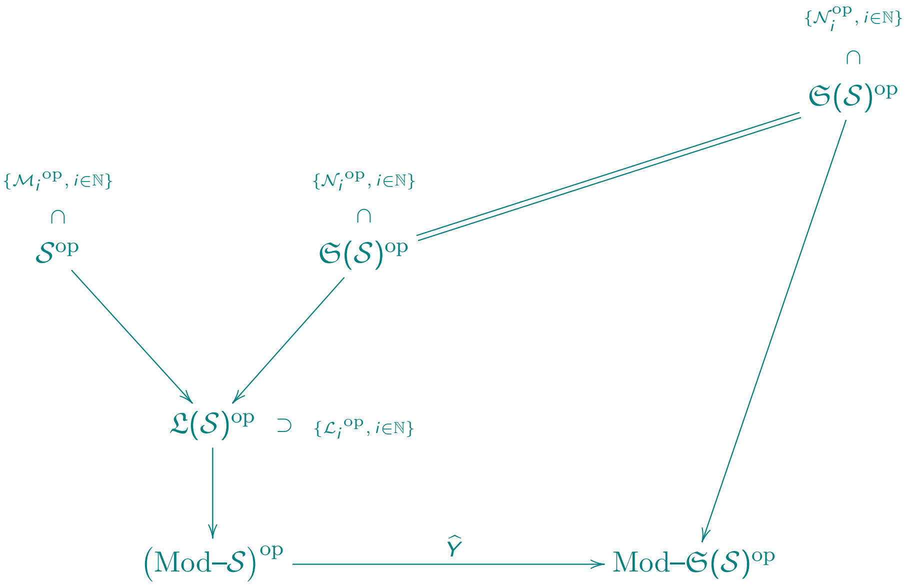
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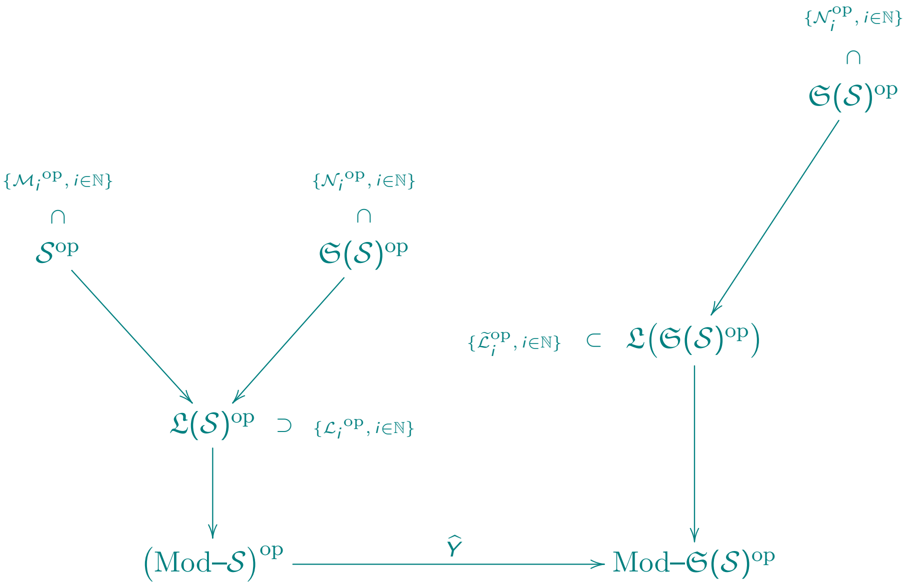

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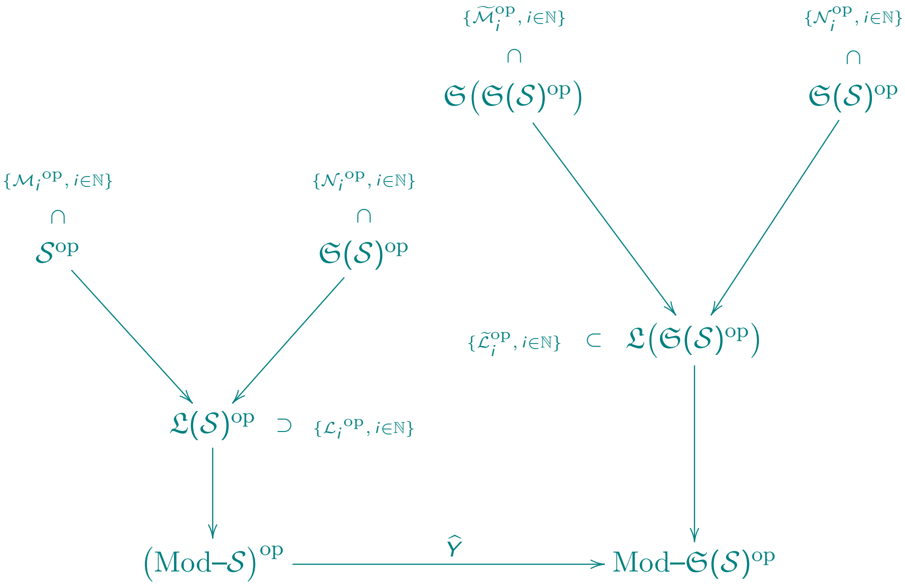
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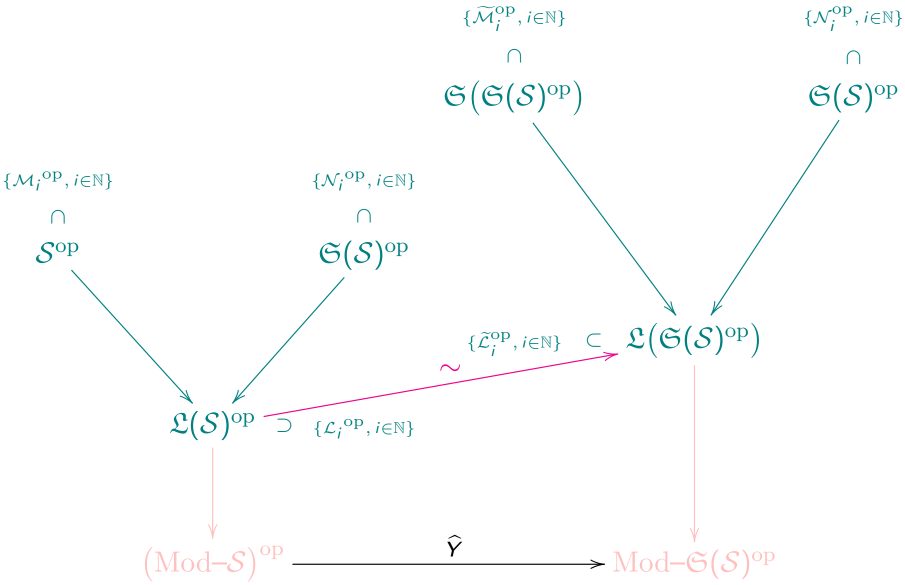
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$$\begin{array}{ccccccc}
A & \xrightarrow{F} & B & \longrightarrow & C & \longrightarrow & A[1] \\
\| & & \| & & \| & & \| \\
\underset{\longrightarrow}{\operatorname{colim}}\, Y(a_*) & \longrightarrow & \underset{\longrightarrow}{\operatorname{colim}}\, Y(b_*) & \longrightarrow & \underset{\longrightarrow}{\operatorname{colim}}\, Y(c_*) & \longrightarrow & \underset{\longrightarrow}{\operatorname{colim}}\, Y\big(a_*[1]\big)
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\end{array}$$

We would like to view $F : A \longrightarrow B$ as **a short morphism** if $C \in \mathcal{L}_n$ for $n \gg 0$.

Problem

The category $\mathfrak{L}(\mathcal{S})$ isn't triangulated, and hence a morphism

$$A \xrightarrow{\quad F \quad} B$$

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2.

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can be completed to a strong triangle in more than one way.

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To solve this problem we make the following

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Let the notation be as above. A **length data** is

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Let the notation be as above. A **length data** is

1. A morphism $F : A \longrightarrow B$, in the category $\mathfrak{L}(\mathcal{S})$.
2. in the category $\mathcal{S}$ a pair of Cauchy sequences a'_* and b'_* with

$$A = \underset{\longrightarrow}{\operatorname{colim}}\, Y(a'_*) , \qquad B = \underset{\longrightarrow}{\operatorname{colim}}\, Y(b'_*) .$$

Lemma

Let $\mathcal{S}$ *be a triangulated category, and let* $\{\mathcal{M}_i,\ i \in \mathbb{N}\}$ *be a good metric on* $\mathcal{S}$*. Suppose we are given a* **length data***, meaning a morphism* $F : A \longrightarrow B$ *in the category* $\mathfrak{L}(\mathcal{S})$*, as well as a pair of Cauchy sequences* a'_* *and* b'_* *in the category* $\mathcal{S}$*, satisfying the requirements.*

Lemma

Let $\mathcal{S}$ be a triangulated category, and let $\{\mathcal{M}_i,\ i \in \mathbb{N}\}$ be a good metric on $\mathcal{S}$. Suppose we are given a **length data**, *meaning a morphism $F : A \longrightarrow B$ in the category $\mathfrak{L}(\mathcal{S})$, as well as a pair of Cauchy sequences a'_* and b'_* in the category $\mathcal{S}$, satisfying the requirements.*

Consider the set $\boldsymbol{\Lambda}$, *of* **all possible** *Cauchy sequence of exact triangles $a_* \longrightarrow b_* \longrightarrow c_* \longrightarrow a_*[1]$ in the category $\mathcal{S}$, with*

1. *a_* is a subsequence of a'_*, and b_* is a subsequence of b'_*.*
2. *The square below commutes*

$$
\begin{array}{ccc}
A & \xrightarrow{\quad F \quad} & B \\
\Big\| & & \Big\| \\
\operatorname*{colim}_{\longrightarrow} Y(a_*) & \xrightarrow{\qquad} & \operatorname*{colim}_{\longrightarrow} Y(b_*)
\end{array}
$$

Lemma

Let $\mathcal{S}$ be a triangulated category, and let $\{\mathcal{M}_i,\ i \in \mathbb{N}\}$ be a good metric on $\mathcal{S}$. Suppose we are given a **length data**, *meaning a morphism $F : A \longrightarrow B$ in the category $\mathfrak{L}(\mathcal{S})$, as well as a pair of Cauchy sequences a'_* and b'_* in the category $\mathcal{S}$, satisfying the requirements.*
Consider the set $\mathbf{\Lambda}$, *of* **all possible** *Cauchy sequence of exact triangles $a_* \longrightarrow b_* \longrightarrow c_* \longrightarrow a_*[1]$ in the category $\mathcal{S}$, with*

1. *a_* is a subsequence of a'_*, and b_* is a subsequence of b'_*.*
2. *The square below commutes*

$$\begin{array}{ccc} A & \xrightarrow{\quad F \quad} & B \\ \Vert & & \Vert \\ \underrightarrow{\mathrm{colim}}\, Y(a_*) & \longrightarrow & \underrightarrow{\mathrm{colim}}\, Y(b_*) \end{array}$$

If one of the Cauchy sequences in $\mathbf{\Lambda}$ *is such that $c_k \in \mathcal{M}_n$ for all $k \gg 0$,* **the same is true for all of them**.

Definition

A morphism $F : A \longrightarrow B$, in the category $\mathfrak{L}(\mathcal{S})$, is declared to be

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$$A = \operatorname*{colim}_{\longrightarrow} Y(a'_*) \ , \qquad B = \operatorname*{colim}_{\longrightarrow} Y(b'_*) \ .$$

3. The length data given by (i) and (ii) above is such that, for any Cauchy sequence of exact triangles $a_* \longrightarrow b_* \longrightarrow c_* \longrightarrow a_*[1]$ in the category $\mathcal{S}$, belonging to the set $\mathbf{\Lambda}$ of the previous slide, we have $c_k \in \mathcal{M}_n$ for all $k \gg 0$.

Remark

Informally: we could consider the category $\mathbf{LD}(\mathcal{S})$, where the objects are pairs (A, a'_*), with

$$A \in \mathfrak{L}(\mathcal{S}) \ , \qquad \text{with } a'_* \text{ a Cauchy sequence in } \mathcal{S}$$

and such that $A = \operatorname*{colim}_{\longrightarrow} Y(a'_*)$.

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With this definition, type-n morphisms should be viewed as morphisms in $\mathbf{LD}(\mathcal{S})$ of length $\leq 2^{-n}$, and this defines a Lawvere metric on $\mathbf{LD}(\mathcal{S})$.

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1. $\mathcal{S} = \cup_{i\in\mathbb{N}}\, {}^{\perp}\mathcal{M}_i$.
2. For every integer $m \in \mathbb{N}$ there exists an integer $n > m$ such that any object $F \in \mathcal{S}$ admits, in the category $\mathcal{S}$, a triangle

$$\begin{array}{ccccccc} E & \longrightarrow & F & \longrightarrow & D & \longrightarrow & E[1] \\ \pitchfork & & & & \pitchfork & & \\ {}^{\perp}\mathcal{M}_n & & & & \mathcal{M}_m & & \end{array}$$

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3. For every integer $m \in \mathbb{N}$ there exists an integer $n > m$ such that any object $F \in \mathcal{S}$ admits, in the category $\mathfrak{L}(\mathcal{S})$, a type-$m$ morphism $Y(F) \longrightarrow D$ with respect to (F, d_*), with $D \in \mathfrak{S}(\mathcal{S}) \cap \mathcal{L}_n^{\perp}$.

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   In the notation (F, d_*), the Cauchy sequences F is taken to be the constant sequence $F \xrightarrow{\mathrm{id}} F \xrightarrow{\mathrm{id}} F \xrightarrow{\mathrm{id}} \cdots$

Theorem

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Then the following holds:

1. *The functor $\widehat{Y}$ below*

$$\left(\text{Mod--}\mathcal{S}\right)^{\text{op}} \xrightarrow{\quad\widehat{Y}\quad} \text{Mod--}\mathfrak{S}(\mathcal{S})^{\text{op}}$$

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2. *The functor $\widehat{Y}$ takes any strong triangle in $\mathfrak{L}(\mathcal{S})$*

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3. *The functor $\widehat{Y}$ takes type-$(n+1)$ morphisms $F:A\longrightarrow B$ in the category $\mathfrak{L}(\mathcal{S})$ to type-n morphisms $\widehat{Y}(F):\widehat{Y}(B)\longrightarrow\widehat{Y}(A)$ in the category $\mathfrak{L}\big(\mathfrak{S}(\mathcal{S})^{\mathrm{op}}\big)$.*

Corollary

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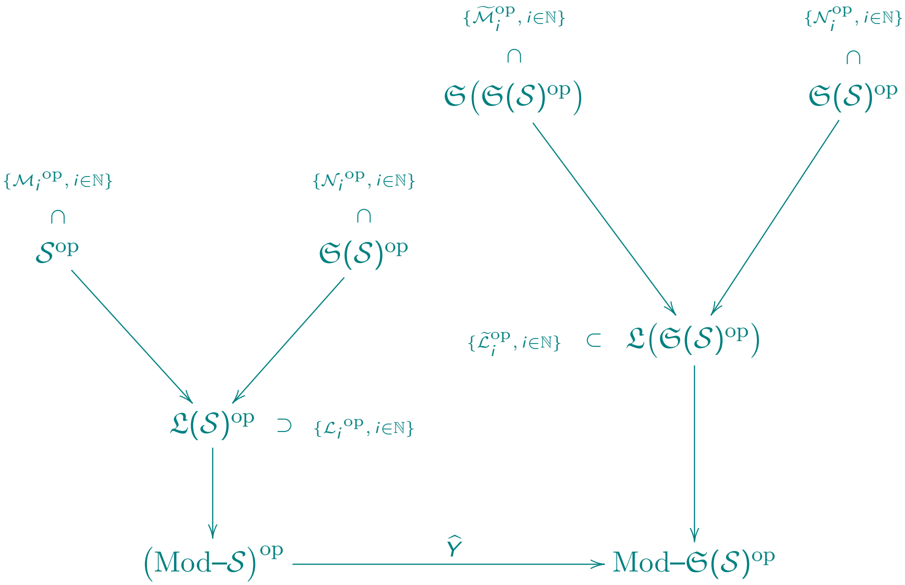
Then the induced metric $\{\mathcal{N}_i^{\mathrm{op}},\ i \in \mathbb{N}\}$ on the triangulated category $\mathfrak{S}(\mathcal{S})^{\mathrm{op}}$ is **also excellent**.


$\{\widetilde{\mathcal{M}}_i^{\mathrm{op}}, i \in \mathbb{N}\}$
$\{\mathcal{N}_i^{\mathrm{op}}, i \in \mathbb{N}\}$
$\cap$
$\mathfrak{S}(\mathfrak{S}(\mathcal{S})^{\mathrm{op}})$
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$\{\widetilde{\mathcal{L}}_i^{\mathrm{op}}, i \in \mathbb{N}\} \subset \mathfrak{L}(\mathfrak{S}(\mathcal{S})^{\mathrm{op}})$
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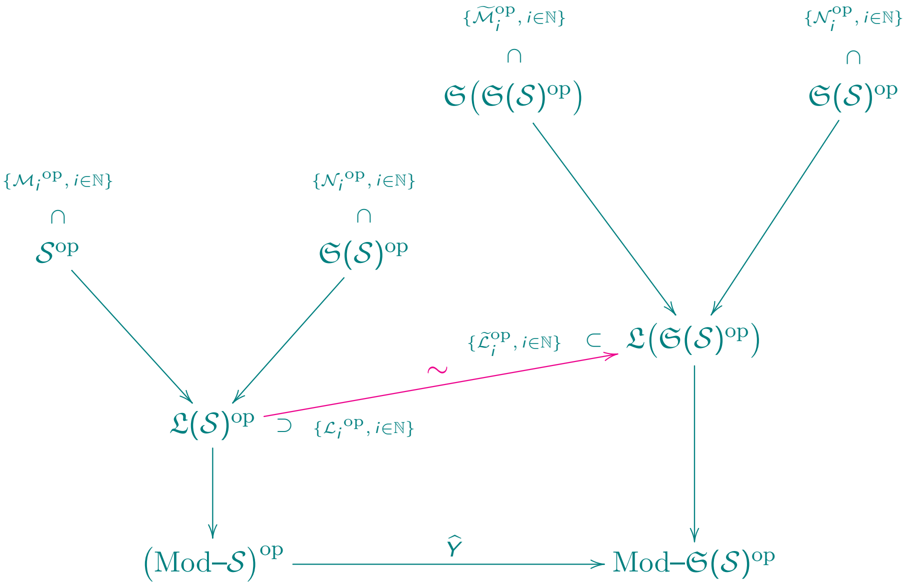

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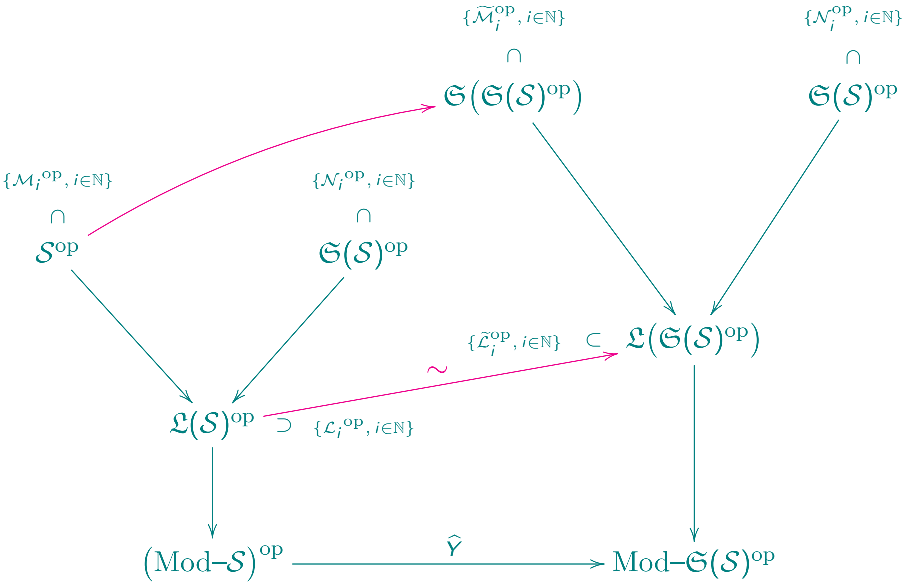

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Example (old)

Let $\mathcal{T}$ be a coherent, weakly approximable triangulated category, let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be a t–structure in the preferred equivalence class, and let the subcategories $\mathcal{T}^c$ and $\mathcal{T}^b_c$ be given the usual meaning.

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$$\mathcal{M}_i = \mathcal{T}^c \cap \mathcal{T}^{\leq -i}, \qquad \mathcal{N}_i = \mathcal{T}^b_c \cap \mathcal{T}^{\leq -i}.$$

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Then the metric $\{\mathcal{M}_i,\, i \in \mathbb{N}\}$ on $\mathcal{T}$ is excellent, with $\mathfrak{S}(\mathcal{S}) = \mathcal{T}^b_c$ having the induced metric $\{\mathcal{N}_i,\, i \in \mathbb{N}\}$.

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If $\mathcal{T} = \mathbf{D}(R\text{–}\mathrm{Mod})$ and the t–structure is the standard one, then $\mathcal{T}^c = \mathbf{D}^b(R\text{–}\mathrm{proj})$ and $\mathcal{T}^b_c = \mathbf{D}^b(R\text{–}\mathrm{mod})$.

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$$\begin{aligned} \mathfrak{S}\Big(\mathbf{D}^b(R\text{–}\mathrm{proj})\Big) &= \mathbf{D}^b(R\text{–}\mathrm{mod})\ , \\ \mathfrak{S}\Big(\mathbf{D}^b(R\text{–}\mathrm{mod})^{\mathrm{op}}\Big) &= D^b(R\text{–}\mathrm{proj})^{\mathrm{op}}\ . \end{aligned}$$

Example (new)

Let R be a ring, let $\mathcal{S} = \mathbf{D}^b(R\text{–}\mathrm{Proj})$, and let $\mathcal{M}_i = \mathbf{D}^b(R\text{–}\mathrm{Proj})^{\leq -i}$. Then the metric $\{\mathcal{M}_i,\ i \in \mathbb{N}\}$ is excellent on the triangulated category $\mathcal{S}$.

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It can be computed that $\mathfrak{S}(\mathcal{S}) = \mathbf{D}^b(R\text{–}\mathrm{Mod})$, and that the metric $\{\mathcal{N}_i,\ i \in \mathbb{N}\}$ on $\mathfrak{S}(\mathcal{S})$ is given by the formula $\mathcal{N}_i = \mathbf{D}^b(R\text{–}\mathrm{Mod})^{\leq -i}$.

Example (new, in gorgeous generality)

More generally: let $\mathcal{T}$ a weakly approximable triangulated category, and let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be a *t*–structure in the preferred equivalence class.

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The category $\mathfrak{S}\Big((\mathcal{T}^b)^{\text{op}}\Big)^{\text{op}}$ can be computed. If $G \in \mathcal{T}$ is a compact generator, then

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And the metric on $\mathfrak{S}\Big(\left(\mathcal{T}^b\right)^{\mathrm{op}}\Big)^{\mathrm{op}}$ is given by the formula

$$\mathcal{M}_i \quad = \quad \mathcal{T}^{\leq -i} \cap \mathfrak{S}\Big(\left(\mathcal{T}^b\right)^{\mathrm{op}}\Big)^{\mathrm{op}} .$$

Example (old and new, schemes)

Let X be a quasicompact, quasiseparated scheme, and let $\mathcal{T}$ be either one of the the pair of triangulated categories below

$$\mathbf{D}^b_{\mathbf{coh}}(X) \quad \subset \quad \mathbf{D}^b_{\mathbf{qc}}(X) \ .$$

With the standard t–structure, define the metric on $\mathcal{T}$ by the formula $\mathcal{N}_i = \mathcal{T}^{\leq -i}$.

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$$\mathbf{D}^b_{\mathbf{coh}}(X) \quad \subset \quad \mathbf{D}^b_{\mathbf{qc}}(X) \ .$$

With the standard t–structure, define the metric on $\mathcal{T}$ by the formula $\mathcal{N}_i = \mathcal{T}^{\leq -i}$.

Then the metrics $\mathcal{N}_i^{\mathrm{op}}$ on $\mathcal{T}^{\mathrm{op}}$ **are both excellent**.

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For X arbitrary, the category

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seems new, although it is easy enough to describe explicitly.

Amnon Neeman, *The categories $\mathcal{T}^c$ and $\mathcal{T}_c^b$ determine each other*, arXiv:1806.06471.

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Amnon Neeman, *Excellent metrics on triangulated categories, and the involutivity of the map taking $\mathcal{S}$ to $\mathfrak{S}(\mathcal{S})^{\mathrm{op}}$*, arXiv:2505.09120.

Thank you!