MUKAI BUNDLES ON FANO THREEFOLDS

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Abstract. We give a proof of Mukai's Theorem on the existence of certain exceptional vector bundles on prime Fano threefolds. To our knowledge this is the first complete proof in the literature. The result is essential for Mukai's biregular classification of prime Fano threefolds, and for the existence of semiorthogonal decompositions in their derived categories.

Our approach is based on Lazarsfeld's construction that produces vector bundles on a variety from globally generated line bundles on a divisor, on Mukai's theory of stable vector bundles on K3 surfaces, and on Brill–Noether properties of curves and (in the sense of Mukai) of K3 surfaces.

Contents

1. Introduction

Let X be a complex smooth prime Fano threefold, i.e., a smooth proper threefold such that

$$\mathrm{Pic}(X) = \mathbb{Z} \cdot K_X$$

and $-K_X$ is ample. The discrete invariant of X is given by the genus $\mathrm{g}(X)$, defined by the equality

$$(-K_X)^3 = 2\mathrm{g}(X) - 2,$$

and by Iskovskikh's fundamental result, see [IP, Theorem 4.3.3], $g = \mathrm{g}(X)$ satisfies

$$2 \leq g \leq 12, \qquad g \neq 11. \tag{1}$$

Following Fano's ideas, Iskovskikh also gave a birational description of all prime Fano threefolds.

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1.1. **Mukai's Theorem.** If $g \le 5$, the image of the anticanonical morphism $X \to \mathbb{P}^{g+1}$ has small codimension, which one can use to describe X as a complete intersection in a weighted projective space, see [IP, Section 12.2]. For $g \ge 6$, the image of the anticanonical morphism is no longer a complete intersection, and an alternative method for the biregular classification of X was needed. This was provided by Mukai, based on a version of the following result.

Theorem 1.1. *Let $\Bbbk$ be an algebraically closed field of characteristic zero. Let X be a smooth prime Fano threefold over $\Bbbk$ of genus $\mathrm{g}(X) = r \cdot s \ge 6$, for $r, s \ge 2$. Then there exists a unique stable vector bundle $\mathcal{U}_r$ on X with*

$$\mathrm{rk}(\mathcal{U}_r) = r, \qquad \mathrm{c}_1(\mathcal{U}_r) = K_X, \qquad \mathrm{H}^\bullet(X, \mathcal{U}_r) = 0 \qquad \textit{and} \qquad \mathrm{Ext}^\bullet(\mathcal{U}_r, \mathcal{U}_r) = \Bbbk.$$

Moreover, $\mathcal{U}_r^\vee$ is globally generated with $\dim \mathrm{H}^0(X, \mathcal{U}_r^\vee) = r + s$ and $\mathrm{H}^{>0}(X, \mathcal{U}_r^\vee) = 0$.

Since $\mathcal{U}_r^\vee$ is globally generated, the anticanonical morphism of X factors, up to a linear projection, through a morphism

$$X \to \mathrm{Gr}(r, r + s),$$

to the Grassmannian of r-dimensional subspaces in a vector space of dimension $r + s$. Using the decompositions $g = 2 \cdot s$ for $g \in \{6, 8, 10\}$ and $g = 3 \cdot s$ for $g \in \{9, 12\}$ and studying this morphism, Mukai found an explicit description of any prime Fano threefold X of genus $g \ge 6$.

Theorem 1.2 (Mukai)**.** *Let $\Bbbk$ be an algebraically closed field of characteristic zero and let X be a smooth prime Fano threefold over $\Bbbk$ of genus $g \ge 6$.*

- *If $g = 6$ then X is a complete intersection of a quadric and three hyperplanes in the cone in $\mathbb{P}^{10}$ over $\mathrm{Gr}(2, 5) \subset \mathbb{P}^9$;*
- *if $g = 7$ then X is a dimensionally transverse linear section of $\mathrm{OGr}_+(5, 10) \subset \mathbb{P}^{15}$, the connected component of the Grassmannian of isotropic 5-dimensional subspaces in a 10-dimensional vector space endowed with a non-degenerate symmetric bilinear form;*
- *if $g = 8$ then X is a dimensionally transverse linear section of $\mathrm{Gr}(2, 6) \subset \mathbb{P}^{14}$;*
- *if $g = 9$ then X is a dimensionally transverse linear section of $\mathrm{LGr}(3, 6) \subset \mathbb{P}^{13}$, the Grassmannian of isotropic 3-dimensional subspaces in a 6-dimensional vector space endowed with a non-degenerate skew-symmetric bilinear form;*
- *if $g = 10$ then X is a dimensionally transverse linear section in $\mathbb{P}^{13}$ of a 5-dimensional homogeneous variety of the simple algebraic group of Dynkin type $\mathbf{G}_2$; and*
- *if $g = 12$ then X is the subvariety in $\mathrm{Gr}(3, 7) \subset \mathbb{P}^{34}$, parameterizing 3-dimensional subspaces isotropic for three skew-symmetric bilinear forms.*

Note that in the case $g = 7$ Theorem 1.1 does not produce a morphism $X \to \mathrm{Gr}(5, 10)$; however, one can use the twist $\mathcal{U}_5 := \mathcal{N}_{X/\mathbb{P}^8}(2K_X)$ of the normal bundle of X in the anticanonical embedding, and the technique of [DK] (where a similar bundle was used in the case $g = 6$).

The significance of Mukai's results for the study of Fano threefolds and K3 surfaces is hard to overestimate. They have applications ranging from computing quantum periods (e.g., [CCGK]), the classification of 2-Fano manifolds (e.g., [AC]), and Picard groups of moduli spaces of K3 surfaces (e.g., [GLT]), to the study of curves (e.g., [CFM, CU]) and automorphism groups (e.g., [KPS]). They were also used to define interesting semiorthogonal decompositions of derived categories of Fano threefolds, see [K].

However, to the best of our knowledge, a complete direct proof of these results still cannot be found in the literature, see Section 1.5 for a detailed discussion. The main goal of our paper is to address this gap by giving a complete proof of Theorem 1.1. We also generalize it to include some singular Fano threefolds, see Theorem 5.3 for the full statement and Section 1.4 for a discussion of possible further generalizations. The proof of Theorem 1.2 has also only been sketched in the literature, and we plan to return to it in a future paper.

1.2. **Lazarsfeld bundles on K3 surfaces.** Our proof of Theorem 1.1 uses ideas of Mukai and two extra ingredients: Lazarsfeld's fundamental observation in [L] that Brill-Noether properties of line bundles on a divisor are naturally encoded by its associated vector bundle on the ambient variety (perhaps first constructed in [M]), and Brill–Noether and Petri generality properties of K3 surfaces and curves, see §2 for a review of what is relevant for us.

In particular, given a curve C with an embedding $j\colon C \hookrightarrow S$ into a K3 surface S, and a globally generated line bundle ξ on C, the vector bundle $\mathbf{L}_S(\xi)$ on S is defined by the exact sequence

$$0 \to \mathbf{L}_S(\xi) \to \mathrm{H}^0(C,\xi) \otimes \mathcal{O}_S \to j_*\xi \to 0.$$

We apply this construction when C is a smooth Brill–Noether–Petri general curve and ξ satisfies

$$\deg(\xi) = (r-1)(s+1) \qquad \text{and} \qquad h^0(\xi) = r. \tag{2}$$

where $\mathrm{g}(C) = r \cdot s$ is a fixed factorization. (The existence of such ξ follows from the Brill–Noether generality of C; however, ξ is not unique, see Remark 2.4.) The vector bundle $\mathbf{L}_S(\xi)$ has the same invariants as the restriction of a Mukai bundle from an ambient Fano threefold, and we show that Petri generality of C implies that $\mathbf{L}_S(\xi)$ is simple and rigid, i.e., spherical.

It is easy to see that $\mathbf{L}_S(\xi)$ is stable if $\mathrm{Pic}(S)$ is generated by the class of C. The first important step in our proof of Theorem 1.1 is the following general stability criterion for $\mathbf{L}_S(\xi)$ in terms of an extension of the class ξ to $\mathrm{Pic}(S)$.

Theorem 1.3 (cf. Theorem 3.4). *Let $\mathbf{L}_S(\xi)$ be the spherical vector bundle associated to a smooth Brill–Noether–Petri general curve C on a K3 surface S and a globally generated line bundle ξ on C satisfying* (2)*, where $r \in \{2, 3\}$ and $rs = \mathrm{g}(C)$. Let H be the class of C in $\mathrm{Pic}(S)$. If $\mathbf{L}_S(\xi)$ is not H-Gieseker stable then ξ extends to S. More precisely, there is a base point free divisor class $\Xi \in \mathrm{Pic}(S)$ such that $\mathcal{O}_S(\Xi)|_C \cong \xi$ and*

$$H \cdot \Xi = (r-1)(s+1), \qquad h^0(\mathcal{O}_S(\Xi)) = r, \qquad \text{and} \qquad h^1(\mathcal{O}_S(\Xi)) = 0. \tag{3}$$

Moreover, if $\mathrm{Pic}(S)$ does not have a class Ξ satisfying (3) *then $\mathbf{L}_S(\xi)$ does not depend on C or ξ.*

We call divisor classes Ξ satisfying (3) special Mukai classes of type (r, s). In addition to Ξ being an extension of ξ from C to S—with both ξ and Ξ being extremal in the sense of the Brill–Noether theory for C and S, respectively—the significance of Ξ is that the analogue $\mathbf{L}_X(\Xi)$ on X of the Lazarsfeld construction $\mathbf{L}_S(\xi)$ on S will produce our sought-after Mukai bundle on X.

The next result characterizes the restriction of Lazarsfeld bundles $\mathbf{L}_S(\xi)$ back to the curve C.

Theorem 1.4 (cf. Theorem 4.13). *Let $\mathbf{L}_S(\xi)$ be the spherical vector bundle associated to a Brill–Noether–Petri general curve C on a K3 surface S and a globally generated line bundle ξ on C satisfying* (2)*, where $rs = \mathrm{g}(C)$. Then there is an exact sequence*

$$0 \to \big(\mathrm{H}^0(C,\xi)^\vee \otimes \mathcal{O}_C\big)/\xi^{-1} \to \mathbf{L}_S(\xi)|_C \to \xi^{-1}(K_C) \to 0. \tag{4}$$

Moreover, if $\operatorname{Pic}(S) = \mathbb{Z} \cdot [C]$ *and* $r \in \{2, 3\}$*, the extension class of* (4) *is uniquely determined by the property that the connecting homomorphism*

$$\mathrm{H}^0(C, \xi^{-1}(K_C)) \to \mathrm{H}^1(C, (\mathrm{H}^0(C, \xi)^\vee \otimes \mathcal{O}_C)/\xi^{-1})$$

in the cohomology exact sequence of (4) *is zero.*

Theorems 1.3 and 1.4 form the technical core of our paper. We prove them for $r \in \{2, 3\}$ (but for arbitrary g), but we expect these results to be true for any r and g, and it would be interesting to find a proof.

Theorem 1.4 is a variation of a long-running theme. When one assumes additionally that the extension class (4) produces a slope-semistable vector bundle on C, it fits into a series of results that stable vector bundles on C with large number of sections are restrictions of vector bundles on S, see, e.g., [Mu4, Section 10], [ABS], [F1]. Closest to our case is [F2], as it covers the case of spherical vector bundles; it treats the case when $g \gg 0$; see [F2, Remark 4.3] for the precise list of conditions. Our proof is more elementary, but its idea is motivated by Feyzbakhsh' approach.

Even closer are the results of [V, Section 3]: the rank 2 case of Theorem 1.4 follows from Voisin's results in the case where S is general. Indeed, Lemma 3.18 of [V] shows the uniqueness of the extension under the assumption [V, 3.1(i)]. Moreover, by the proof of [V, Proposition 4.1], this assumption holds when C is a general curve on a general K3 surface with $\operatorname{Pic}(S) = \mathbb{Z} \cdot [C]$.

Meanwhile, Theorem 1.3 is reminiscent to results on the Donagi–Morrison Conjecture; see, e.g., [L-C, AH, H]. Often, such results show that line bundles on C with negative Brill–Noether number are, after adding an effective divisor, obtained by restriction of a line bundle on S, and take advantage of the fact that the associated Lazarsfeld bundle is not simple. While we obtain a precise extension for a Brill–Noether extremal line bundle ξ under the assumption that the Lazarsfeld bundle (which is simple in our case) is not stable.

1.3. **Our argument.** We use Theorems 1.3 and 1.4 to prove Theorem 1.1 as follows. Let X be a prime Fano threefold of genus $g \geq 6$ and let $S \subset X$ be a smooth hyperplane section such that

$$\operatorname{Pic}(S) = \mathbb{Z} \cdot H_S, \tag{5}$$

where H_S stands for the restriction of $H \coloneqq -K_X$ to S; for smooth X this assumption is automatically satisfied by the Noether–Lefschetz theorem [Mo, Theorem 7.5]. First, we apply [L, Theorem] to find a Brill–Noether–Petri general curve $C \subset S$ in the linear system $|H_S|$ such that the pencil $\{S_t\}_{t \in \mathbb{P}^1}$ of anticanonical divisors of X through C is a Lefschetz pencil. This pencil contains S, hence $\operatorname{Pic}(S_t) \cong \mathbb{Z} \cdot H_{S_t}$ for very general t.

Next, we consider the blowup $\widetilde{X} \coloneqq \mathrm{Bl}_C(X)$ as a family of K3 surfaces over $\mathbb{P}^1$, containing the fixed curve C as the exceptional divisor $E = C \times \mathbb{P}^1$. We choose a globally generated line bundle $\xi \in \operatorname{Pic}(C)$ satisfying (2), where $rs = \mathrm{g}(C)$, and apply a relative version of the Lazarsfeld construction to the line bundle $\xi \boxtimes \mathcal{O}_{\mathbb{P}^1}$ on E. As $\operatorname{Pic}(S_t) = \mathbb{Z} \cdot H_{S_t}$ for very general $t \in \mathbb{P}^1$, the restriction $\mathbf{L}_{\widetilde{X}/\mathbb{P}^1}(\xi \boxtimes \mathcal{O})|_{S_t}$ of the Lazarsfeld bundle $\mathbf{L}_{\widetilde{X}/\mathbb{P}^1}(\xi \boxtimes \mathcal{O})$ on $\widetilde{X}$ is stable in this case. Then Theorem 1.4 implies that the extension classes of the sequences (4) associated with the restrictions

$$(\mathbf{L}_{\widetilde{X}/\mathbb{P}^1}(\xi \boxtimes \mathcal{O})|_{S_t})|_C \cong (\mathbf{L}_{S_t}(\xi))|_C$$

are all proportional to a fixed class. It follows that this extension class vanishes for some $t_0 \in \mathbb{P}^1$, hence the Lazarsfeld bundle $\mathbf{L}_{\widetilde{X}/\mathbb{P}^1}(\xi \boxtimes \mathcal{O})|_{S_{t_0}}$ is not stable and, by Theorem 1.3, that the surface S_{t_0}

has a special Mukai class Ξ of type (r, s). More precisely, since S_{t_0} may be singular, we consider the minimal resolution $\sigma\colon \widetilde{S}_{t_0} \to S_{t_0}$ and apply Theorem 3.4, a more precise version of Theorem 1.3, to deduce the existence of the divisor class Ξ on the minimal resolution $\widetilde{S}_{t_0}$ of S_{t_0}.

Finally, we apply the Lazarsfeld construction to the sheaf $\sigma_*\mathcal{O}_{S_{t_0}}(\Xi)$ to obtain a vector bundle $\mathbf{L}_X(\Xi)$ on X, and check that, if $g \geq 6$ it satisfies the requirements of Theorem 1.1, see Corollaries 5.6 and 5.9 for details. To prove exceptionality (and uniqueness) of $\mathbf{L}_X(\Xi)$ we use an extra cohomology vanishing for the Lazarsfeld bundle on S proved in Proposition 4.2.

1.4. **Further generalizations.** It would be interesting and useful to extend our argument (and thus Theorems 1.1 and 1.2) to more general situations.

Factorial prime Fano threefolds with a single ordinary double point were shown in [KP] to satisfy the description of Theorem 1.2 (based on a completely different approach); in particular they have Mukai bundles. It is natural to expect (though we cannot prove this) that these threefolds have a smooth anticanonical divisor satisfying (5). If this is the case then our construction proves the existence of a reflexive sheaf $\mathcal{U}_r$ satisfying all the properties of Theorem 1.1 except for the property $\operatorname{Ext}^\bullet(\mathcal{U}_r, \mathcal{U}_r) = \Bbbk$. We think it makes sense to extend our methods to more general singular Fano threefolds: factorial prime Fano threefolds, or even *Brill–Noether general* Fano threefolds in the sense of [Mu5, Definition 6.4], see [Mu5, Theorem 6.5]. Moreover, even on nonfactorial 1-nodal Fano threefolds one can often construct an "almost Mukai" bundle, i.e., a vector bundle $\mathcal{U}_r$ that satisfies all the properties of Theorem 1.1 except possibly for the global generation of $\mathcal{U}_r^\vee$, see [KS, Proposition 3.3 and Remark 3.5]. This suggests that Theorem 1.1 can be generalized even further. However, Theorem 1.2 fails for nonfactorial threefolds, see [Mu6].

The second possible direction of generalization is to positive characteristic.

In both cases the most problematic step of our argument is the existence of a smooth anticanonical divisor $S \subset X$ satisfying (5). Recall that the existence of such S is used twice in the proof: first to apply [L] and conclude that X contains a Brill–Noether–Petri general curve C, and second to prove the uniqueness of the extension class of the sequence (4) in Theorem 4.13. Condition (5) cannot be satisfied in positive characteristic (because the Picard number of any K3 surface is even), nor for nonfactorial X (because the Picard number of an anticanonical divisor of X is greater or equal than the rank of the class group of X). However, we hope that (5) can be replaced by the weaker Brill–Noether generality property of S.

Another interesting question is the following. Note that Theorems 1.3 and 1.4 are concerned with special vector bundles on K3 surfaces and curves and do not involve a Fano threefold. Therefore, they make sense for any value of g, not only for the Fano range (1), and consequently it is interesting to ask if the corresponding results (Theorems 3.4 and 4.13) stay true for $r > 3$. Again, we currently have no approach towards proofs in this generality.

1.5. **History of Theorem 1.1.** Theorem 1.1 was first announced by Mukai in [Mu2]. The argument sketched there, expanded upon in [IP, §5.2], relies on Fujita's extension theorem in [Fu] for sheaves $\mathcal{F}$ on an ample divisor D to the ambient variety X. However, Fujita's theorem does not apply when D is a surface, as it relies on the vanishing of $\mathrm{H}^2(D, \mathcal{E}nd(\mathcal{F}) \otimes \mathcal{O}_X(-nD)|_D)$ for $n \geq 1$.

We are also aware of a sketch of an argument written by Mukai in [Mu5] below Theorem 6.5, which relies on the analogue [Mu5, Theorem 4.7] of Theorem 1.2 for K3 surfaces, the proof of which in turn is also only given as a sketch.

An attempt at a different proof was given in [BLMS, Theorem 6.2]. However, the proof given there is also incomplete at best. It claims that for a K3 surface $S \subset X$, there is a *stable* bundle $\mathcal{U}$ of the right invariants; for higher Picard rank of S, this is not clear a priori. More crucially, it claims that as S varies in a pencil, the restriction of $\mathcal{U}$ to the base locus remains constant; we are not able to reproduce the deformation argument alluded to there.

Finally, in the cases where $g \in \{6, 8\}$, Theorem 1.1 (and even Theorem 1.2) was deduced by Gushel′ in [Gu1, Gu2] from the existence of special elliptic curves on smooth hyperplane sections $S \subset X$ (these elliptic curves are precisely special Mukai classes of types $(2, 3)$ and $(2, 4)$, respectively). The existence of elliptic curves on other Fano threefolds of even genus and a proof of Theorem 1.1 for $r = 2$ was recently given in [CFK]. This argument seems to be independent of Mukai's classification results. It is more indirect, as it relies on the irreducibility of the moduli space of Fano threefolds, proved in [CLM, Theorem 7].

1.6. **Notation and conventions.** We work over an algebraically closed field $\Bbbk$ of characteristic zero; all schemes and morphisms are assumed to be $\Bbbk$-linear. Given a morphism f we let f_* and f^* denote the *underived* pushforward and pullback functors; the corresponding derived functors are denoted by $\mathrm{R}f_*$ and $\mathrm{L}f^*$, respectively. We write $[1]$ for the shift functor in the derived category.

For a coherent sheaf $\mathcal{F}$ on a scheme S we write

$$h^i(\mathcal{F}) \coloneqq \dim \mathrm{H}^i(S, \mathcal{F}).$$

Furthermore, we write

$$\chi(\mathcal{F}) \coloneqq \sum (-1)^i h^i(\mathcal{F}) \qquad \text{and} \qquad \chi(\mathcal{F}_1, \mathcal{F}_2) \coloneqq \sum (-1)^i \dim \mathrm{Ext}^i(\mathcal{F}_1, \mathcal{F}_2)$$

for the Euler characteristic of a sheaf $\mathcal{F}$ and the Euler bilinear form, respectively.

For a sheaf $\mathcal{F}$ of finite projective dimension on a K3 surface S with du Val singularities we write

$$\mathbf{v}(\mathcal{F}) \coloneqq (\mathrm{rk}(\mathcal{F}), \mathrm{c}_1(\mathcal{F}), \mathrm{ch}_2(\mathcal{F}) + \mathrm{rk}(\mathcal{F})) \in \mathbb{Z} \oplus \mathrm{Pic}(S) \oplus \mathbb{Z}$$

for the Mukai vector of $\mathcal{F}$. The Mukai pairing on $\mathbb{Z} \oplus \mathrm{Pic}(S) \oplus \mathbb{Z}$ is given by

$$\langle (r_1, D_1, s_1), (r_2, D_2, s_2) \rangle \coloneqq -r_1 s_2 + D_1 \cdot D_2 - s_1 r_2.$$

The Riemann–Roch Theorem translates then into the equality

$$\chi(\mathcal{F}_1, \mathcal{F}_2) = -\langle \mathbf{v}(\mathcal{F}_1), \mathbf{v}(\mathcal{F}_2) \rangle.$$

In particular, since $\mathbf{v}(\mathcal{O}_S) = (1, 0, 1)$, for $\mathbf{v}(\mathcal{F}) = (r, D, s)$ we obtain

$$\chi(\mathcal{F}) = r + s, \qquad \chi(\mathcal{F}, \mathcal{F}) = 2rs - D^2. \tag{6}$$

Finally, given an ample divisor A on S, the reduced Hilbert polynomial with respect to A of a sheaf $\mathcal{F}$ of positive rank has the form

$$\mathbf{p}_A(\mathcal{F})(t) = \tfrac{1}{2} A^2 t^2 + \mu_A(\mathcal{F}) t + \delta(\mathcal{F}),$$

where

$$\mu_A(\mathcal{F}) \coloneqq \tfrac{A \cdot \mathrm{c}_1(\mathcal{F})}{\mathrm{rk}(\mathcal{F})}, \qquad \delta(\mathcal{F}) \coloneqq \tfrac{\chi(\mathcal{F})}{\mathrm{rk}(\mathcal{F})} \tag{7}$$

is the slope and the reduced Euler characteristic of $\mathcal{F}$, respectively.

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2. Brill–Noether theory

In this section we recall some facts from Brill–Noether theory for curves and K3 surfaces and introduce Mukai special classes that play an important role in the rest of the paper.

2.1. **Curves.** We refer to [ACGH, Ch. IV–V] for a general treatment of Brill–Noether theory for curves.

Let C be a smooth proper curve of genus g. For $1 \le d \le 2g-3$ and $r \ge 0$ the Brill–Noether locus $\mathrm{W}^r_g(C) \subset \mathrm{Pic}^d(C)$ is defined as

$$\mathrm{W}^r_d(C) \coloneqq \{\mathcal{L} \in \mathrm{Pic}^d(C) \mid h^0(\mathcal{L}) \ge r+1,\ h^1(\mathcal{L}) \ge 1\}.$$

If a line bundle $\mathcal{L}$ corresponds to a point of $\mathrm{W}^r_d(C) \setminus \mathrm{W}^{r+1}_d(C)$ the cotangent space to $\mathrm{W}^r_d(C)$ at this point is equal to the cokernel of the Petri map

$$\mathrm{H}^0(C,\mathcal{L}) \otimes \mathrm{H}^0(C,\mathcal{L}^{-1}(K_C)) \to \mathrm{H}^0(C,\mathcal{O}_C(K_C)).$$

It is well known (see, e.g., [ACGH, Theorem V.1.1]) that for $r \ge d-g$ one has

$$\dim \mathrm{W}^r_d(C) \ge g-(r+1)(g-d+r),$$

and that if the right-hand side (the expected dimension of $\mathrm{W}^r_d(C)$) is nonnegative, $\mathrm{W}^r_d(C) \ne \varnothing$.

Definition 2.1. A smooth proper curve C is Brill–Noether general (BN-general) if the locus $\mathrm{W}^r_d(C)$ is nonempty if and only if $(r+1)(g-d+r) \le g$; in other words,

$$h^0(\mathcal{L}) \cdot h^1(\mathcal{L}) \le g$$

for any line bundle $\mathcal{L}$ on C. Furthermore, we will say that C is Brill–Noether–Petri general (BNP-general) if it is BN-general and the Petri map is injective for any line bundle $\mathcal{L}$ on C.

A general curve of genus g is known to be BNP-general, see [G] and [L]. For BNP-general curves the locus $\mathrm{W}^r_d(C) \setminus \mathrm{W}^{r+1}_d(C)$ is smooth and nonempty of dimension $g-(r+1)(g-d+r)$ whenever this number is nonnegative ([ACGH, Proposition IV.4.2 and Theorem V.1.7]).

The following definition axiomatizes the situation studied in [Mu3, §3].

Definition 2.2. A pair of line bundles (ξ,η) on a curve C of genus $g = r \cdot s$ is a Mukai pair of type (r,s) if $\xi \otimes \eta \cong \mathcal{O}_C(K_C)$,

$$\deg(\xi) = (r-1)(s+1), \qquad h^0(\xi) = r, \tag{8}$$

and both ξ and η are globally generated.

A simple Riemann–Roch computation shows that if (ξ, η) is a Mukai pair of type (r, s) then

$$\deg(\eta) = (r+1)(s-1), \qquad h^0(\eta) = s,$$

and (η, ξ) is a Mukai pair of type (s, r). Thus, the definition is symmetric.

The following lemma deduces the existence of Mukai pairs from Brill–Noether theory.

Lemma 2.3. *Any BN-general curve C of genus $g = r \cdot s$ has a Mukai pair of type (r, s). If, moreover, C is BNP-general, the Petri maps for ξ and η are isomorphisms.*

Proof. Since $r \cdot s = g$, the expected dimension of $\mathrm{W}^{r-1}_{(r-1)(s+1)}(C)$ is 0, hence $\mathrm{W}^{r-1}_{(r-1)(s+1)}(C) \neq \varnothing$ by [ACGH, Theorem V.1.1]. This means that there is a line bundle ξ such that

$$\deg(\xi) = (r-1)(s+1), \qquad \text{and} \qquad h^0(\xi) \geq r.$$

Note that $\chi(\xi) = (r-1)(s+1)+1-rs = r-s$ by Riemann–Roch. So, if $h^0(\xi) > r$ then $h^1(\xi) > s$ which contradicts to BN-generality of C, hence $h^0(\xi) = r$ and $h^1(\xi) = s$.

Similarly, if ξ is not globally generated at a point P then $\xi(-P)$ violates the BN property. Therefore, ξ is globally generated, and by the symmetry of the definition the same holds for η.

The Petri map for ξ is injective by the BNP-generality of C, and as the dimensions of its source and target are the same, it is an isomorphism; again, the same property for η follows. ☐

Remark 2.4. If C is BNP-general, the Brill–Noether locus $\mathrm{W}^{r-1}_{(r-1)(s+1)}(C)$ is reduced and zero-dimensional of length equal to the degree of the Grassmannian $\mathrm{Gr}(r, r+s)$, see [EH]. Therefore, this degree is exactly the number of Mukai pairs of type (r, s) on a BNP-general curve.

The following proposition will be important for the study of Lazarsfeld bundles. Part (a), for any r, is attributed to Castelnuovo in [C, Theorem 1.11].

Proposition 2.5. *Let C be a BNP-general curve of genus $g = r \cdot s$ with $s \geq r$ and $r \in \{2, 3\}$.*

(a) *If (ξ, η) is a Mukai pair on C of type (r, s) then the natural morphism*

$$\mathrm{H}^0(C, \xi) \otimes \mathrm{H}^0(C, \mathcal{O}_C(K_C)) \to \mathrm{H}^0(C, \xi(K_C))$$

is surjective.

(b) *If (ξ_1, η_1) and (ξ_2, η_2) are Mukai pairs of type (r, s) with $\xi_1 \not\cong \eta_2$ then the natural morphism*

$$\mathrm{H}^0(C, \xi_1) \otimes \mathrm{H}^0(C, \eta_2(K_C)) \to \mathrm{H}^0(C, \xi_1 \otimes \eta_2(K_C))$$

is surjective.

Proof. (a) First, assume $r = 2$. As ξ is globally generated by definition, we have an exact sequence

$$0 \to \xi^{-1} \to \mathrm{H}^0(C, \xi) \otimes \mathcal{O}_C \to \xi \to 0.$$

Twisting it by K_C, we obtain

$$0 \to \eta \to \mathrm{H}^0(C, \xi) \otimes \mathcal{O}_C(K_C) \to \xi(K_C) \to 0.$$

As $h^1(\eta) = h^0(\xi) = h^1(\mathrm{H}^0(C, \xi) \otimes \mathcal{O}_C(K_C))$ by Serre duality and $h^1(\xi(K_C)) = 0$, this exact sequence induces a surjection of global sections, which is our claim.

Now assume $r = 3$, and let $V = \mathrm{H}^0(C, \xi)$. Since ξ is globally generated, it induces a morphism $C \to \mathbb{P}^2 = \mathbb{P}(V^\vee)$. The pullback of the Koszul complex on $\mathbb{P}^2$ is an exact sequence

$$0 \to \wedge^3 V \otimes \xi^{-3} \to \wedge^2 V \otimes \xi^{-2} \to V \otimes \xi^{-1} \to \mathcal{O}_C \to 0.$$

Twisting it by $\xi(K_C)$, we obtain an exact sequence

$$0 \to \wedge^3 V \otimes \xi^{-2}(K_C) \to \wedge^2 V \otimes \xi^{-1}(K_C) \to V \otimes \mathcal{O}_C(K_C) \to \xi(K_C) \to 0.$$

The $\mathbf{E}_1$ page of its hypercohomology spectral sequence has the form

$$\begin{array}{cccc} \wedge^3 V \otimes \mathrm{H}^1(C, \xi^{-2}(K_C)) \to \wedge^2 V \otimes \mathrm{H}^1(C, \xi^{-1}(K_C)) & \longrightarrow V & & 0 \\ \wedge^3 V \otimes \mathrm{H}^0(C, \xi^{-2}(K_C)) \to \wedge^2 V \otimes \mathrm{H}^0(C, \xi^{-1}(K_C)) & \to V \otimes \mathrm{H}^0(C, \mathcal{O}_C(K_C)) & \to & \mathrm{H}^0(C, \xi(K_C)). \end{array}$$

Since it converges to zero, surjectivity of the last arrow in the lower row is equivalent to the dashed arrow in the $\mathbf{E}_2$ page being zero. To show this, it is enough to check that the upper row is exact in the second term. Tensoring it with $\wedge^3 V^\vee$, using the natural identifications $V \otimes \wedge^3 V^\vee = \wedge^2 V^\vee$ and $\wedge^2 V \otimes \wedge^3 V^\vee = V^\vee$, and dualizing it afterwards, we obtain the sequence

$$\wedge^2 V \to V \otimes V \to \mathrm{H}^0(C, \xi^2),$$

where the first arrow is the natural embedding. Therefore it is enough to check that the natural map $\mathrm{Sym}^2 V = \mathrm{Sym}^2 \mathrm{H}^0(C, \xi) \to \mathrm{H}^0(C, \xi^2)$ is injective. But if it is not injective, the image of the map $C \to \mathbb{P}^2$ given by ξ is contained in a conic in $\mathbb{P}^2$, hence this map factors through a map $C \to \mathbb{P}^1$ of degree $\frac{1}{2}\deg(\xi) = \frac{1}{2}(r-1)(s+1) = s+1$, hence $\mathrm{W}^1_{s+1}(C) \neq \varnothing$, which contradicts the BN-generality of the curve C as $g - 2(g-s) = 2s - g < 0$.

(b) As before, consider the morphism $C \to \mathbb{P}(\mathrm{H}^0(C, \xi_1)^\vee)$ and the pullback of the Koszul exact sequence

$$\dots \to \wedge^2 \mathrm{H}^0(C, \xi_1) \otimes \xi_1^{-2} \to \mathrm{H}^0(C, \xi_1) \otimes \xi_1^{-1} \to \mathcal{O}_C \to 0.$$

Tensoring it by $\xi_1 \otimes \eta_2(K_C)$, we obtain an exact sequence

$$\dots \to \wedge^2 \mathrm{H}^0(C, \xi_1) \otimes \xi_1^{-1} \otimes \eta_2(K_C) \to \mathrm{H}^0(C, \xi_1) \otimes \eta_2(K_C) \to \xi_1 \otimes \eta_2(K_C) \to 0.$$

Now note that $\deg(\xi_1) = r \le s = \deg(\eta_2)$, and if this is an equality, $\xi_1^{-1} \otimes \eta_2$ is a nontrivial (by the assumption $\xi_1 \not\cong \eta_2$) line bundle of degree zero, hence $\mathrm{H}^1(C, \xi_1^{-1} \otimes \eta_2(K_C)) = 0$. Therefore, the hypercohomology spectral sequence proves the surjectivity of the required map. □

2.2. **Quasipolarized K3 surfaces.** In Section 5 we will have to deal with polarized K3 surfaces with du Val singularities; they will appear as special members of generic pencils of hyperplane sections of terminal Fano threefolds. A minimal resolution of such surface is a smooth quasipolarized K3 surface. In this subsection we explain this relation and prove a few useful results. For readers only interested in the case where X is smooth, it is enough to consider the case of K3 surfaces that have at most one ordinary double point, since in this case a generic pencil is a Lefschetz pencil.

So, let $(\bar{S}, \bar{H})$ be a K3 surface with du Val singularities and an ample class $\bar{H} \in \mathrm{Pic}(\bar{S})$. Let

$$\sigma \colon S \to \bar{S}, \qquad H \coloneqq \sigma^*(\bar{H})$$

be the minimal resolution of singularites and the pullback of the ample class. Then S is a smooth K3 surface and $H \in \mathrm{Pic}(S)$ is a quasipolarization, i.e., a big and nef divisor class. The exceptional locus of σ is formed by a finite number of smooth rational curves $R_i \subset S$ that are characterized by the condition $H \cdot R_i = 0$ and form an ADE configuration that we denote by $\mathfrak{R}(S, H)$.

Conversely, if (S, H) is a quasipolarized K3 surface, all irreducible curves orthogonal to H are smooth and rational and they form an ADE configuration $\mathfrak{R}(S, H)$, see [S-D, (4.2)]; moreover, a multiple of H defines a proper birational morphism $\sigma \colon S \to \bar{S}$ onto a K3 surface $\bar{S}$ with du

Val singularities contracting the configuration $\mathfrak{R}(S,H)$ so that H is the pullback of an ample class $\bar{H} \in \operatorname{Pic}(\bar{S})$. The morphism σ will be referred to as the contraction of $\mathfrak{R}(S,H)$.

The two above constructions are mutually inverse, so the language of polarized du Val K3 surfaces is equivalent to the language of quasipolarized smooth K3 surfaces. Below we use the language that is more convenient, depending on the situation.

If H is a quasipolarization of a smooth (or a polarization of a du Val K3 surface) S then

$$H^2 = 2g-2, \qquad h^0(\mathcal{O}_S(H)) = g+1, \qquad \text{and} \qquad h^1(\mathcal{O}_S(H)) = h^2(\mathcal{O}_S(H)) = 0, \tag{9}$$

where $g \geq 2$ is an integer called the genus of (S,H). The line bundle $\mathcal{O}_S(H)$ is globally generated if and only if the linear system $|H|$ contains a smooth curve $C \subset S$ of genus g; one direction follows from Bertini's Theorem, the other is [S-D, Theorem 3.1].

If (S,H) is a quasipolarized smooth K3 surface, the irreducible curves R_i that form the ADE configuration $\mathfrak{R}(S,H)$ are called the simple roots; they satisfy $R_i^2 = -2$ and generate a root system in $H^\perp \subset \operatorname{Pic}(S) \otimes \mathbb{R}$; its positve roots are nonnegative linear combinations $R = \sum a_i R_i$ such that $R^2 = -2$, and the reflections in R_i generate an action of the corresponding Weyl group.

A basic fact about Weyl groups (see, e.g., Corollary 2 to Proposition IV.17 in [B]) tells us that for any non-simple positive root R there is a simple root R_j such that $R \cdot R_j = -1$. Then $R' = R - R_j$ is also a positive root and $R' \cdot R_j = 1$. Therefore, considering R and R' as Cartier divisors on S, we obtain exact sequences

$$0 \to \mathcal{O}_{R_j}(-1) \to \mathcal{O}_R \to \mathcal{O}_{R'} \to 0. \tag{10}$$

and

$$0 \to \mathcal{O}_{R'}(-R_j) \to \mathcal{O}_R \to \mathcal{O}_{R_j} \to 0. \tag{11}$$

These sequences are useful for inductive arguments about positive roots.

Lemma 2.6. *If $D \in \operatorname{Pic}(S)$ and $R = \sum a_i R_i$ is a positive root in $\mathfrak{R}(S,H)$ then*

$$\chi(\mathcal{O}_R(D)) = 1 + D \cdot R.$$

Moreover, if $D \cdot R_i \leq 0$ for all R_i with $a_i > 0$ and $D \cdot R < 0$ then $h^0(\mathcal{O}_R(D)) = 0$.

Proof. The formula for $\chi(\mathcal{O}_R(D))$ follows immediately from the Riemann–Roch Theorem.

The second statement is clear if R is a simple root. Otherwise, we consider exact sequences (10) and (11), where R_j is a simple root such that $R \cdot R_j = -1$ and $R' = R - R_j$, so that $R' \cdot R_j = 1$.

If $D \cdot R' < 0$, twisting (10) by D and noting that $h^0(\mathcal{O}_{R_j}(D \cdot R_j - 1)) = 0$ because $D \cdot R_j \leq 0$, we see that $h^0(\mathcal{O}_R(D)) \leq h^0(\mathcal{O}_{R'}(D))$, which is zero by induction, hence $h^0(\mathcal{O}_R(D)) = 0$.

Similarly, if $D \cdot R' \geq 0$ the hypotheses of the lemma imply that $D \cdot R' = 0$ and $D \cdot R_j < 0$, hence $h^0(\mathcal{O}_{R_j}(D)) = 0$. Therefore, twisting (11) by D we obtain $h^0(\mathcal{O}_R(D)) = h^0(\mathcal{O}_{R'}(D - R_j))$, and since $(D - R_j) \cdot R' = -R' \cdot R_j = -1$, this is zero, again by induction. □

Using further the terminology of Weyl groups, we will say that a divisor class D on a quasipolarized K3 surface (S,H) is minuscule, is

$$D \cdot R \in \{-1, 0, 1\}$$

for any positive root R in the ADE configuration $\mathfrak{R}(S,H)$.

If D is minuscule, by [B, Exercises 23, 24 to Ch. VI.2] there is a sequence of minuscule divisor classes $D_0 = D, D_1, \dots, D_n$ such that for each $1 \le k \le n$ the difference $R_{i_k} := D_{k-1} - D_k$ is a simple root satisfying $D_k \cdot R_{i_k} = -D_{k-1} \cdot R_{i_k} = 1$, so that there are exact sequences

$$0 \to \mathcal{O}_S(D_k) \to \mathcal{O}_S(D_{k-1}) \to \mathcal{O}_{R_{i_k}}(-1) \to 0, \tag{12}$$

and $D_n \cdot R \in \{0, 1\}$ for all positive roots R in $\mathfrak{R}(S, H)$. The sequence $\{D_k\}$ as above is not unique, but the last element D_n is unique, we will call it the dominant replacement for D and denote D_+. Note that

$$D_+^2 = D^2, \qquad H \cdot D_+ = H \cdot D, \qquad \text{and} \qquad \mathrm{H}^\bullet(S, \mathcal{O}_S(D_+)) \cong \mathrm{H}^\bullet(S, \mathcal{O}_S(D)). \tag{13}$$

Moreover, if $\sigma \colon S \to \bar{S}$ is the contraction of $\mathfrak{R}(S, H)$ then

$$\mathrm{R}\sigma_*\mathcal{O}_S(D) \cong \mathrm{R}\sigma_*\mathcal{O}_S(D_+). \tag{14}$$

Indeed, both (13) and (14) follow from the definition and sequences (12) by induction.

Lemma 2.7. *Let (S, H) be a globally generated quasipolarized $K3$ surface of genus $g = r \cdot s$. Let $\sigma \colon S \to \bar{S}$ be the contraction of the ADE configuration $\mathfrak{R}(S, H)$. If $D \in \mathrm{Pic}(S)$ is a minuscule divisor class on S then $\mathrm{R}^1\sigma_*\mathcal{O}_S(D) = 0$ and*

$$(\sigma_*\mathcal{O}_S(D))^\vee \cong \sigma_*\mathcal{O}_S(-D). \tag{15}$$

In particular, $\sigma_*\mathcal{O}_S(D)$ is a maximal Cohen–Macaulay sheaf on $\bar{S}$. If, moreover, the dominant replacement D_+ is globally generated, then so is the sheaf $\sigma_*\mathcal{O}_S(D)$.

Proof. The dominant replacement D_+ for D is nef over $\bar{S}$; hence we have $\mathrm{R}^1\sigma_*\mathcal{O}_S(D_+) = 0$ by the Kawamata–Viehweg vanishing theorem. Applying (14), we obtain $\mathrm{R}^1\sigma_*\mathcal{O}_S(D) = 0$. Furthermore,

$$\mathrm{R}\mathcal{H}om(\sigma_*\mathcal{O}_S(D), \mathcal{O}_{\bar{S}}) \cong \mathrm{R}\sigma_*\,\mathrm{R}\mathcal{H}om(\mathcal{O}_S(D), \sigma^!\mathcal{O}_{\bar{S}}) \cong \mathrm{R}\sigma_*\,\mathrm{R}\mathcal{H}om(\mathcal{O}_S(D), \mathcal{O}_S) \cong \mathrm{R}\sigma_*\mathcal{O}_S(-D)$$

(where the first isomorphism is the Grothendieck duality and the second is crepancy of σ). Clearly, the divisor $-D$ is also minuscule, hence the right-hand side is a pure sheaf, hence $\sigma_*\mathcal{O}_S(D)$ is a maximal Cohen–Macaulay sheaf and (15) holds.

If D is minuscule and D_+ is globally generated then a general pair of global sections of $\mathcal{O}_S(D_+)$ generates it on the union $\cup R_i$ of all simple roots in $\mathfrak{R}(S, H)$, hence the sequence

$$0 \to \mathcal{O}_S(-D_+) \to \mathcal{O}_S \oplus \mathcal{O}_S \to \mathcal{O}_S(D_+) \to 0$$

is exact in a neighborhood of $\cup R_i$. Pushing it forward to $\bar{S}$ and taking into account that $-D_+$ is also minuscule, and hence $\mathrm{R}^1\sigma_*\mathcal{O}_S(-D_+) = 0$, we see that $\sigma_*\mathcal{O}_S(D_+)$ is globally generated in a neighborhood of $\mathrm{Sing}(\bar{S})$. On the other hand, over the complement of $\mathrm{Sing}(\bar{S})$ the morphism σ is an isomorphism, hence $\sigma_*\mathcal{O}_S(D_+)$ is globally generated away from $\mathrm{Sing}(\bar{S})$, hence everywhere. Since $\sigma_*\mathcal{O}_S(D) \cong \sigma_*\mathcal{O}_S(D_+)$ by (14), it is also globally generated. ☐

2.3. **Brill–Noether theory for K3 surfaces.** The following definition was introduced in [Mu5, Definition 3.8] for polarized K3 surfaces; we generalize it to the case of quasipolarizations.

Definition 2.8. A quasipolarized K3 surface (S, H) is called Brill–Noether general (BN-general) if

$$h^0(\mathcal{O}_S(D)) \cdot h^0(\mathcal{O}_S(H - D)) < h^0(\mathcal{O}_S(H)) = g + 1$$

for all $D \notin \{0, H\}$.

Remark 2.9. It follows from Saint-Donat's fundamental results about linear systems on K3 surfaces that if (S, H) is a BN-general quasipolarized $K3$ surface, then H is globally generated. Indeed, if H is not globally generated, then by [S-D, Corollary 3.2] the linear system $|H|$ has a fixed component D. In particular, $h^0(\mathcal{O}_S(H-D)) = h^0(\mathcal{O}_S(H)) = g+1$ and

$$h^0(\mathcal{O}_S(D)) \cdot h^0(\mathcal{O}_S(H-D)) = 1 \cdot (g+1) = g+1,$$

contradicting the assumption that (S, H) is BN-general.

The following theorem relates Brill–Noether properties of curves and K3 surfaces.

Theorem 2.10. *Let $C \subset S$ be a smooth curve of genus $g \geq 2$ on a $K3$ surface.*

(a) *If C is BN-general then (S, C) is BN-general.*
(b) *If (S, H) is BN-general and every curve in $|H|$ is reduced and irreducible then a general curve in $|H|$ is BNP-general.*

Proof. Part (a) is easy. Assume $h^0(\mathcal{O}_S(D)) \cdot h^0(\mathcal{O}_S(C-D)) \geq g+1$. Using the exact sequence

$$0 \to \mathcal{O}_S(D-C) \to \mathcal{O}_S(D) \to \mathcal{O}_C(D|_C) \to 0$$

and effectivity of $C-D$ we obtain $h^0(\mathcal{O}_C(D|_C)) \geq h^0(\mathcal{O}_S(D))$. Using the similar inequality for $C-D$, we obtain

$$h^0(\mathcal{O}_C(D|_C)) \cdot h^0(\mathcal{O}_C((C-D)|_C)) \geq g+1$$

which contradicts BN-generality of C.

Part (b) is much harder; it is proved in [L]. □

Remark 2.11. Part (b) has been partially extended in [H, Theorem 1] to BN-general K3 surfaces of genus $g \leq 19$, showing that in these cases a smooth curve in $|C|$ is Brill-Noether general.

The following definition is analogous to Definition 2.2 of a Mukai pair on a curve; it plays an important role in the rest of the paper.

Definition 2.12. Let $r, s \geq 2$. A special Mukai class of type (r, s) on a quasipolarized smooth K3 surface (S, H) of genus $g = r \cdot s$ is a globally generated class $\Xi \in \mathrm{Pic}(S)$ such that

$$\Xi \cdot H = (r-1)(s+1), \qquad h^0(\mathcal{O}_S(\Xi)) = r, \qquad \text{and} \qquad h^1(\mathcal{O}_S(\Xi)) = h^2(\mathcal{O}_S(\Xi)) = 0. \tag{16}$$

Recall from Section 2.2 the definition of minuscule divisor classes.

Lemma 2.13. *If Ξ is a special Mukai class of type (r, s) on a BN-general quasipolarized $K3$ surface (S, H) of genus $g = r \cdot s$ then Ξ is minuscule.*

Proof. Let R be a positive root. We have $\Xi \cdot R \geq 0$ because Ξ is globally generated, hence nef, and R is effective, and it remains to show that $\Xi \cdot R \leq 1$. So, assume $\Xi \cdot R \geq 2$. Then $(\Xi + R) \cdot R \geq 0$, hence $\chi(\mathcal{O}_R(\Xi + R)) > 0$ by Lemma 2.6, hence $h^0(\mathcal{O}_R(\Xi + R)) > 0$, and the exact sequence

$$0 \to \mathcal{O}_S(\Xi) \to \mathcal{O}_S(\Xi + R) \to \mathcal{O}_R(\Xi + R) \to 0$$

implies that $h^0(\mathcal{O}_S(\Xi + R)) > h^0(\mathcal{O}_S(\Xi)) = r$.

On the other hand, $(H - \Xi) \cdot R_i = -\Xi \cdot R_i \leq 0$ for any simple root R_i, because Ξ is nef, and $(H - \Xi) \cdot R = -\Xi \cdot R \leq -2$, hence we have $h^0(\mathcal{O}_R(H - \Xi)) = 0$, again by Lemma 2.6. Now, the exact sequence

$$0 \to \mathcal{O}_S(H - \Xi - R) \to \mathcal{O}_S(H - \Xi) \to \mathcal{O}_R(H - \Xi) \to 0$$

implies that $h^0(\mathcal{O}_S(H-\Xi-R)) = h^0(\mathcal{O}_S(H-\Xi)) = s$. We conclude that

$$h^0(\mathcal{O}_S(\Xi+R)) \cdot h^0(\mathcal{O}_S(H-\Xi-R)) > rs = g,$$

contradicting Brill–Noether generality of (S, H). Thus, Ξ is minuscule. □

The next lemma gives a precise relation between special Mukai classes and Mukai pairs. Recall from Section 2.2 the definition of the dominant replacement D_+ of a minuscule divisor class D.

Proposition 2.14. *Let Ξ be a special Mukai class of type (r, s) on a BN-general quasipolarized K3 surface (S, H) of genus $g = r \cdot s \geq 4$. If (S, H) contains a BNP-general curve $C \in |H|$ then $(H - \Xi)_+$ is a Mukai special class of type (s, r) and $(\mathcal{O}_S(\Xi)|_C, \mathcal{O}_S(H-\Xi)|_C)$ is a Mukai pair.*

Proof. We have $H \cdot (H-\Xi) = 2rs-2-(r-1)(s+1) = (r+1)(s-1) > 0$, hence $h^0(\mathcal{O}_S(\Xi-H)) = 0$ and the natural exact sequence

$$0 \to \mathcal{O}_S(\Xi - H) \to \mathcal{O}_S(\Xi) \to \mathcal{O}_C(\Xi|_C) \to 0$$

shows that $h^0(\mathcal{O}_C(\Xi|_C)) \geq h^0(\mathcal{O}_S(\Xi)) = r$. Since, on the other hand $\deg(\mathcal{O}_S(\Xi)|_C) = (r-1)(s+1)$, applying the argument of Lemma 2.3 we see that $(\mathcal{O}_S(\Xi)|_C, \mathcal{O}_S(H-\Xi)|_C)$ is a Mukai pair of type (r, s). In particular, $h^0(\mathcal{O}_C(\Xi|_C)) = r$ and $h^1(\mathcal{O}_C(\Xi|_C)) = s$, hence

$$h^0(\mathcal{O}_S(\Xi - H)) = h^1(\mathcal{O}_S(\Xi - H)) = 0 \qquad \text{and} \qquad h^2(\mathcal{O}_S(\Xi - H)) = s.$$

Using Serre duality we see that $H - \Xi$ satisfies the conditions (16) (where the role of r and s is swapped). By (13) the same holds for $(H-\Xi)_+$, so it remains to show that it is globally generated.

Assume to the contrary that $(H-\Xi)_+$ is not globally generated. Since $(H - \Xi)_+|_C = (H - \Xi)|_C$ is a part of a Mukai pair, it is globally generated. Since $h^1(\mathcal{O}_S(-\Xi)) = 0$, the restriction morphism $\mathrm{H}^0(S, \mathcal{O}_S(H-\Xi)) \to \mathrm{H}^0(C, \mathcal{O}_C((H-\Xi)|_C))$ is surjective, hence $(H-\Xi)_+$ is globally generated in a neighborhood of C. Since C intersects any curve on S except for those in $\mathfrak{R}(S, H)$ and since the base locus of a divisor on a smooth K3 surface is a Cartier divisor (see [S-D, Corollary 3.2]), we conclude that the base locus of $(H - \Xi)_+$ is contained in the ADE configuration $\mathfrak{R}(S, H)$. Thus,

$$(H - \Xi)_+ \cdot \Gamma \geq 0$$

for any irreducible curve not contained in $\mathfrak{R}(S, H)$. But the same inequality also holds for any Γ contained in $\mathfrak{R}(S, H)$ by the definition of dominant replacement. Therefore, $(H-\Xi)_+$ is nef. Since it is not globally generated by our assumption, it follows from [R2, Theorem 3.8] that

$$(H - \Xi)_+ = (s-1)E + R_i,$$

where R_i is a simple root, $|E|$ is an elliptic pencil, $E \cdot R_i = 1$, and $s \geq 3$. Now, let $\zeta := \mathcal{O}_S(E)|_C$. Then $\mathcal{O}_S((H - \Xi)_+|_C) \cong \zeta^{(s-1)}$, and ζ is a line bundle with $h^0(\zeta) \geq 2$ of degree

$$\deg(\zeta) = \tfrac{1}{s-1}(H - \Xi)_+ \cdot H = \tfrac{1}{s-1}(r+1)(s-1) = r + 1.$$

Then $\chi(\zeta) = r + 2 - rs$, hence $h^1(\zeta) \geq r(s-1)$, hence $h^0(\zeta)h^1(\zeta) \geq 2r(s-1)$; since $s \geq 3$, this is larger than the genus $g = rs$ of C. Thus, we obtain a contradiction with BN generality of C. □

Corollary 2.15. *Let Ξ be a special Mukai class of type (r, s) on a BN-general quasipolarized K3 surface (S, H) of genus $g = r \cdot s \geq 4$. If $\sigma\colon S \to \bar{S}$ is the contraction of $\mathfrak{R}(S, H)$ then*

$$(\sigma_* \mathcal{O}_S(\Xi))^\vee \cong \sigma_* \mathcal{O}_S(-\Xi).$$

Moreover, $\sigma_*\mathcal{O}_S(\Xi)$ *and* $\sigma_*\mathcal{O}_S(H-\Xi)$ *are globally generated maximal Cohen–Macaulay sheaves on* $\bar{S}$ *with* $h^0(\sigma_*\mathcal{O}_S(\Xi)) = r$, $h^0(\sigma_*\mathcal{O}_S(H-\Xi)) = s$, *and no higher cohomology.*

Proof. This is a combination of Lemma 2.7 and (13) with Lemma 2.13 and Proposition 2.14. □

3. Mukai bundles on K3 surfaces

The goal of this section is to prove Theorem 3.4, a criterion for stability of Lazarsfeld bundles on K3 surfaces in terms of the existence of special Mukai classes.

Recall from Section 1.6 that, for a sheaf $\mathcal{F}$ of finite projective dimension on a K3 surface with du Val singularities, we write $\upsilon(\mathcal{F})$ for its Mukai vector. Also recall that the Euler bilinear form $\chi(\mathcal{F},\mathcal{F})$ is even, and that a sheaf $\mathcal{F}$ is

- simple, if $\mathrm{Hom}(\mathcal{F},\mathcal{F}) = \Bbbk$,
- rigid, if $\mathrm{Ext}^1(\mathcal{F},\mathcal{F}) = 0$, and
- spherical, if it is simple and rigid, i.e., $\mathrm{Ext}^\bullet(\mathcal{F},\mathcal{F}) = \Bbbk \oplus \Bbbk[-2]$.

In particular, if $\mathcal{F}$ is rigid then $\chi(\mathcal{F},\mathcal{F}) \geq 2$, and a rigid sheaf is spherical if and only if $\chi(\mathcal{F},\mathcal{F}) < 4$. Similarly, if $\mathcal{F}$ is simple then $\chi(\mathcal{F},\mathcal{F}) \leq 2$, and a simple sheaf is spherical if and only if $\chi(\mathcal{F},\mathcal{F}) > 0$.

Definition 3.1. Let $(\bar{S},\bar{H})$ be a polarized K3 surface of genus $g = r\cdot s$ with du Val singularities. Let $\sigma\colon S \to \bar{S}$ be the minimal resolution and let $H = \sigma^*(\bar{H})$ be the induced quasipolarization.

(i) A vector bundle $\bar{\mathcal{U}}$ on $(\bar{S},\bar{H})$ with Mukai vector

$$\upsilon(\bar{\mathcal{U}}) = (r, -\bar{H}, s), \tag{17}$$

is called a Mukai bundle of type (r,s) if it is $\bar{H}$-Gieseker stable.

(ii) A vector bundle $\mathcal{U}$ on (S,H) is called a Mukai bundle of type (r,s) if $\mathcal{U} \cong \sigma^*(\bar{\mathcal{U}})$, where $\bar{\mathcal{U}}$ is a Mukai bundle on $(\bar{S},\bar{H})$ of type (r,s).

Note that (17) is equivalent to the equalities $\mathrm{rk}(\bar{\mathcal{U}}) = r$, $\mathrm{c}_1(\bar{\mathcal{U}}) = -\bar{H}$, and $\chi(\bar{\mathcal{U}}) = r+s$. Moreover, if $\mathcal{U} \cong \sigma^*(\bar{\mathcal{U}})$ is a Mukai bundle on (S,H) then

$$\upsilon(\mathcal{U}) = (r, -H, s)$$

and $\bar{\mathcal{U}} \cong \sigma_*(\mathcal{U})$ by the projection formula.

A standard argument proves the uniqueness of a Mukai bundle.

Lemma 3.2. *If a Mukai bundle of type* (r,s) *on* (S,H) *or* $(\bar{S},\bar{H})$ *exists, it is unique.*

Proof. By our definitions, it is enough to prove uniqueness of the latter, which is standard. Indeed, if $\bar{\mathcal{U}}_1$ and $\bar{\mathcal{U}}_2$ are two Mukai bundles on $(\bar{S},\bar{H})$ of type (r,s), we have

$$\chi(\bar{\mathcal{U}}_1,\bar{\mathcal{U}}_2) = -\langle\upsilon(\bar{\mathcal{U}}_1),\upsilon(\bar{\mathcal{U}}_2)\rangle = -\langle\upsilon(\bar{\mathcal{U}}_1),\upsilon(\bar{\mathcal{U}}_1)\rangle = \chi(\bar{\mathcal{U}}_1,\bar{\mathcal{U}}_1) = 2,$$

hence either $\mathrm{Hom}(\bar{\mathcal{U}}_1,\bar{\mathcal{U}}_2) \neq 0$ or $\mathrm{Hom}(\bar{\mathcal{U}}_2,\bar{\mathcal{U}}_1) \cong \mathrm{Ext}^2(\bar{\mathcal{U}}_1,\bar{\mathcal{U}}_2)^\vee \neq 0$. So we may assume that there is a morphism $\bar{\mathcal{U}}_1 \to \bar{\mathcal{U}}_2$, and since $\bar{\mathcal{U}}_1$ and $\bar{\mathcal{U}}_2$ are $\bar{H}$-stable, it is an isomorphism. □

If $\bar{\mathcal{U}}$ is a Mukai bundle of type (r,s) and the dual Mukai bundle $\bar{\mathcal{U}}^\vee$ is globally generated and satisfies $\mathrm{H}^{>0}(\bar{S},\bar{\mathcal{U}}^\vee) = 0$ (which is often the case) then the bundle

$$\bar{\mathcal{U}}^\perp := \mathrm{Ker}(\mathrm{H}^0(\bar{S},\bar{\mathcal{U}}^\vee)\otimes\mathcal{O}_{\bar{S}} \twoheadrightarrow \bar{\mathcal{U}}^\vee)$$

has Mukai vector $(s, -\bar{H}, r)$, so if it is $\bar{H}$-stable (which is also often the case), it is a Mukai bundle of type (s, r). Thus, the definition is almost symmetric. Moreover, the Mukai bundle of type $(1, g)$ is isomorphic to $\mathcal{O}_{\bar{S}}(-\bar{H})$, so from now on we assume

$$s \geq r \geq 2. \tag{18}$$

3.1. **The Lazarsfeld bundle.** In this section, we explain a construction, due to Lazarsfeld, of a vector bundle on $\bar{S}$ with Mukai vector (17) and state a criterion for stability of this bundle. This will give us a construction of a Mukai bundle.

Note that if $(\bar{S}, \bar{H})$ is a polarized du Val K3 surface, a smooth curve $C \subset \bar{S}$ in $|\bar{H}|$ is contained in the smooth locus of $\bar{S}$ (because it is a Cartier divisor). Similarly, if (S, H) is a quasipolarized smooth K3 surface, a smooth curve $C \subset S$ in $|H|$ does not intersect the simple roots R_i. Thus, if $\sigma\colon S \to \bar{S}$ is the contraction of $\mathfrak{R}(S, H)$ and $H = \sigma^*(\bar{H})$ we have a bijection between smooth curves on $\bar{S}$ in $|\bar{H}|$ and smooth curves on S in $|H|$. Given such a curve C we will denote its embedding into both S and $\bar{S}$ by j, hoping that this will not cause any confusion.

Now assume that C is a smooth BNP-general curve. We fix a factorization $g = r\cdot s$, where r and s are as in (18). By Lemma 2.3 the curve C has a Mukai pair (ξ, η) of type (r, s). If $j\colon C \to S$ is an embedding into a smooth or du Val K3 surface, we define, following [L] the Lazarsfeld bundle $\mathbf{L}_S(\xi)$ by the exact sequence

$$0 \to \mathbf{L}_S(\xi) \longrightarrow \mathrm{H}^0(C, \xi) \otimes \mathcal{O}_S \xrightarrow{\mathrm{ev}} j_*\xi \to 0, \tag{19}$$

where ev is the evaluation morphism (note that ξ is globally generated by Definition 2.2). Note that $\mathbf{L}_S(\xi) := \sigma^*\mathbf{L}_{\bar{S}}(\xi)$.

The most important properties of $\mathbf{L}_{\bar{S}}(\xi)$ are summarized below.

Lemma 3.3. *Let (S, H) be a smooth quasipolarized or du Val polarized $K3$ surface of genus $g = r\cdot s$ and let C be a smooth BNP-general curve in $|H|$. If (ξ, η) is a Mukai pair of type (r, s) on C then the sheaf $\mathbf{L}_S(\xi)$ defined by* (19) *is locally free, spherical, has Mukai vector $(r, -H, s)$ and satisfies*

$$h^0(\mathbf{L}_S(\xi)) = h^1(\mathbf{L}_S(\xi)) = 0 \qquad \text{and} \qquad h^2(\mathbf{L}_S(\xi)) = r + s. \tag{20}$$

The dual Lazarsfeld bundle $\mathbf{L}_S(\xi)^\vee$ fits into an exact sequence

$$0 \to \mathrm{H}^0(C, \xi)^\vee \otimes \mathcal{O}_S \to \mathbf{L}_S(\xi)^\vee \to j_*\eta \to 0. \tag{21}$$

It is globally generated and satisfies

$$h^0(\mathbf{L}_S(\xi)^\vee) = r + s \qquad \text{and} \qquad h^1(\mathbf{L}_S(\xi)^\vee) = h^2(\mathbf{L}_S(\xi)^\vee) = 0. \tag{22}$$

Proof. The sheaf $\mathbf{L}_S(\xi)$ is locally free because $j_*\xi$ has projective dimension 1 and the evaluation morphism is surjective. The computation of the Mukai vector of $\mathbf{L}_S(\xi)$ is obvious.

The equalities (20) follow from the cohomology exact sequence of (19) and (8). Applying Serre duality we deduce (22). The sequence (21) follows from (19) by dualizing, and the global generation of $\mathbf{L}_S(\xi)^\vee$ follows from $\mathrm{H}^1(S, \mathcal{O}_S) = 0$ and the global generation of $\mathcal{O}_S$ and η.

To compute $\mathrm{Ext}^\bullet(\mathbf{L}_S(\xi), \mathbf{L}_S(\xi))$ and check that $\mathbf{L}_S(\xi)$ is spherical, we use the defining short exact sequence (19). Taking into account the Grothendieck duality isomorphism

$$\mathrm{Ext}^p(j_*\xi, \mathcal{O}_S) \cong \mathrm{Ext}^p(\xi, j^!\mathcal{O}_S) \cong \mathrm{Ext}^p(\xi, \mathcal{O}_C(K_C)[-1]) \cong \mathrm{Ext}^{p-1}(\xi, \mathcal{O}_C(K_C)),$$

we obtain a self-dual spectral sequence with the $\mathbf{E}_1$ page

$$\begin{array}{ccccc}
\mathrm{H}^0(\xi)\otimes\mathrm{Ext}^1(\xi,\mathcal{O}_C(K_C)) & \longrightarrow & \mathrm{H}^0(\xi)\otimes\mathrm{H}^0(\xi)^\vee\oplus\mathrm{Ext}^2(j_*\xi,j_*\xi) & & 0\\
\mathrm{H}^0(\xi)\otimes\mathrm{Hom}(\xi,\mathcal{O}_C(K_C)) & \longrightarrow & \mathrm{Ext}^1(j_*\xi,j_*\xi) & \longrightarrow & \mathrm{H}^1(\xi)\otimes\mathrm{H}^0(\xi)^\vee\\
0 & & \mathrm{H}^0(\xi)\otimes\mathrm{H}^0(\xi)^\vee\oplus\mathrm{Hom}(j_*\xi,j_*\xi) & \longrightarrow & \mathrm{H}^0(\xi)\otimes\mathrm{H}^0(\xi)^\vee
\end{array}$$

that converges to $\mathrm{Ext}^\bullet(\mathbf{L}_S(\xi),\mathbf{L}_S(\xi))$. Obviously, the arrow of the bottom row induces an isomorphism $\mathrm{H}^0(\xi)\otimes\mathrm{H}^0(\xi)^\vee \xrightarrow{\sim} \mathrm{H}^0(\xi)\otimes\mathrm{H}^0(\xi)^\vee$. Since $\mathrm{Hom}(j_*\xi,j_*\xi)\cong\mathrm{Hom}(\xi,\xi)\cong\Bbbk$, to check that $\mathbf{L}_S(\xi)$ is simple, it is enough to show that the first arrow in the middle row is injective. For this note that we have a natural self-dual exact sequence

$$0\to\mathrm{Ext}^1(\xi,\xi)\to\mathrm{Ext}^1(j_*\xi,j_*\xi)\to\mathrm{Hom}(\xi,\xi(K_C))\to 0$$

and the composition

$$\mathrm{H}^0(\xi)\otimes\mathrm{Hom}(\xi,\mathcal{O}_C(K_C))\to\mathrm{Ext}^1(j_*\xi,j_*\xi)\to\mathrm{Hom}(\xi,\xi(K_C))$$

of the first arrow in the middle row of the spectral sequence with the second arrow in the exact sequence above is the Petri map of ξ, hence it is an isomorphism by Lemma 2.3. Therefore, the first arrow in the middle row of the spectral sequence is injective, hence $\mathrm{Hom}(\mathbf{L}_S(\xi),\mathbf{L}_S(\xi))=\Bbbk$. Finally, by (6) we have $\chi(\mathbf{L}_S(\xi),\mathbf{L}_S(\xi))=2rs-H^2=2$, hence $\mathbf{L}_S(\xi)$ is spherical. ☐

Lemma 3.3 shows that $\mathbf{L}_{\bar S}(\xi)$ is a Mukai bundle if and only if it is $\bar H$-Gieseker stable. The main result of this section is a criterion for stability of $\mathbf{L}_{\bar S}(\xi)$ in terms of special Mukai classes on the minimal resolution S of $\bar S$ defined in §2.3 (see Definition 2.12).

Theorem 3.4. *Let $(\bar S,\bar H)$ be a polarized du Val K3 surface of genus $g=r\cdot s$ with*

$$r\in\{2,3\}\qquad\text{and}\qquad s\ge r.$$

Let $\sigma\colon S\to\bar S$ be the minimal resolution and let $H\coloneqq\sigma^*(\bar H)$. Assume $|\bar H|$ contains a BNP-general curve $C\subset\bar S$. Then the following conditions are equivalent:

- (a) *The Lazarsfeld bundle $\mathbf{L}_{\bar S}(\xi)$ associated with a Mukai pair (ξ,η) of type (r,s) on C is not $\bar H$-Gieseker stable.*
- (b) *The surface (S,H) has a special Mukai class Ξ of type (r,s) such that $\mathcal{O}_S(\Xi)|_C\cong\xi$.*

Moreover, if (S,H) does not have special Mukai classes of type (r,s) then $\mathbf{L}_{\bar S}(\xi)$ does not depend on the choice of the curve C and Mukai pair (ξ,η) and it is a Mukai bundle of type (r,s) on $\bar S$.

We will prove this theorem in §3.4 after some preparation.

Remark 3.5. We expect that Theorem 3.4 holds for any $r,s\ge 2$.

3.2. **Multispherical filtration.** In this and the next subsections we work on a smooth quasipolarized K3 surface S. We start with a few well-known results.

Lemma 3.6. *Let $0\to\mathcal{F}_1\to\mathcal{F}\to\mathcal{F}_2\to 0$ be an exact sequence. If $\mathcal{F}$ is a simple sheaf then* $\mathrm{Hom}(\mathcal{F}_2,\mathcal{F}_1)=0$.

Lemma 3.7. *Let S be a smooth projective surface.*

- (a) *If $\mathcal{F}\hookrightarrow\mathcal{O}_S^{\oplus n}$ is a monomorphism from a sheaf $\mathcal{F}$ with $\mathrm{c}_1(\mathcal{F})\ge 0$ then $\mathcal{F}^{\vee\vee}\cong\mathcal{O}_S^{\oplus m}$.*
- (b) *If $\mathcal{O}_S^{\oplus n}\twoheadrightarrow\mathcal{F}$ is an epimorphism onto a torsion free sheaf $\mathcal{F}$ with $\mathrm{c}_1(\mathcal{F})\le 0$ then $\mathcal{F}\cong\mathcal{O}_S^{\oplus m}$.*

Proof. (a) Set $m := \mathrm{rk}(\mathcal{F})$. The embedding $\mathcal{F} \hookrightarrow \mathcal{O}_S^{\oplus n}$ induces an embedding $\mathcal{F}^{\vee\vee} \hookrightarrow \mathcal{O}_S^{\oplus n}$. Composing it with a sufficiently general projection $\mathcal{O}_S^{\oplus n} \twoheadrightarrow \mathcal{O}_S^{\oplus m}$ we obtain an injective morphism $\varphi\colon \mathcal{F}^{\vee\vee} \hookrightarrow \mathcal{O}_S^{\oplus m}$ of locally free sheaves of rank m. Its determinant $\det(\varphi)\colon \det(\mathcal{F}^{\vee\vee}) \to \mathcal{O}_S$ is then also injective, and since $\mathrm{c}_1(\mathcal{F}^{\vee\vee}) = \mathrm{c}_1(\mathcal{F}) \ge 0$, it follows that $\det(\varphi)$ is an isomorphism, hence φ is an isomorphism as well.

(b) Dualizing the epimorphism, we obtain a monomorphism $\mathcal{F}^\vee \hookrightarrow \mathcal{O}_S^{\oplus n}$. Since $\mathcal{F}$ is torsion free, we have $\mathrm{c}_1(\mathcal{F}^\vee) = -\mathrm{c}_1(\mathcal{F}) \ge 0$. Therefore, (a) implies that $\mathcal{F}^\vee \cong \mathcal{O}_S^{\oplus m}$, hence $\mathcal{F}^{\vee\vee} \cong \mathcal{O}_S^{\oplus m}$. The composition

$$\mathcal{O}_S^{\oplus n} \twoheadrightarrow \mathcal{F} \to \mathcal{F}^{\vee\vee} \cong \mathcal{O}_S^{\oplus m}$$

is generically surjective, hence it is surjective everywhere. Finally, the middle map is injective because $\mathcal{F}$ is torsion free, hence it is an isomorphism, and therefore $\mathcal{F} \cong \mathcal{O}_S^{\oplus m}$. □

Corollary 3.8. *Let $\mathcal{F}$ be a globally generated sheaf on a smooth $K3$ surface. If $\mathcal{F} \to \mathcal{F}'$ is an epimorphism of vector bundles, the line bundle* $\det(\mathcal{F}')$ *is globally generated and* $h^2(\det(\mathcal{F}')) = 0$ *unless $\mathcal{F}' \cong \mathcal{O}^{\oplus m}$ and the surjection $\mathcal{F} \twoheadrightarrow \mathcal{F}'$ splits.*

Proof. By assumptions we have epimorphisms $\mathcal{O}^{\oplus n} \twoheadrightarrow \mathcal{F} \twoheadrightarrow \mathcal{F}'$. If $\mathrm{rk}(\mathcal{F}') = m$, taking the m-th wedge power of the composition, we conclude that $\det(\mathcal{F}')$ is globally generated.

Assume $h^2(\det(\mathcal{F}')) \ne 0$. By Serre duality there is a generically injective morphism $\det(\mathcal{F}') \to \mathcal{O}$, hence $\mathrm{c}_1(\mathcal{F}') \le 0$. Applying Lemma 3.7(b) we conclude that $\mathcal{F}' \cong \mathcal{O}^{\oplus m}$. Finally, since the composition $\mathcal{O}^{\oplus n} \twoheadrightarrow \mathcal{F} \twoheadrightarrow \mathcal{F}' \cong \mathcal{O}^{\oplus m}$ is surjective, there is a splitting $\mathcal{O}^{\oplus m} \hookrightarrow \mathcal{O}^{\oplus n}$ of the composition, which induces a splitting $\mathcal{F}' \hookrightarrow \mathcal{F}$ of the epimorphism $\mathcal{F} \twoheadrightarrow \mathcal{F}'$. □

We will also need the following three fundamental results of Mukai.

Proposition 3.9 ([Mu1, Proposition 3.1])**.** *Let S be a smooth $K3$ surface of Picard rank one. Then every spherical bundle on S is slope-stable, hence also Gieseker-stable.*

Proposition 3.10 ([Mu1, Proposition 3.3])**.** *Any rigid torsion free sheaf on a smooth $K3$ surface is locally free.*

Lemma 3.11 ([Mu1, Proposition 2.7 and Corollary 2.8])**.** *Let $\mathcal{F}$ be a rigid sheaf on a smooth $K3$ surface. If*

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$

is an exact sequence and $\mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2) = 0$ *then $\mathcal{F}_1$ and $\mathcal{F}_2$ are both rigid.*

Moreover, if $\varepsilon \in \mathrm{Ext}^1(\mathcal{F}_2, \mathcal{F}_1)$ is the extension class of the sequence then

(a) *if $\mathcal{F}_1$ is simple then the map* $\mathrm{Hom}(\mathcal{F}_2, \mathcal{F}_2) \xrightarrow{\varepsilon} \mathrm{Ext}(\mathcal{F}_2, \mathcal{F}_1)$ *is surjective;*
(b) *if $\mathcal{F}_2$ is simple then the map* $\mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_1) \xrightarrow{\varepsilon} \mathrm{Ext}(\mathcal{F}_2, \mathcal{F}_1)$ *is surjective.*

The main goal of this subsection is to construct a filtration of any rigid sheaf on a smooth quasipolarized K3 surface (S, H) into stable spherical sheaves. Since Gieseker-stability for a quasipolarisation is not well–defined, we use a notion of stability that is equivalent to Gieseker stability with respect to a small ample perturbation $H + \epsilon A$ of the quasipolarization H. Recall from Section 1.6 our notation for the reduced Hilbert polynomial (and for its coefficients $\upmu$ and $\updelta$; see (7)). Consequently, stability for $0 < \epsilon \ll 1$ can be rephrased in the following terms.

Definition 3.12. Let S be a smooth K3 surface, let H be a quasipolarization, let $\sigma\colon S \to \bar{S}$ be the contraction of $\mathfrak{R}(S,H)$, and let A be a σ-ample divisor class on S.

Given two sheaves $\mathcal{F}_1, \mathcal{F}_2$ on S of positive rank we write $\mathbf{p}_{H,A}(\mathcal{F}_1) \prec \mathbf{p}_{H,A}(\mathcal{F}_2)$ if

- $\mu_H(\mathcal{F}_1) < \mu_H(\mathcal{F}_2)$, or
- $\mu_H(\mathcal{F}_1) = \mu_H(\mathcal{F}_2)$ and $\mu_A(\mathcal{F}_1) < \mu_A(\mathcal{F}_2)$, or
- $\mu_H(\mathcal{F}_1) = \mu_H(\mathcal{F}_2)$ and $\mu_A(\mathcal{F}_1) = \mu_A(\mathcal{F}_2)$ and $\delta(\mathcal{F}_1) < \delta(\mathcal{F}_2)$.

Similarly, we write $\mathbf{p}_{H,A}(\mathcal{F}_1) = \mathbf{p}_{H,A}(\mathcal{F}_2)$ if

$$\mu_H(\mathcal{F}_1) = \mu_H(\mathcal{F}_2), \qquad \mu_A(\mathcal{F}_1) = \mu_A(\mathcal{F}_2), \qquad \text{and} \qquad \delta(\mathcal{F}_1) = \delta(\mathcal{F}_2).$$

We say that a torsion free sheaf $\mathcal{F}$ is (H,A)-(semi)stable if $\mathbf{p}_{H,A}(\mathcal{F}') \prec (\preceq)\ \mathbf{p}_{H,A}(\mathcal{F})$ for every subsheaf $\mathcal{F}' \subset \mathcal{F}$.

Remark 3.13. If $\mathcal{F}_1 \subset \mathcal{F}_2$ is a subsheaf of positive rank such that $\mathcal{F}_2/\mathcal{F}_1$ is torsion and nonzero, then $\mathbf{p}_{H,A}(\mathcal{F}_1) \prec \mathbf{p}_{H,A}(\mathcal{F}_2)$. Indeed, we have $H \cdot \mathrm{c}_1(\mathcal{F}_2/\mathcal{F}_1) \geq 0$ and $A \cdot \mathrm{c}_1(\mathcal{F}_2/\mathcal{F}_1) \geq 0$, and if both $H \cdot \mathrm{c}_1(\mathcal{F}_2/\mathcal{F}_1) = 0$ and $A \cdot \mathrm{c}_1(\mathcal{F}_2/\mathcal{F}_1) = 0$ then $\mathcal{F}_2/\mathcal{F}_1$ has 0-dimensional support, and therefore $\chi(\mathcal{F}_2/\mathcal{F}_1) > 0$. Thus, in the definition of (H,A)-(semi)stability it is enough to assume that the respective inequality holds only for saturated subsheaves of $\mathcal{F}$, i.e., subsheaves $\mathcal{F}'$ such that $\mathcal{F}/\mathcal{F}'$ is torsion free.

The following proposition is the main result of this section.

Proposition 3.14. *Let S be a smooth K3 surface, let H be a quasipolarization, let $\sigma\colon S \to \bar{S}$ be the contraction of $\mathfrak{R}(S,H)$, and let A be a σ-ample divisor class on S. Every rigid torsion free sheaf $\mathcal{F}$ on S has a filtration such that its factors $\mathcal{F}_1, \dots, \mathcal{F}_n$ have the following properties:*

(a) *for all i we have $\mathcal{F}_i \cong \mathcal{G}_i^{\oplus m_i}$, where $\mathcal{G}_i$ is an (H,A)-stable spherical sheaf, and*
(b) *for all $i \leq j$ we have $\mu_H(\mathcal{F}_i) \geq \mu_H(\mathcal{F}_j)$ and* $\mathrm{Hom}(\mathcal{F}_i, \mathcal{F}_{i+1}) = 0$.

Moreover, $\sum m_i = 1$ if and only if $\mathcal{F}$ is (H,A)-stable.

We will call a filtration as in Proposition 3.14 a multispherical filtration of $\mathcal{F}$. Note that we do not claim $\mathrm{Hom}(\mathcal{F}_i, \mathcal{F}_j) = 0$ for $j > i+1$; indeed, a priori the same stable factor could appear multiple times in the filtration.

Proof. We first treat the case where the sheaf $\mathcal{F}$ is (H,A)-semistable. If $\mathcal{F}$ is (H,A)-stable, there is nothing to prove. Otherwise, let $\mathcal{G} \subsetneq \mathcal{F}$ be an (H,A)-stable subsheaf such that $\mathbf{p}_{H,A}(\mathcal{G}) = \mathbf{p}_{H,A}(\mathcal{F})$.

Now among all short exact sequences

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

with the property that $\mathcal{F}'$ is an iterated extension of $\mathcal{G}$ (hence $\mathbf{p}_{H,A}(\mathcal{F}') = \mathbf{p}_{H,A}(\mathcal{G}) = \mathbf{p}_{H,A}(\mathcal{F})$), we consider one where $\mathcal{F}'$ has maximal possible rank.

Note that $\mathcal{F}''$ is torsion free: otherwise, by Remark 3.13 the saturation $\widetilde{\mathcal{F}}'$ in $\mathcal{F}$ of the subsheaf $\mathcal{F}'$ has $\mathbf{p}_{H,A}(\widetilde{\mathcal{F}}') \succ \mathbf{p}_{H,A}(\mathcal{F}') = \mathbf{p}_{H,A}(\mathcal{F})$ and destabilizes $\mathcal{F}$. It is also easy to see that $\mathcal{F}''$ is (H,A)-semistable. We claim that $\mathrm{Hom}(\mathcal{G}, \mathcal{F}'') = 0$. Indeed, as $\mathcal{G}$ is (H,A)-stable with $\mathbf{p}_{H,A}(\mathcal{G}) = \mathbf{p}_{H,A}(\mathcal{F}'')$ any morphism $\mathcal{G} \to \mathcal{F}''$ would be injective, hence the preimage of $\mathcal{G}$ under the surjection $\mathcal{F} \twoheadrightarrow \mathcal{F}''$ would be an extension of $\mathcal{G}$ by $\mathcal{F}'$ of bigger rank than $\mathcal{F}'$, in contradiction to our assumption on $\mathcal{F}'$.

It follows that $\mathrm{Hom}(\mathcal{F}', \mathcal{F}'') = 0$, and by Lemma 3.11 both $\mathcal{F}'$ and $\mathcal{F}''$ are rigid. This implies that $\chi(\mathcal{F}', \mathcal{F}') > 0$. Since $\mathcal{F}'$ is an iterated extension of $\mathcal{G}$, its Mukai vector $v(\mathcal{F}')$ is proportional to $\mathbf{v}(\mathcal{G})$, hence we also have $\chi(\mathcal{G}, \mathcal{G}) > 0$. Since $\mathcal{G}$ is (H, A)-stable, hence simple, it must be spherical. In particular, $\mathrm{Ext}^1(\mathcal{G}, \mathcal{G}) = 0$, hence $\mathcal{F}' \cong \mathcal{G}^{\oplus m}$.

By induction on the rank, this concludes the case where $\mathcal{F}$ is (H, A)-semistable.

Now assume that $\mathcal{F}$ is not (H, A)-semistable. The same arguments as for Gieseker stability (see, e.g., [HL, Theorem 1.3.4]) prove the existence of a Harder–Narasimhan filtration for $\mathcal{F}$ with respect to (H, A)-stability. Applying Lemma 3.11 again, we see that every Harder–Narasimhan filtration factor of $\mathcal{F}$ is rigid. Combined with the filtrations of the semistable factors proven in the previous case, this immediately gives a filtration satisfying (a).

For part (b), just note that our construction in fact implies $\mathbf{p}_{H,A}(\mathcal{F}_i) \succeq \mathbf{p}_{H,A}(\mathcal{F}_{i+1})$, hence a fortiori $\mu_H(\mathcal{F}_i) \geq \mu_H(\mathcal{F}_{i+1})$, and, moreover, $\mathrm{Hom}(\mathcal{F}_i, \mathcal{F}_{i+1}) = 0$. □

3.3. **Brill–Noether inequality.** The following is the key result for the proof of Theorem 3.4.

Proposition 3.15. *Let (S, H) be a Brill–Noether general quasipolarized smooth K3 surface. Let $\mathcal{F}$ be a globally generated spherical bundle on S with $\mathbf{v}(\mathcal{F}) = (r, H, s)$. If*

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$

is an exact sequence of nontrivial spherical bundles with $\mu_H(\mathcal{F}_1) \geq \mu_H(\mathcal{F}) \geq \mu_H(\mathcal{F}_2)$ then

$$\mathrm{rk}(\mathcal{F}_1) = \mathrm{ch}_2(\mathcal{F}_2) + \mathrm{rk}(\mathcal{F}_2) = 1.$$

Moreover, the class $\Xi \coloneqq \mathrm{c}_1(\mathcal{F}_2)$ is a special Mukai class of type (r, s).

Proof. Let $\mathbf{v}(\mathcal{F}_i) = (r_i, D_i, s_i)$, so that $\mathbf{v}(\mathcal{F}) = (r_1+r_2, D_1+D_2, s_1+s_2)$; in particular $D_1+D_2 = H$. Since $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}$ are spherical, (6) implies that

$$D_1^2 = 2r_1s_1 - 2, \qquad D_2^2 = 2r_2s_2 - 2, \qquad (D_1 + D_2)^2 = 2(r_1 + r_2)(s_1 + s_2) - 2. \tag{23}$$

Subtracting the sum of the first two equalities from the last one and dividing by 2, we obtain

$$D_1 \cdot D_2 = r_1s_2 + r_2s_1 + 1.$$

This implies $H \cdot D_1 = 2r_1s_1 + r_1s_2 + r_2s_1 - 1$ and $H \cdot D_2 = 2r_2s_2 + r_1s_2 + r_2s_1 - 1$, hence

$$2s_1 + s_2 + \frac{r_2s_1 - 1}{r_1} = \frac{1}{r_1}H \cdot D_1 = \mu_H(\mathcal{F}_1) \geq \mu_H(\mathcal{F}_2) = \frac{1}{r_2}H \cdot D_2 = s_1 + 2s_2 + \frac{r_1s_2 - 1}{r_2}.$$

Equivalently,

$$s_1 - s_2 \geq \frac{r_1s_2 - 1}{r_2} - \frac{r_2s_1 - 1}{r_1}. \tag{24}$$

This inequality will be used a few times below.

By Corollary 3.8 the line bundle $\det(\mathcal{F}_2) \cong \mathcal{O}_S(D_2)$ is globally generated. Therefore, $D_2^2 \geq 0$, hence $r_2s_2 \geq 1$ by (23), and since $r_2 \geq 1$, we have $s_2 \geq 1$.

Assume $s_1 \leq 0$. Since r_1 and r_2 are positive, we obtain $r_1s_2 - 1 \geq 0$, $r_2s_1 - 1 < 0$, hence the right side of (24) is positive, while the left side is negative, which is absurd. Therefore, $s_1 \geq 1$.

Further, $h^2(\mathcal{O}_S(D_2)) = 0$ by Corollary 3.8. Moreover, $h^2(\mathcal{O}_S(D_1)) = 0$; indeed, otherwise Serre duality shows that $-D_1$ is effective, hence $\mu_H(\mathcal{F}_1) \leq 0$, contradicting to the assumption. Therefore,

$$h^0(\mathcal{O}_S(D_i)) \geq \chi(\mathcal{O}_S(D_i)) = \tfrac{1}{2}D_i^2 + 2 = r_is_i + 1.$$

Since $r_i \geq 1$ and $s_i \geq 1$, this is positive, and since $D_1 + D_2 = H$ and $g = (r_1 + r_2)(s_1 + s_2)$ by the last equality in (23), the Brill–Noether property of S implies that

$$(r_1 s_1 + 1)(r_2 s_2 + 1) \leq h^0(\mathcal{O}_S(D_1)) \cdot h^0(\mathcal{O}_S(D_2)) \leq (r_1 + r_2)(s_1 + s_2).$$

Expanding the products we obtain $r_1 r_2 s_1 s_2 + r_1 s_1 + r_2 s_2 + 1 \leq r_1 s_1 + r_2 s_2 + r_1 s_2 + r_2 s_1$, hence

$$(r_1 s_2 - 1)(r_2 s_1 - 1) = r_1 r_2 s_1 s_2 - r_1 s_2 - r_2 s_1 + 1 \leq 0.$$

But as we have shown above, both factors in the left side are nonnegative, therefore one of them is zero. If $r_1 s_2 - 1 \geq 1$, hence $r_2 s_1 - 1 = 0$, then $r_2 = s_1 = 1$, the right side in (24) is positive, hence $s_2 < s_1$, in contradiction to $s_2 \geq 1$. Therefore, $r_1 s_2 - 1 = 0$, hence $r_1 = s_2 = 1$, i.e.,

$$\mathrm{rk}(\mathcal{F}_1) = \mathrm{ch}_2(\mathcal{F}_2) + \mathrm{rk}(\mathcal{F}_2) = 1.$$

Moreover, $h^0(\mathcal{O}_S(D_1)) \geq r_1 s_1 + 1 = s_1 + s_2 = s$ and $h^0(\mathcal{O}_S(D_2)) \geq r_2 s_2 + 1 = r_2 + r_1 = r$, so the Brill–Noether property of S implies that these are equalities, hence $h^1(\mathcal{O}_S(D_2)) = 0$.

Finally, it follows that

$$D_2 \cdot H = 2r_2 s_2 + r_1 s_2 + r_2 s_1 - 1 = 2r_2 + 1 + r_2 s_1 - 1 = r_2(s_1 + 2) = (r-1)(s+1),$$

hence the divisor class $\Xi \coloneqq D_2$ is a special Mukai class of type (r, s). ☐

We will also need the following similar result.

Lemma 3.16. *Let (S, H) be a Brill–Noether general quasipolarized smooth $K3$ surface. Let $\mathcal{F}$ be a globally generated spherical bundle on S with $\mathrm{rk}(\mathcal{F}) = 3$ and $\mathrm{c}_1(\mathcal{F}) = H$. Then $\mathcal{F}$ cannot fit into an exact sequence*

$$0 \to \mathcal{F}_1^{\oplus 2} \to \mathcal{F} \to \mathcal{F}_2 \to 0 \qquad \text{or} \qquad 0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2^{\oplus 2} \to 0,$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are line bundles and $\mu_H(\mathcal{F}_1) \geq \mu_H(\mathcal{F}_2)$.

Proof. Let $\upsilon(\mathcal{F}_i) = (1, D_i, s_i)$, so that $D_i^2 = 2s_i - 2$.

Assume we have the first sequence. Then $\upsilon(\mathcal{F}) = (3, 2D_1 + D_2, 2s_1 + s_2)$, and since $\mathcal{F}$ is spherical, we have $(2D_1 + D_2)^2 = 12s_1 + 6s_2 - 2$, which implies

$$D_1 \cdot D_2 = s_1 + s_2 + 2. \tag{25}$$

Moreover, as in Proposition 3.15 the global generation of $\mathcal{F}_2$ implies $s_2 \geq 1$, and since

$$2(2s_1 - 2) + (s_1 + s_2 + 2) = \mu_H(\mathcal{F}_1) \geq \mu_H(\mathcal{F}_2) = 2(s_1 + s_2 + 2) + (2s_2 - 2),$$

we see that $3s_1 \geq 3s_2 + 4$, hence $s_1 \geq 3$. Finally, we have $h^2(\mathcal{O}_S(D_1 + D_2)) = h^2(\mathcal{O}_S(D_1)) = 0$ by the argument of Proposition 3.15, hence $h^0(\mathcal{O}_S(D_1 + D_2)) \geq \chi(\mathcal{O}_S(D_1 + D_2)) = 2s_1 + 2s_2 + 2$ and $h^0(\mathcal{O}_S(D_1)) \geq \chi(\mathcal{O}_S(D_1)) = s_1 + 1$, and the Brill–Noether inequality gives

$$2(s_1 + s_2 + 1)(s_1 + 1) \leq 6s_1 + 3s_2.$$

This can be rewritten as $(2s_1 + 2s_2 - 1)(2s_1 - 1) + 3 \leq 0$, contradicting $s_1 \geq 3$ and $s_2 \geq 1$.

Similarly, assuming the second sequence, we obtain $\upsilon(\mathcal{F}) = (3, D_1 + 2D_2, s_1 + 2s_2)$, which again implies (25). Arguing as before, we obtain $s_2 \geq 1$ and $3s_1 \geq 3s_2 - 4$, hence $s_1 \geq 0$, and this time the Brill–Noether inequality gives

$$2(s_1 + s_2 + 1)(s_2 + 1) \leq 3s_1 + 6s_2$$

which can be rewritten as $(2s_2 - 1)(2s_1 + 2s_2 - 1) + 3 \leq 0$, contradicting $s_1 \geq 0$ and $s_2 \geq 1$. ☐

3.4. **Proof of the theorem.** Now we can finally start proving Theorem 3.4, so we return to the setup of Section 3.1. We use the notation introduced therein. Also recall from Section 1.6 our notation for the reduced Hilbert polynomial (and for its coefficients μ and δ; see (7)).

Lemma 3.17. *If the Lazarsfeld bundle $\mathbf{L}_{\bar{S}}(\xi)$ is not $\bar{H}$-stable then there is a σ-ample divisor class $A \in \operatorname{Pic}(S)$ such that $\mathbf{L}_S(\xi) \cong \sigma^*(\mathbf{L}_{\bar{S}}(\xi))$ is not (H, A)-stable.*

Proof. Note that the functor σ_* defines a bijection between the set of all saturated subsheaves in $\mathbf{L}_S(\xi)$ and the set of all saturated subsheaves in $\mathbf{L}_{\bar{S}}(\xi)$ (this can be seen directly; for the general result, see [E, Lemma and definition (2.2)]). Therefore, if $\mathbf{L}_{\bar{S}}(\xi)$ is not $\bar{H}$-stable there is a saturated subsheaf $\mathcal{F} \subset \mathbf{L}_S(\xi)$ such that $\sigma_*\mathcal{F} \subset \mathbf{L}_{\bar{S}}(\xi)$ is a destabilizing saturated subsheaf. Let $A = \sum a_i R_i$ be a σ-ample class such that $a_i < 0$ for all i. We will show that $\mathcal{F}$ destabilizes $\mathbf{L}_S(\xi)$ with respect to (H, A)-stability.

First, we have $H = \sigma^*(\bar{H})$ and $\sigma_*(\mathrm{c}_1(\mathcal{F})) = \mathrm{c}_1(\sigma_*\mathcal{F})$, hence the projection formula implies

$$\mu_H(\mathcal{F}) = \tfrac{1}{\operatorname{rk}(\mathcal{F})}\mathrm{c}_1(\mathcal{F}) \cdot \sigma^*(\bar{H}) = \tfrac{1}{\operatorname{rk}(\sigma_*\mathcal{F})}\mathrm{c}_1(\sigma_*\mathcal{F}) \cdot \bar{H} = \mu_{\bar{H}}(\sigma_*\mathcal{F}) \geq \mu_{\bar{H}}(\mathbf{L}_{\bar{S}}(\xi)) = \mu_H(\mathbf{L}_S(\xi)).$$

Moreover, if this is an equality then we must have $\delta(\sigma_*\mathcal{F}) \geq \delta(\mathbf{L}_{\bar{S}}(\xi)) = \delta(\mathbf{L}_S(\xi))$.

Next, for any curve R_i in $\mathfrak{R}(S, H)$ we have $H \cdot R_i = 0$, hence $\mathrm{c}_1(\mathbf{L}_S(\xi)) \cdot A = 0$. Note also that $\mathbf{L}_S(\xi)$ is trivial on R_i, because it is a pullback along σ. Thus, $\mathcal{F}|_{R_i}$ is a subsheaf in a trivial vector bundle, hence we have $\mathrm{c}_1(\mathcal{F}) \cdot R_i \leq 0$ for all i. Therefore, we have $\mathrm{c}_1(\mathcal{F}) \cdot A \geq 0$, hence

$$\mu_A(\mathcal{F}) = \tfrac{1}{\operatorname{rk}(\mathcal{F})}\mathrm{c}_1(\mathcal{F}) \cdot A \geq 0 = \tfrac{1}{\operatorname{rk}(\mathbf{L}_S(\xi))}\mathrm{c}_1(\mathbf{L}_S(\xi)) \cdot A = \mu_A(\mathbf{L}_S(\xi)). \tag{26}$$

Finally, if (26) is an equality, we have $\mathrm{c}_1(\mathcal{F}) \cdot R_i = 0$ for each R_i, hence $\mathcal{F}$ is trivial on each of these curves, hence $\mathcal{F} \cong \sigma^*(\sigma_*\mathcal{F})$, hence $\chi(\mathcal{F}) = \chi(\sigma_*\mathcal{F})$, hence $\delta(\mathcal{F}) = \delta(\sigma_*\mathcal{F})$, and we conclude that $\mathbf{p}_{H,A}(\mathcal{F}) \succeq \mathbf{p}_{H,A}(\mathbf{L}_S(\xi))$, so that $\mathbf{L}_S(\xi)$ is not (H, A)-stable. □

Lemma 3.18. *If $r \in \{2, 3\}$ and the Lazarsfeld bundle $\mathbf{L}_{\bar{S}}(\xi)$ is not $\bar{H}$-Gieseker stable, then its pullback $\mathbf{L}_S(\xi) \cong \sigma^*\mathbf{L}_{\bar{S}}(\xi)$ fits into an exact sequence*

$$0 \to \mathcal{G}_1^{\oplus m_1} \to \mathbf{L}_S(\xi) \to \mathcal{G}_2^{\oplus m_2} \to 0,$$

where $\mathcal{G}_1$ and $\mathcal{G}_2$ are spherical bundles with $\mu_H(\mathcal{G}_1) \geq \mu_H(\mathcal{G}_2)$ and $(m_1, m_2) \in \{(1, 1), (1, 2), (2, 1)\}$.

Proof. By Lemma 3.17 the sheaf $\mathbf{L}_S(\xi)$ is not (H, A)-stable for appropriate A, hence the multispherical filtration of $\mathbf{L}_S(\xi)$ provided by Proposition 3.14 is not trivial. In the case $r = 2$, therefore, it has the above form with $(m_1, m_2) = (1, 1)$.

Now assume $r = 3$. The only case where the conclusion is not immediately obvious is where the multispherical filtration of $\mathbf{L}_S(\xi)$ has three spherical factors $\mathcal{G}_1 \not\cong \mathcal{G}_2 \not\cong \mathcal{G}_3$, By Proposition 3.10 the sheaves $\mathcal{G}_i$ are locally free, hence they are line bundles. Note also that

$$\operatorname{Hom}(\mathcal{G}_1, \mathcal{G}_2) = \operatorname{Hom}(\mathcal{G}_2, \mathcal{G}_3) = \operatorname{Hom}(\mathcal{G}_1, \mathcal{G}_3) = 0.$$

Indeed, the first two spaces are zero by Proposition 3.14. Moreover, if $\mu_H(\mathcal{G}_1) > \mu_H(\mathcal{G}_3)$, the third space is zero by (H, A)-stability of $\mathcal{G}_1$ and $\mathcal{G}_3$, and otherwise we would have $\mu_H(\mathcal{G}_i) = \mu_H(\mathbf{L}_S(\xi))$ for all i, hence $\mu_H(\mathbf{L}_S(\xi))$ would be integral, while in fact $\mu_H(\mathbf{L}_S(\xi)) = \frac{1}{3}H^2 = 2s - \frac{2}{3}$.

Now, consider the sheaves $\mathcal{G}_{1,2} := \operatorname{Ker}(\mathbf{L}_S(\xi) \to \mathcal{G}_3)$ and $\mathcal{G}_{2,3} := \mathbf{L}_S(\xi)/\mathcal{G}_1$. It follows that

$$\operatorname{Hom}(\mathcal{G}_{1,2}, \mathcal{G}_3) = \operatorname{Hom}(\mathcal{G}_1, \mathcal{G}_{2,3}) = 0$$

hence $\mathcal{G}_{1,2}$ and $\mathcal{G}_{2,3}$ are rigid by Lemma 3.11. Furthermore, Lemma 3.11 implies that the maps

$$\Bbbk = \operatorname{Hom}(\mathcal{G}_i, \mathcal{G}_i) \to \operatorname{Ext}^1(\mathcal{G}_i, \mathcal{G}_{i-1}) \qquad \text{and} \qquad \Bbbk = \operatorname{Hom}(\mathcal{G}_i, \mathcal{G}_i) \to \operatorname{Ext}^1(\mathcal{G}_{i+1}, \mathcal{G}_i)$$

are surjective, in particular, the spaces $\operatorname{Ext}^1(\mathcal{G}_2, \mathcal{G}_1)$ and $\operatorname{Ext}^1(\mathcal{G}_3, \mathcal{G}_2)$ are at most 1-dimensional.

If $\operatorname{Hom}(\mathcal{G}_2, \mathcal{G}_1) \neq 0$ and $\operatorname{Hom}(\mathcal{G}_3, \mathcal{G}_2) \neq 0$, the composition of nontrivial morphisms $\mathcal{G}_3 \to \mathcal{G}_2$ and $\mathcal{G}_2 \to \mathcal{G}_1$ is nontrivial, hence $\operatorname{Hom}(\mathcal{G}_3, \mathcal{G}_1) \neq 0$, which is impossible by Lemma 3.6 because $\mathbf{L}_S(\xi)$ is spherical. Thus, one of the above spaces must be zero.

Assume $\operatorname{Hom}(\mathcal{G}_2, \mathcal{G}_1) = 0$ and $\operatorname{Hom}(\mathcal{G}_3, \mathcal{G}_2) \neq 0$. If $\operatorname{Ext}^1(\mathcal{G}_2, \mathcal{G}_1) = 0$ then $\mathcal{G}_{1,2} \cong \mathcal{G}_1 \oplus \mathcal{G}_2$, hence we have $\operatorname{Hom}(\mathcal{G}_3, \mathcal{G}_{1,2}) \neq 0$, which is impossible by Lemma 3.6. Therefore, $\operatorname{Ext}^1(\mathcal{G}_2, \mathcal{G}_1) = \Bbbk$, hence we have $\chi(\mathcal{G}_{1,2}, \mathcal{G}_{1,2}) = 2$, and since the sheaf $\mathcal{G}_{1,2}$ is rigid, it is spherical. Therefore the filtration $0 \to \mathcal{G}_{1,2} \to \mathbf{L}_S(\xi) \to \mathcal{G}_3 \to 0$ has the required properties.

The case where $\operatorname{Hom}(\mathcal{G}_2, \mathcal{G}_1) \neq 0$ and $\operatorname{Hom}(\mathcal{G}_3, \mathcal{G}_2) = 0$ is considered analogously; in this case the filtration $0 \to \mathcal{G}_1 \to \mathbf{L}_S(\xi) \to \mathcal{G}_{2,3} \to 0$ has the required properties.

Finally, assume $\operatorname{Hom}(\mathcal{G}_2, \mathcal{G}_1) = \operatorname{Hom}(\mathcal{G}_3, \mathcal{G}_2) = 0$. If also $\operatorname{Ext}^1(\mathcal{G}_2, \mathcal{G}_1) = \operatorname{Ext}^1(\mathcal{G}_3, \mathcal{G}_2) = 0$ then it is easy to see that $\mathcal{G}_2$ is a direct summand of $\mathbf{L}_S(\xi)$, which is impossible because $\mathbf{L}_S(\xi)$ is spherical. Therefore, $\operatorname{Ext}^1(\mathcal{G}_i, \mathcal{G}_{i-1}) = \Bbbk$ for $i = 2$ or $i = 3$, hence $\mathcal{G}_{i-1,i}$ is spherical and as in one of the two previous cases we obtain a filtration of $\mathbf{L}_S(\xi)$ with the required properties. □

Proof of Theorem 3.4. Assume $\mathbf{L}_{\bar{S}}(\xi)$ is not $\bar{H}$-Gieseker stable. Dualizing the sequence produced by Lemma 3.18, we obtain an exact sequence

$$0 \to \mathcal{F}_1 \to \mathbf{L}_S(\xi)^\vee \to \mathcal{F}_2 \to 0$$

of multispherical sheaves, and we deduce from Lemma 3.16 that both $\mathcal{F}_1$ and $\mathcal{F}_2$ must be spherical. On the other hand, since S contains a BNP-general curve, it is BN-general (see Theorem 2.10(a)). Therefore, Proposition 3.15 proves that $\Xi \coloneqq \mathrm{c}_1(\mathcal{F}_2)$ is a special Mukai class of type (r, s) on S and $\mathcal{F}_1$ is a line bundle, hence $\mathcal{F}_1 \cong \mathcal{O}_S(H - \Xi)$. Consider the composition

$$\mathcal{O}_S(H - \Xi) \cong \mathcal{F}_1 \hookrightarrow \mathbf{L}_S(\xi)^\vee \to j_*\eta,$$

where the last arrow comes from exact sequence (21). If the composition vanishes, sequence (21) implies that the embedding $\mathcal{O}_S(H - \Xi) \hookrightarrow \mathbf{L}_S(\xi)^\vee$ factors through $\mathrm{H}^0(C, \xi)^\vee \otimes \mathcal{O}_S$, which is absurd because $H \cdot (H - \Xi) = (r + 1)(s - 1) > 0$. Therefore, the composition is nonzero, so it factors through a nonzero morphism

$$\mathcal{O}_S(H - \Xi)|_C \to \eta.$$

The source and target are line bundles of the same degree $(r+1)(s-1)$, hence the above morphism is an isomorphism, and we conclude that $\mathcal{O}_S(\Xi)|_C \cong \eta^{-1}(K_C) \cong \xi$.

Conversely, assume Ξ is a special Mukai class of type (r, s) on S. Let $C \subset S$ be a BNP-general curve in $|H|$ and let

$$\xi \coloneqq \mathcal{O}_S(\Xi)|_C, \qquad \eta \coloneqq \mathcal{O}_S(H - \Xi)|_C.$$

We know from Proposition 2.14 that (ξ, η) is a Mukai pair of type (r, s) and the restriction induces an isomorphism $\mathrm{H}^0(S, \mathcal{O}_S(\Xi)) \cong \mathrm{H}^0(C, \xi)$. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mathrm{H}^0(S, \mathcal{O}_S(\Xi)) \otimes \mathcal{O}_S & = & \mathrm{H}^0(C, \xi) \otimes \mathcal{O}_S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_S(\Xi - H) & \longrightarrow & \mathcal{O}_S(\Xi) & \longrightarrow & j_*\xi & \longrightarrow & 0.
\end{array}$$

The middle vertical arrow is surjective, because Ξ is globally generated, and the kernel of the right vertical arrow is $\sigma^*\mathbf{L}_S(\xi)$. It follows that there is an epimorphism $\mathbf{L}_S(\xi) \twoheadrightarrow \mathcal{O}_S(\Xi - H)$ and

$$\operatorname{Ker}\bigl(\mathbf{L}_S(\xi) \twoheadrightarrow \mathcal{O}_S(\Xi - H)\bigr) \cong \operatorname{Ker}\bigl(\mathrm{H}^0(S, \mathcal{O}_S(\Xi)) \otimes \mathcal{O}_S \twoheadrightarrow \mathcal{O}_S(\Xi)\bigr).$$

Moreover, as $\sigma_*\mathcal{O}_S(\Xi)$ is globally generated by Corollary 2.15, the first direct image of the right-hand side vanishes, hence the same is true for the first direct image of the left-hand side, hence the induced morphism $\mathbf{L}_{\bar{S}}(\xi) \cong \sigma_*\mathbf{L}_S(\xi) \to \sigma_*\mathcal{O}_S(\Xi - H)$ is surjective, and it is easy to see that such an epimorphism violates stability of $\mathbf{L}_{\bar{S}}(\xi)$.

Now assume that (S, H) does not have special Mukai classes of type (r, s). Then for any BNP-general curve C and any Mukai pair (ξ, η) on it the bundle $\mathbf{L}_{\bar{S}}(\xi)$ is stable, hence it is a Mukai bundle of type (r, s). Finally, since a Mukai bundle is unique by Lemma 3.2, $\mathbf{L}_{\bar{S}}(\xi)$ does not depend on C or ξ. □

4. Mukai extension classes

In this section we define and study Mukai extension classes on curves $C \subset S$. In particular, when $\operatorname{Pic}(S) = \mathbf{Z} \cdot H$ we will show that the defining property of a Mukai extension class (see Definition 4.9) uniquely characterises the restriction of the (dual of) the Lazarsfeld bundle on S, see Definition 4.6 and Theorem 4.13.

Throughout this section we work on a smooth quasipolarized K3 surface S.

4.1. **Restriction of Lazarsfeld bundles.** Here we establish some cohomology vanishings for the Lazarsfeld bundles and deduce some corollaries about their restrictions to curves.

The first result is elementary.

Lemma 4.1. *Let (S, H) be a quasipolarized $K3$ surface of genus $g = r \cdot s \geq 4$ and let $C \subset S$ be a BNP-general curve in $|H|$. Assume $s \geq r \in \{2, 3\}$. If (ξ, η) is a Mukai pair of type (r, s) on C then*

$$\mathrm{H}^2(S, \mathbf{L}_S(\xi) \otimes \mathcal{O}_S(H)) = \mathrm{H}^1(S, \mathbf{L}_S(\xi) \otimes \mathcal{O}_S(H)) = 0 \quad \textit{and}$$
$$\mathrm{H}^1(S, \mathbf{L}_S(\xi)^\vee \otimes \mathcal{O}_S(-H)) = \mathrm{H}^0(S, \mathbf{L}_S(\xi)^\vee \otimes \mathcal{O}_S(-H)) = 0.$$

Proof. Twisting (19) by $\mathcal{O}_S(H)$ we obtain an exact sequence

$$0 \to \mathbf{L}_S(\xi) \otimes \mathcal{O}_S(H) \longrightarrow \mathrm{H}^0(C, \xi) \otimes \mathcal{O}_S(H) \xrightarrow{\mathrm{ev}} j_*(\xi(K_C)) \to 0.$$

Since $\mathrm{H}^1(S, \mathcal{O}_S(H)) = \mathrm{H}^2(S, \mathcal{O}_S(H)) = 0$ by (9), the vanishing of $\mathrm{H}^2(S, \mathbf{L}_S(\xi) \otimes \mathcal{O}_S(H))$ is obvious, and to prove the vanishing of $\mathrm{H}^1(S, \mathbf{L}_S(\xi) \otimes \mathcal{O}_S(H))$ it is enough to check that the morphism

$$\mathrm{H}^0(C, \xi) \otimes \mathrm{H}^0(C, \mathcal{O}_C(K_C)) \to \mathrm{H}^0(C, \xi(K_C))$$

is surjective, which is proved in Proposition 2.5(a). The remaining two vanishings follow from the first two by Serre duality. □

The second vanishing result is more complicated; in particular, it relies on Theorem 3.4.

Proposition 4.2. *Let (S, H) be a quasipolarized $K3$ surface of genus $g = r \cdot s \geq 6$ and let $C \subset S$ be a BNP-general curve in $|H|$. Assume $s \geq r \in \{2, 3\}$. If S does not have special Mukai classes of type (r, s), then for any Mukai pair (ξ, η) of type (r, s) on C we have*

$$\mathrm{H}^1(S, \mathbf{L}_S(\xi)^\vee \otimes \mathbf{L}_S(\xi) \otimes \mathcal{O}_S(-H)) = \mathrm{H}^1(S, \mathbf{L}_S(\xi)^\vee \otimes \mathbf{L}_S(\xi) \otimes \mathcal{O}_S(H)) = 0.$$

Proof. First, tensoring (21) by $\mathbf{L}_S(\xi) \otimes \mathcal{O}_S(H)$ we obtain an exact sequence

$$0 \to \mathrm{H}^0(C, \xi)^\vee \otimes \mathbf{L}_S(\xi) \otimes \mathcal{O}_S(H) \to \mathbf{L}_S(\xi)^\vee \otimes \mathbf{L}_S(\xi) \otimes \mathcal{O}_S(H) \to j_*(j^*\mathbf{L}_S(\xi) \otimes \eta(K_C)) \to 0.$$

Its first term has no first cohomology by Lemma 4.1, so to prove the vanishing of the first cohomology of the middle term, it is enough to check that $\mathrm{H}^1(C, j^*\mathbf{L}_S(\xi) \otimes \eta(K_C)) = 0$.

Now we note that by Remark 2.4 the number of Mukai pairs on C of type (r, s) is equal to the degree of $\mathrm{Gr}(r, r + s)$. Since $r, s \geq 2$ and $(r, s) \neq (2, 2)$, this number is greater than 2; in particular, we can choose a Mukai pair (ξ', η') of type (r, s) such that $\xi \not\cong \xi'$, $\eta \not\cong \xi'$. Applying Theorem 3.4 we obtain an isomorphism

$$\mathbf{L}_S(\xi) \cong \mathbf{L}_S(\xi'),$$

hence $j^*\mathbf{L}_S(\xi) \cong j^*\mathbf{L}_S(\xi')$, so it is enough to check that $\mathrm{H}^1(C, j^*\mathbf{L}_S(\xi') \otimes \eta(K_C)) = 0$.

Now consider sequence (19) for the line bundle ξ' and its restriction to C:

$$0 \to \xi'(-K_C) \to j^*\mathbf{L}_S(\xi') \to \mathrm{H}^0(C, \xi') \otimes \mathcal{O}_C \to \xi' \to 0.$$

Tensoring this by $\eta(K_C)$, we obtain the following exact sequence

$$0 \to \xi' \otimes \eta \to j^*\mathbf{L}_S(\xi') \otimes \eta(K_C) \to \mathrm{H}^0(C, \xi') \otimes \eta(K_C) \to \xi' \otimes \eta(K_C) \to 0.$$

We have $\mathrm{H}^1(C, \xi' \otimes \eta) \cong \mathrm{H}^1(C, \xi' \otimes \xi^{-1} \otimes \mathcal{O}_C(K_C))$, and since $\xi' \otimes \xi^{-1}$ is a nontrivial line bundle of degree 0, this space vanishes. Moreover, $\mathrm{H}^1(C, \eta(K_C)) = 0$ because $\deg(\eta) > 0$. Therefore,

$$\mathrm{H}^1(C, j^*\mathbf{L}_S(\xi') \otimes \eta(K_C)) \cong \mathrm{Coker}\Big(\mathrm{H}^0(C, \xi') \otimes \mathrm{H}^0(C, \eta(K_C)) \to \mathrm{H}^0(C, \xi' \otimes \eta(K_C))\Big),$$

and it remains to note that the morphism in the right side is surjective by Proposition 2.5(b). Therefore, $\mathrm{H}^1(S, \mathbf{L}_S(\xi)^\vee \otimes \mathbf{L}_S(\xi) \otimes \mathcal{O}_S(H)) = 0$, and the other vanishing follows by Serre duality. □

4.2. **Lazarsfeld extension class.** For any globally generated line bundle ξ on a smooth curve C we consider the evaluation morphism $\mathrm{ev}\colon \mathrm{H}^0(C, \xi) \otimes \mathcal{O}_C \twoheadrightarrow \xi$. Its dual gives an exact sequence

$$0 \to \xi^{-1} \xrightarrow{\mathrm{ev}^\vee} \mathrm{H}^0(C, \xi)^\vee \otimes \mathcal{O}_C \longrightarrow \mathbf{R}_C(\xi^{-1}) \to 0, \tag{27}$$

defining a vector bundle $\mathbf{R}_C(\xi^{-1})$ on C.

Lemma 4.3. *For any globally generated line bundle ξ on a smooth curve C the bundle $\mathbf{R}_C(\xi^{-1})$ is globally generated, $\mathrm{H}^0(C, \mathbf{R}_C(\xi^{-1})^\vee) = 0$, and $\mathrm{H}^1(C, \mathbf{R}_C(\xi^{-1}) \otimes \mathcal{O}_C(K_C)) = 0$.*

Proof. First, the sheaf $\mathbf{R}_C(\xi^{-1})^\vee$ is the kernel of the evaluation morphism $\mathrm{H}^0(C, \xi) \otimes \mathcal{O}_C \xrightarrow{\mathrm{ev}} \xi$, hence $\mathrm{H}^0(C, \mathbf{R}_C(\xi^{-1})^\vee) = 0$. Applying Serre duality we obtain $\mathrm{H}^1(C, \mathbf{R}_C(\xi^{-1}) \otimes \mathcal{O}_C(K_C)) = 0$. Global generation of $\mathbf{R}_C(\xi^{-1})$ is immediate from (27). □

If (ξ, η) is a Mukai pair, the sheaf $\mathbf{R}_C(\xi^{-1})$ has stronger properties.

Lemma 4.4. *If (ξ, η) is a Mukai pair of type (r, s) with $s \geq r \in \{2, 3\}$ on a BNP-general curve C of genus $g = r \cdot s$, then*

$$\mathrm{H}^0(C, \mathbf{R}_C(\xi^{-1})) \cong \mathrm{H}^0(C, \xi)^\vee \qquad \textit{and} \qquad \mathrm{Hom}(\mathbf{R}_C(\xi^{-1}), \eta) = 0.$$

Moreover, the sheaf $\mathbf{R}_C(\xi^{-1})$ is simple.

Proof. Applying the functor $\mathrm{Hom}(-,\eta)$ to (27) we obtain a left-exact sequence

$$0 \to \mathrm{Hom}(\mathbf{R}_C(\xi^{-1}),\eta) \to \mathrm{H}^0(C,\xi)\otimes\mathrm{H}^0(C,\eta) \to \mathrm{H}^0(C,\xi\otimes\eta).$$

Its second arrow is induced by the evaluation morphism of ξ, hence it coincides with the Petri map. Since the curve C is BNP-general and (ξ,η) is a Mukai pair, this map is an isomorphism (see Lemma 2.3), hence $\mathrm{Hom}(\mathbf{R}_C(\xi^{-1}),\eta) = 0$.

Furthermore, since $\mathrm{H}^0(C,\xi^{-1}) = 0$, the long exact sequence of cohomology of (27) looks like

$$0 \to \mathrm{H}^0(C,\xi)^\vee \to \mathrm{H}^0(C,\mathbf{R}_C(\xi^{-1})) \to \mathrm{H}^1(C,\xi^{-1}) \to \mathrm{H}^0(C,\xi)^\vee\otimes\mathrm{H}^1(C,\mathcal{O}_C) \to \mathrm{H}^1(C,\mathbf{R}_C(\xi^{-1})) \to 0.$$

We need to check that the map $\mathrm{H}^0(C,\mathbf{R}_C(\xi^{-1})) \to \mathrm{H}^1(C,\xi^{-1})$ is zero, i.e., that the connecting map $\mathrm{H}^1(C,\xi^{-1}) \to \mathrm{H}^0(C,\xi)^\vee\otimes\mathrm{H}^1(C,\mathcal{O}_C)$ is injective, i.e., that its dual map is surjective. But the dual map is nothing but the map

$$\mathrm{H}^0(C,\xi)\otimes\mathrm{H}^0(C,\mathcal{O}_C(K_C)) \to \mathrm{H}^0(C,\xi(K_C))$$

whose surjectivity was established in Proposition 2.5(a). Thus, $\mathrm{H}^0(C,\mathbf{R}_C(\xi^{-1})) \cong \mathrm{H}^0(C,\xi)^\vee$.

Finally, since the second arrow in (27) induces an isomorphism of global sections, it is the evaluation morphism for $\mathbf{R}_C(\xi^{-1})$, hence any endomorphism of $\mathbf{R}_C(\xi^{-1})$ extends to an endomorphism of the exact sequence (27). On the other hand, since $\mathrm{Hom}(\mathbf{R}_C(\xi^{-1}),\mathcal{O}_C) = 0$ by Lemma 4.3, an endomorphism of the exact sequence (27) is determined uniquely by its endomorphism of ξ^{-1}. As ξ^{-1} is simple, the same holds for $\mathbf{R}_C(\xi^{-1})$. □

The sheaf $\mathbf{R}_C(\xi^{-1})$ is naturally related to the restriction $j^*\mathbf{L}_S(\xi)$ of the Lazarsfeld bundle.

Lemma 4.5. *Let (ξ,η) be a Mukai pair on a BNP-general curve C which lies on a smooth quasipolarized $K3$ surface S. If $\phi\colon \mathbf{L}_S(\xi)^\vee \to j_*\eta$ is the epimorphism from* (21) *there is a canonical exact sequence*

$$0 \to \mathbf{R}_C(\xi^{-1}) \longrightarrow j^*\mathbf{L}_S(\xi)^\vee \xrightarrow{j^*(\phi)} \eta \to 0. \tag{28}$$

Proof. Pulling back (21) and using an isomorphism $\eta(-K_C) \cong \xi^{-1}$, we obtain an exact sequence

$$0 \to \xi^{-1} \longrightarrow \mathrm{H}^0(C,\xi)^\vee\otimes\mathcal{O}_C \longrightarrow j^*\mathbf{L}_S(\xi)^\vee \xrightarrow{j^*(\phi)} \eta \to 0,$$

in particular $j^*(\phi)$ is surjective. Moreover, this sequence is dual to the exact sequence

$$0 \to \eta^{-1} \longrightarrow j^*\mathbf{L}_S(\xi) \longrightarrow \mathrm{H}^0(C,\xi)\otimes\mathcal{O}_C \xrightarrow{\mathrm{ev}} \xi \to 0$$

obtained by pulling back (19) (because their middle maps are mutually dual), hence the first map in the first sequence is the dual evaluation morphism $\mathrm{ev}^\vee$. Therefore, its cokernel is isomorphic to $\mathbf{R}_C(\xi^{-1})$, hence $\mathrm{Ker}(j^*(\phi)) \cong \mathbf{R}_C(\xi^{-1})$, and we obtain (28). □

Definition 4.6. We call the extension class

$$\epsilon_{\mathbf{L}_S(\xi)} \in \mathrm{Ext}^1(\eta,\mathbf{R}_C(\xi^{-1})).$$

of the exact sequence (28) the Lazarsfeld extension class.

Remark 4.7. Since the sheaves η and $\mathbf{R}_C(\xi^{-1})$ are both simple, the extension class $\epsilon_{\mathbf{L}_S(\xi)}$ of (28) is well-defined up to rescaling.

The main result of this subsection is the following.

Proposition 4.8. *Let (S, H) be a smooth quasipolarized K3 surface of genus $g = r \cdot s \geq 6$ and let $C \subset S$ be a BNP-general curve in $|H|$. Assume $s \geq r \in \{2, 3\}$. If S does not have special Mukai classes of type (r, s) then for any Mukai pair (ξ, η) of type (r, s) on C the Lazarsfeld extension class $\epsilon_{\mathbf{L}_S(\xi)}$ is nonzero, but the connecting morphism*

$$\epsilon_{\mathbf{L}_S(\xi)} \colon \mathrm{H}^0(C, \eta) \to \mathrm{H}^1(C, \mathbf{R}_C(\xi^{-1}))$$

of the exact sequence (28) vanishes.

Proof. Consider the exact sequence

$$0 \to \mathbf{L}_S(\xi)^\vee \otimes \mathbf{L}_S(\xi) \otimes \mathcal{O}_S(-H) \to \mathbf{L}_S(\xi)^\vee \otimes \mathbf{L}_S(\xi) \to j_*j^*(\mathbf{L}_S(\xi)^\vee \otimes \mathbf{L}_S(\xi)) \to 0.$$

We have $h^0(\mathbf{L}_S(\xi)^\vee \otimes \mathbf{L}_S(\xi) \otimes \mathcal{O}_S(-H)) = 0$ because $\mathbf{L}_S(\xi)$ is spherical, and in particular simple, by Lemma 3.3, and $h^1(\mathbf{L}_S(\xi)^\vee \otimes \mathbf{L}_S(\xi) \otimes \mathcal{O}_S(-H)) = 0$ by Proposition 4.2. Therefore

$$\mathrm{End}(j^*\mathbf{L}_S(\xi)) = \mathrm{H}^0(C, j^*(\mathbf{L}_S(\xi)^\vee \otimes \mathbf{L}_S(\xi))) = \mathrm{H}^0(S, \mathbf{L}_S(\xi)^\vee \otimes \mathbf{L}_S(\xi)) = \mathrm{End}(\mathbf{L}_S(\xi)).$$

Since $\mathbf{L}_S(\xi)$ is simple, so is $j^*\mathbf{L}_S(\xi)$, and a fortiori $j^*\mathbf{L}_S(\xi)$ is indecomposable. Thus the exact sequence (28) does not split, hence $\epsilon_{\mathbf{L}_S(\xi)} \neq 0$.

Furthermore, consider the exact sequence

$$0 \to \mathbf{L}_S(\xi)^\vee \otimes \mathcal{O}_S(-H) \to \mathbf{L}_S(\xi)^\vee \to j_*j^*(\mathbf{L}_S(\xi)^\vee) \to 0.$$

We have $h^0(\mathbf{L}_S(\xi)^\vee \otimes \mathcal{O}_S(-H)) = h^1(\mathbf{L}_S(\xi)^\vee \otimes \mathcal{O}_S(-H)) = 0$ by Lemma 4.1, hence

$$h^0(j^*\mathbf{L}_S(\xi)^\vee) = h^0(\mathbf{L}_S(\xi)^\vee) = r + s,$$

where the second equality uses (22). Finally, consider the cohomology exact sequence of (28):

$$0 \to \mathrm{H}^0(C, \mathbf{R}_C(\xi^{-1})) \to \mathrm{H}^0(C, j^*\mathbf{L}_S(\xi)^\vee) \to \mathrm{H}^0(C, \eta) \to \mathrm{H}^1(C, \mathbf{R}_C(\xi^{-1})).$$

Its first three terms have dimension r (by Lemma 4.4), $r + s$ (proved above), and s, respectively, and therefore the last map vanishes. □

4.3. **Mukai extension class.** In this subsection we show that the properties of the Lazarsfeld extension class established in Proposition 4.8 characterize it uniquely if $\mathrm{Pic}(S) = \mathbb{Z} \cdot H$. We axiomatize these properties in the following

Definition 4.9. Let (ξ, η) be a Mukai pair. A Mukai extension is a non-split exact sequence

$$0 \to \mathbf{R}_C(\xi^{-1}) \to \mathcal{G} \to \eta \to 0 \tag{29}$$

such that the connecting morphism $\mathrm{H}^0(C, \eta) \to \mathrm{H}^1(C, \mathbf{R}_C(\xi^{-1}))$ vanishes. The extension class

$$\epsilon \in \mathrm{Ext}^1(\eta, \mathbf{R}_C(\xi^{-1}))$$

of a Mukai extension is called a Mukai extension class.

To study Mukai extension classes we use the following generalization of the Lazarsfeld construction. In §3.1 we defined the Lazarsfeld bundle $\mathbf{L}_S(\xi)$ of a globally generated line bundle ξ. The same construction can be applied to any globally generated vector bundle $\mathcal{G}$ on C; it defines a vector bundle $\mathbf{L}_S(\mathcal{G})$ that fits into an exact sequence

$$0 \to \mathbf{L}_S(\mathcal{G}) \longrightarrow \mathrm{H}^0(C, \mathcal{G}) \otimes \mathcal{O}_S \xrightarrow{\mathrm{ev}} j_*\mathcal{G} \to 0.$$

We will apply this construction to all bundles in (29).

Lemma 4.10. *If* (ξ,η) *is a Mukai pair on a BNP-general curve* $C \subset S$ *then*

$$\mathbf{L}_S(\mathbf{R}_C(\xi^{-1})) \cong \mathbf{L}_S(\xi)^\vee \otimes \mathcal{O}_S(-H).$$

Proof. Recall that $\mathbf{R}_C(\xi^{-1})$ is globally generated by Lemma 4.3, hence $\mathbf{L}_S(\mathbf{R}_C(\xi^{-1}))$ is well-defined. Moreover, $\mathrm{H}^0(C,\mathbf{R}_C(\xi^{-1})) \cong \mathrm{H}^0(C,\xi)^\vee$ by Lemma 4.4. Therefore, twisting the defining exact sequence of $\mathbf{L}_S(\mathbf{R}_C(\xi^{-1}))$ by $\mathcal{O}_S(H)$ we obtain

$$0 \to \mathbf{L}_S(\mathbf{R}_C(\xi^{-1})) \otimes \mathcal{O}_S(H) \to \mathrm{H}^0(C,\xi)^\vee \otimes \mathcal{O}_S(H) \to j_*(\mathbf{R}_C(\xi^{-1}) \otimes \mathcal{O}_C(K_C)) \to 0.$$

By (9), we have $\mathrm{H}^2(S,\mathbf{L}_S(\mathbf{R}_C(\xi^{-1})) \otimes \mathcal{O}_S(H)) = \mathrm{H}^1(C,\mathbf{R}_C(\xi^{-1}) \otimes \mathcal{O}_C(K_C))$, and by Lemma 4.3 this is zero. Applying Serre duality we deduce $\mathrm{H}^0(S,\mathbf{L}_S(\mathbf{R}_C(\xi^{-1}))^\vee \otimes \mathcal{O}_S(-H)) = 0$.

Furthermore, consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \mathrm{H}^0(C,\xi)^\vee \otimes \mathcal{O}_S & = & \mathrm{H}^0(C,\mathbf{R}_C(\xi^{-1})) \otimes \mathcal{O}_S & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \scriptstyle\mathrm{ev} & & \\ 0 & \longrightarrow & j_*(\xi^{-1}) & \longrightarrow & \mathrm{H}^0(C,\xi)^\vee \otimes j_*\mathcal{O}_C & \xrightarrow{\mathrm{ev}} & j_*\mathbf{R}_C(\xi^{-1}) & \longrightarrow & 0 \end{array}$$

The middle vertical arrow is induced by the natural morphism $\mathcal{O}_S \to j_*\mathcal{O}_C$, hence it is surjective, therefore the exact sequence of kernels and cokernels takes the form

$$0 \to \mathrm{H}^0(C,\xi)^\vee \otimes \mathcal{O}_S(-H) \to \mathbf{L}_S(\mathbf{R}_C(\xi^{-1})) \to j_*(\xi^{-1}) \to 0,$$

and its dual sequence twisted by $\mathcal{O}_S(-H)$ takes the form

$$0 \to \mathbf{L}_S(\mathbf{R}_C(\xi^{-1}))^\vee \otimes \mathcal{O}_S(-H) \to \mathrm{H}^0(C,\xi) \otimes \mathcal{O}_S \to j_*\xi \to 0.$$

As we checked before, $\mathrm{H}^0(S,\mathbf{L}_S(\mathbf{R}_C(\xi^{-1}))^\vee \otimes \mathcal{O}_S(-H)) = 0$, hence the second arrow is the evaluation morphism, hence its kernel is isomorphic to the Lazarsfeld bundle $\mathbf{L}_S(\xi)$ and the required isomorphism $\mathbf{L}_S(\mathbf{R}_C(\xi^{-1})) \cong \mathbf{L}_S(\xi)^\vee \otimes \mathcal{O}_S(-H)$ follows. ☐

Proposition 4.11. *Let* (S,H) *be a quasipolarized* $K3$ *surface of genus* $g = r \cdot s \geq 6$ *and let* $C \subset S$ *be a BNP-general curve in* $|H|$. *Assume* $s \geq r \in \{2,3\}$. *If* S *does not have special Mukai classes of type* (r,s) *then for any Mukai pair* (ξ,η) *of type* (r,s) *on* C *and any Mukai extension* (29) *there is an exact sequence*

$$0 \to \mathbf{L}_S(\xi)^\vee \otimes \mathcal{O}_S(-H) \to \mathbf{L}_S(\mathcal{G}) \to \mathbf{L}_S(\eta) \to 0. \tag{30}$$

of the corresponding Lazarsfeld bundles.

Proof. If (29) is a Mukai extension, we have an exact sequence of global sections

$$0 \to \mathrm{H}^0(C,\xi)^\vee \to \mathrm{H}^0(C,\mathcal{G}) \to \mathrm{H}^0(C,\eta) \to 0$$

(where we use Lemma 4.4 to identify the first term and Definition 4.9 to prove exactness on the right). Moreover, since η and $\mathbf{R}_C(\xi^{-1})$ are globally generated (by definition of a Mukai pair and Lemma 4.3, respectively), we conclude that $\mathcal{G}$ is also globally generated, hence we have a commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{L}_S(\xi)^\vee \otimes \mathcal{O}_S(-H) & \longrightarrow & \mathbf{L}_S(\mathcal{G}) & \longrightarrow & \mathbf{L}_S(\eta) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{H}^0(C,\xi)^\vee \otimes \mathcal{O}_S & \longrightarrow & \mathrm{H}^0(C,\mathcal{G}) \otimes \mathcal{O}_S & \longrightarrow & \mathrm{H}^0(C,\eta) \otimes \mathcal{O}_S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & j_*\mathbf{R}_C(\xi^{-1}) & \longrightarrow & j_*\mathcal{G} & \longrightarrow & j_*\eta & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{31}
$$

with exact rows and columns, where the first term of the left column is identified in Lemma 4.10. Now the top row of the diagram gives (30). □

From now on we work under the assumption that $\mathrm{Pic}(S) = \mathbb{Z} \cdot H$.

Lemma 4.12. *Assume* $\mathrm{Pic}(S) = \mathbb{Z} \cdot H$ *and* $g \geq 6$. *Let*

$$0 \to \mathbf{L}_S(\xi)^\vee \otimes \mathcal{O}_S(-H) \to \mathcal{F} \to \mathbf{L}_S(\eta) \to 0 \tag{32}$$

be an exact sequence. If the sheaf $\mathcal{F}$ *is not* H*-Gieseker stable then the sequence splits.*

Proof. By Lemma 3.3 and Proposition 3.9, the subbundle $\mathbf{L}_S(\xi)^\vee \otimes \mathcal{O}_S(-H)$ and quotient bundle $\mathbf{L}_S(\eta)$ of $\mathcal{F}$ are stable with Mukai vectors

$$\mathrm{v}(\mathbf{L}_S(\xi)^\vee \otimes \mathcal{O}_S(-H)) = (r, -(r-1)H, s + (rs-1)(r-2)), \qquad \mathrm{v}(\mathbf{L}_S(\eta)) = (s, -H, r) \tag{33}$$

and slopes

$$\mu_H(\mathbf{L}_S(\xi)^\vee \otimes \mathcal{O}_S(-H)) = \tfrac{1}{r} - 1, \qquad \mu_H(\mathbf{L}_S(\eta)) = -\tfrac{1}{s}.$$

Note that $\frac{1}{r} - 1 \leq -\frac{1}{2} < -\frac{1}{s}$ because $r \geq 2$ and $s \geq 3$.

Assume $\mathcal{F}$ is not H-Gieseker stable. Consider the Harder–Narasimhan filtration of a sheaf $\mathcal{F}$ and refine it to a filtration with H-Gieseker stable factors $\mathcal{F}_1, \dots, \mathcal{F}_m$. We will show that $m = 2$, $\mathcal{F}_1 = \mathbf{L}_S(\eta)$, and $\mathcal{F}_2 = \mathbf{L}_S(\xi)^\vee \otimes \mathcal{O}_S(-H)$, so that this filtration is opposite to filtration (32).

First, (32) implies that the slopes of $\mathcal{F}_i$ are bounded by the slopes of $\mathbf{L}_S(\xi)^\vee \otimes \mathcal{O}_S(-H)$ and $\mathbf{L}_S(\eta)$:

$$-\tfrac{1}{s} \geq \mu_H(\mathcal{F}_1) \geq \dots \geq \mu_H(\mathcal{F}_m) \geq \tfrac{1}{r} - 1. \tag{34}$$

Furthermore, since $\mathrm{Pic}(S) = \mathbb{Z} \cdot H$, we can write $\mathrm{v}(\mathcal{F}_i) = (x_i, -y_i H, z_i)$. Then $\mu_H(\mathcal{F}_i) \leq -\frac{1}{s} < 0$ implies $y_i > 0$ and $\mu_H(\mathcal{F}_1) \geq \frac{1}{r} - 1 > -1$ implies $x_i > y_i$. Moreover, a combination of (33) and (34) gives the relations

$$\sum x_i = r + s, \qquad \sum y_i = r, \qquad \sum z_i = (r-1)^2 s + 2, \qquad x_i > y_i \geq 1.$$

The inequalities $x_i > y_i \geq 1$ with the equality $\sum y_i = r$ imply, in particular, that

$$m \leq r \qquad \text{and} \qquad x_1, \dots, x_m \geq 2. \tag{35}$$

On the other hand, by Riemann–Roch we have $\chi(\mathcal{F}_i, \mathcal{F}_i) = 2x_i z_i - (2rs-2)y_i^2$, and since the sheaves $\mathcal{F}_i$ are stable, they are simple, hence $\chi(\mathcal{F}_i, \mathcal{F}_i) \leq 2$, hence

$$x_i z_i \leq (rs-1)y_i^2 + 1, \qquad 1 \leq i \leq m.$$

Dividing the i-th inequality by x_i and summing them up, we obtain

$$(r-1)^2 s + 2 = \sum z_i \le \sum \tfrac{(rs-1)y_i^2+1}{x_i}. \tag{36}$$

From now on we assume $r \in \{2,3\}$. If $m = 3$ then (35) implies $r = 3$ and $y_i = 1$ for all i, hence the right-hand side of (36) equals $3s \sum \frac{1}{x_i}$, and we obtain

$$\sum \tfrac{1}{x_i} \ge \tfrac{4s+2}{3s} = \tfrac{4}{3} + \tfrac{2}{3s} > \tfrac{4}{3}. \tag{37}$$

Since $x_i \ge 2$ by (35), we must have $x_1 = x_2 = x_3 = 2$, but then $s = 3$, and (37) fails. Therefore, we must have $m = 2$; we assume this from now on.

As $m = 2$, the inequality (36) can be rewritten as

$$(rs-1)(y_1^2 x_2 + y_2^2 x_1) + (x_1 + x_2) \ge ((r-1)^2 s + 2) x_1 x_2.$$

Since $r \in \{2,3\}$, one of the y_i is 1 and the other is $r-1$. If we denote the x_i that corresponds to $y_i = 1$ by x, so that the other is $r+s-x$, then $y_1^2 x_2 + y_2^2 x_1 = (r-1)^2 x + (r+s-x)$ and the above inequality takes the form

$$f(x) \coloneqq (rs-1)((r-1)^2 x + (r+s-x)) + (r+s) - ((r-1)^2 s + 2) x (r+s-x) \ge 0.$$

Note that

$$\begin{aligned} f(s) &= (rs-1)((r-1)^2 s + r) + (r+s) - ((r-1)^2 s + 2) rs = 0, \\ f(2) &= (rs-1)(2(r-1)^2 + (r+s-2)) + (r+s) - 2((r-1)^2 s + 2)(r+s-2) \\ &= -s^2 (r-2)(2r-1) - 5rs(r-2) - 2(r^2-4) \le 0. \end{aligned}$$

and the second inequality is strict unless $r = 2$. On the other hand, the function f is convex, hence the inequality $f(x) \ge 0$ implies

$$x \ge s \qquad \text{or} \qquad x \le 2$$

and the second inequality is strict unless $r = 2$. Now, if $x \ge s$ the slope of $\mathcal{F}_i$ is greater or equal than $-\frac{1}{s}$, if $x \le 1$ then the slope of $\mathcal{F}_i$ is less than $\frac{1}{r} - 1$, and if $x = 2 = r$, it is equal to $\frac{1}{r} - 1$. In either case, (34) implies that the slopes of the $\mathcal{F}_i$ are $-\frac{1}{s}$ and $\frac{1}{r} - 1$. Since these coincide with the slopes of the stable bundles $\mathbf{L}_S(\eta)$ and $\mathbf{L}_S(\xi)^\vee \otimes \mathcal{O}_S(-H)$, we conclude that

$$\mathcal{F}_1 \cong \mathbf{L}_S(\eta) \qquad \text{and} \qquad \mathcal{F}_2 \cong \mathbf{L}_S(\xi)^\vee \otimes \mathcal{O}_S(-H).$$

As we noticed above, $\mu_H(\mathcal{F}_1) = -\frac{1}{s} > \frac{1}{r} - 1 = \mu_H(\mathcal{F}_2)$, and since both $\mathcal{F}_1$ and $\mathcal{F}_2$ are stable, we have $\operatorname{Hom}(\mathcal{F}_1, \mathcal{F}_2) = 0$, hence the exact sequence (32) splits. □

Combining the above observations we obtain the main result of this section.

Theorem 4.13. *Let (S,H) be a polarized $K3$ surface of genus $g = r \cdot s \ge 6$ with $\operatorname{Pic}(S) = \mathbb{Z} \cdot H$. If $C \in |H|$ is a BNP-general curve, (ξ, η) is a Mukai pair of type (r,s) on C with $s \ge r \in \{2,3\}$, and $\epsilon \in \operatorname{Ext}^1(\eta, \mathbf{R}_C(\xi^{-1}))$ is a Mukai extension class then $\epsilon = \epsilon_{\mathbf{L}_S(\xi)}$ is the Lazarsfeld class. In particular, the Mukai extension class is unique.*

Proof. Let $\epsilon \ne 0$ be a Mukai extension class and let (29) be the corresponding extension. By Proposition 4.11 we obtain the exact sequence (30). Now a simple Riemann–Roch computation shows that $\chi(\mathbf{L}_S(\mathcal{G}), \mathbf{L}_S(\mathcal{G})) > 2$, hence the bundle $\mathbf{L}_S(\mathcal{G})$ is not stable. Applying Lemma 4.12 we conclude that (30) splits, hence

$$\mathbf{L}_S(\mathcal{G}) \cong \big(\mathbf{L}_S(\xi)^\vee \otimes \mathcal{O}_S(-H)\big) \oplus \mathbf{L}_S(\eta).$$

Let $\psi \colon \mathbf{L}_S(\eta) \to \mathbf{L}_S(\mathcal{G})$ be a splitting of the projection from (30). Consider the upper right square

$$\begin{array}{ccc} \mathbf{L}_S(\mathcal{G}) & \underset{\psi}{\overset{\longrightarrow}{\dashleftarrow}} & \mathbf{L}_S(\eta) \\ \downarrow & \overset{\tilde{\psi}}{\swarrow} & \downarrow \\ \mathrm{H}^0(C, \mathcal{G}) \otimes \mathcal{O}_S & \longrightarrow & \mathrm{H}^0(C, \eta) \otimes \mathcal{O}_S \end{array}$$

of the diagram (31), where $\tilde{\psi}$ is defined as the composition $\mathbf{L}_S(\eta) \xrightarrow{\psi} \mathbf{L}_S(\mathcal{G}) \hookrightarrow \mathrm{H}^0(C, \mathcal{G}) \otimes \mathcal{O}_S$. Let $\mathcal{F} := \mathrm{Coker}(\tilde{\psi})$. It is easy to see that it fits into a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{L}_S(\xi)^\vee \otimes \mathcal{O}_S(-H) & \longrightarrow & \mathcal{F} & \longrightarrow & j_*\mathcal{G} & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{H}^0(C, \xi)^\vee \otimes \mathcal{O}_S & \longrightarrow & \mathcal{F} & \longrightarrow & j_*\eta & \longrightarrow & 0 \end{array} \tag{38}$$

where the left and right vertical arrows coincide with the arrows in (31). If the bottom row of (38) splits, then the composition $j_*\eta \to \mathcal{F} \to j_*\mathcal{G}$ provides a splitting of the morphism $j_*\mathcal{G} \to j_*\eta$ from the bottom row of (31), hence also a splitting of (29), which contradicts to the assumption $\epsilon \neq 0$; therefore, the bottom row of (38) does not split.

Now, pulling back the bottom row of (38) to C, we obtain an exact sequence

$$0 \to \mathrm{L}_1 j^*\mathcal{F} \to \eta(-K_C) \to \mathrm{H}^0(C, \xi)^\vee \otimes \mathcal{O}_C \to j^*\mathcal{F} \to \eta \to 0.$$

The natural isomorphism $\mathrm{Ext}^1(j_*\eta, \mathrm{H}^0(C, \xi)^\vee \otimes \mathcal{O}_S) \cong \mathrm{Hom}(\eta(-K_C), \mathrm{H}^0(C, \xi)^\vee \otimes \mathcal{O}_C)$ shows that the morphism $\eta(-K_C) \to \mathrm{H}^0(C, \xi)^\vee \otimes \mathcal{O}_C$ in it is nontrivial, hence it is injective, hence $\mathrm{L}_1 j^*\mathcal{F} = 0$. Therefore, using the bottom row of (38) to compute the invariants of $\mathcal{F}$, we obtain

$$\mathrm{rk}(j^*\mathcal{F}) = \mathrm{rk}(\mathcal{F}) = r = \mathrm{rk}(\mathcal{G}) \qquad \text{and} \qquad \mathrm{c}_1(j^*\mathcal{F}) = \mathrm{c}_1(\mathcal{F})|_C = H|_C = K_C = \mathrm{c}_1(\mathcal{G}).$$

Now, restricting the top row of (38) to C, we obtain

$$0 \to \mathcal{G}(-K_C) \to j^*\mathbf{L}_S(\xi)^\vee \otimes \mathcal{O}_C(-K_C) \to j^*\mathcal{F} \to \mathcal{G} \to 0.$$

The last arrow in it is an epimorphism of sheaves with the same rank and degree, hence it is an isomorphism. Therefore, the first arrow is also an isomorphism, i.e., $j^*\mathbf{L}_S(\xi)^\vee \cong \mathcal{G}$.

Finally, since $\mathrm{Hom}(\mathbf{R}_C(\xi^{-1}), \eta) = 0$ by Lemma 4.4, it follows from (29) that the space

$$\mathrm{Hom}(j^*\mathbf{L}_S(\xi)^\vee, \eta) \cong \mathrm{Hom}(\mathcal{G}, \eta)$$

is 1-dimensional, hence the composition $j^*\mathbf{L}_S(\xi)^\vee \xrightarrow{\sim} \mathcal{G} \twoheadrightarrow \eta$ of the constructed isomorphism and the surjection of (29) coincides with the map $j^*(\phi)$ in sequence (28), hence $\epsilon = \epsilon_{\mathbf{L}_S(\xi)}$. $\square$

5. Mukai bundles on Fano threefolds

In this section we prove Theorem 1.1. In Section 5.1 we verify a weaker statement, Theorem 5.3, that holds for Fano threefolds with terminal Gorenstein singularities, and in Section 5.2 we consider the case where X is smooth.

Given a prime Fano threefold X with terminal Gorenstein singularities we write

$$H = -K_X,$$

for the ample generator of $\mathrm{Pic}(X)$. Recall that the genus g of X is defined by the equality

$$H^3 = 2g - 2.$$

We will assume in this section that $g \geq 6$ (see Remark 5.10 for a discussion of the case $g = 4$). Note that a general anticanonical divisor $S \subset X$ is smooth by [Me, Theorem 1]. Set

$$H_S \coloneqq H|_S.$$

Then (S, H_S) is a smooth polarized K3 surface of genus g.

Definition 5.1. Let X be a prime Fano threefold of index one with terminal Gorenstein singularities of genus $g = r \cdot s$ with $r, s \geq 2$. A Mukai bundle $\mathcal{U}_X$ on X of type (r, s) is a vector bundle such that

(a) $\mathrm{rk}(\mathcal{U}_X) = r$ and $\mathrm{c}_1(\mathcal{U}_X) = -H$,
(b) $\mathrm{H}^\bullet(X, \mathcal{U}_X) = 0$, and
(c) $\mathrm{Ext}^\bullet(\mathcal{U}_X, \mathcal{U}_X) = \Bbbk$.

Similarly, a maximal Cohen–Macaulay sheaf satisfying (a) and (b) is called a weak Mukai sheaf if it is H-Gieseker stable.

It is clear that Mukai bundles (or weak Mukai sheaves) with $r = 1$ (hence $s = g$) do not exist; indeed, property (a) implies that $\mathcal{U}_X \cong \mathcal{O}_X(-H) \cong \mathcal{O}_X(K_X)$, and then property (b) fails. For this reason we exclude the case $r = 1$ and the symmetric case $s = 1$ from the definition.

In contrast to Definition 3.1 we do not include stability in the definition of a Mukai bundle, because it is automatic.

Lemma 5.2. *Let X be a prime Fano threefold with terminal Gorenstein singularities which has a smooth anticanonical divisor $S \subset X$ with $\mathrm{Pic}(S) = \mathbb{Z} \cdot H_S$. If $\mathcal{U}_X$ is a Mukai bundle of type (r, s) on X then $\mathcal{U}_X$ is H-Gieseker stable and $\mathcal{U}_X|_S$ is a Mukai bundle on S.*

Proof. Since $\mathcal{U}_X$ is exceptional (Definition 5.1(c)), the restriction $\mathcal{U} \coloneqq \mathcal{U}_X|_S$ is spherical; this follows from the cohomology exact sequence of the standard restriction sequence

$$0 \to \mathcal{U}_X^\vee \otimes \mathcal{U}_X(-H) \to \mathcal{U}_X^\vee \otimes \mathcal{U}_X \to i_*(\mathcal{U}^\vee \otimes \mathcal{U}) \to 0$$

(where $i\colon S \hookrightarrow X$ is the embedding) and Serre duality on X; in particular, $\chi(\mathcal{U}, \mathcal{U}) = 2$ and, moreover, $\mathcal{U}$ is stable by Proposition 3.9. Since $\mathrm{rk}(\mathcal{U}) = r$ and $\mathrm{c}_1(\mathcal{U}) = -H_S$ by Definition 5.1(a), it follows from (6) that $\upsilon(\mathcal{U}) = (r, -H_S, s)$, hence $\mathcal{U}$ is a Mukai bundle on S. Now, stability of $\mathcal{U}_X$ easily follows: if $\mathcal{F} \subset \mathcal{U}_X$ is a saturated destabilizing subsheaf then $\mathcal{F}|_S \subset \mathcal{U}$ is a destabilizing subsheaf for $\mathcal{U}$, which is impossible because, as we observed above, $\mathcal{U}$ is stable. □

5.1. **Weak Mukai sheaves on a singular Fano threefold.** Throughout this subsection we assume that X is a prime Fano threefold of genus $g \geq 6$ with terminal Gorenstein singularities such that $\mathrm{Cl}(X) = \mathrm{Pic}(X) = \mathbb{Z}\cdot H$ and X has a smooth anticanonical divisor with $\mathrm{Pic}(S) = \mathbb{Z}\cdot H_S$. The main result of this section is the following theorem (it will be used to deduce Theorem 1.1).

Theorem 5.3. *Let X be a prime Fano threefold of genus $g \geq 4$ with terminal Gorenstein singularities such that $\mathrm{Cl}(X) = \mathrm{Pic}(X) = \mathbb{Z} \cdot H$ and X has a smooth anticanonical divisor with $\mathrm{Pic}(S) = \mathbb{Z} \cdot H_S$. If $g = r \cdot s$ with $s \geq r \geq 2$ then there exists a weak Mukai sheaf $\mathcal{U}_X$ on X with the property that the restriction $\mathcal{U}_X|_S$ is a Mukai bundle on S and the dual sheaf $\mathcal{U}_X^\vee$ is globally generated with $h^0(\mathcal{U}_X^\vee) = r + s$ and $h^{>0}(\mathcal{U}_X^\vee) = 0$.*

To prove this, we will show that in any nice pencil of anticanonical divisors on X there is a du Val K3 surface S_0 such that its minimal resolution $\widetilde{S}_0$ carries a special Mukai class.

Lemma 5.4. *There is a pencil $\mathcal{S} \subset |H|$ of anticanonical divisors in X such that*

(a) *the base locus $C \coloneqq \mathrm{Bs}(\mathcal{S})$ is a smooth BNP-general curve;*
(b) *a very general member S_t of $\mathcal{S}$ has $\mathrm{Pic}(S_t) = \mathbb{Z} \cdot H_{S_t}$;*
(c) *a general member S_t of $\mathcal{S}$ is smooth;*
(d) *any singular member S_t of $\mathcal{S}$ has at worst du Val singularities.*

Proof. Since $\mathrm{Cl}(X) = \mathrm{Pic}(X)$ and $g \geq 4$, by [P, Theorem 4.2 and Example 4.3] the anticanonical morphism of X is a closed embedding $X \subset \mathbb{P}^{g+1}$. Consider the dual projective space $\check{\mathbb{P}}^{g+1}$, the projectively dual variety $X^\vee \subset \check{\mathbb{P}}^{g+1}$, and the hyperplanes $\mathcal{P}_1, \dots, \mathcal{P}_n \subset \check{\mathbb{P}}^{g+1}$ that correspond to the singular points $p_1, \dots, p_n \in X$. Let $\mathcal{P}_i^{\mathrm{ndV}} \subset \mathcal{P}_i$ be the closed subset that corresponds to hyperplanes $\mathbb{P}^g \subset \mathbb{P}^{g+1}$ that contain the point p_i and the singularity of $X \cap \mathbb{P}^g$ at p_i is worse than du Val. Since $p_i \in X$ is a terminal Gorenstein singularity, $\mathcal{P}_i^{\mathrm{ndV}}$ is a proper subset of $\mathcal{P}_i$, see [R1, Main Theorem]. Thus, the subsets

$$\mathrm{Sing}(X^\vee) \subset X^\vee \subset \check{\mathbb{P}}^{g+1} \qquad \text{and} \qquad \mathcal{P}_i^{\mathrm{ndV}} \subset \mathcal{P}_i \subset \check{\mathbb{P}}^{g+1} \tag{39}$$

all have codimension at least 2 in $\check{\mathbb{P}}^{g+1}$.

On the other hand, let $S \subset X$ be a very general smooth anticanonical divisor. By the hypothesis of Theorem 5.3 we have $\mathrm{Pic}(S) = \mathbb{Z} \cdot H_S$, hence by Theorem 2.10(b) a general curve $C \in |H_S|$ is BNP-general. Now consider a general line in $\check{\mathbb{P}}^{g+1}$ passing through the point that corresponds to S and avoiding all subsets in (39) and let $\mathcal{S} \subset |H|$ be the corresponding pencil. Then it is clear that all the properties of the lemma are satisfied. □

Let $\mathcal{S}$ be a pencil of hyperplane sections of X as in Lemma 5.4. Consider the blowup

$$\pi \colon \widetilde{X} \coloneqq \mathrm{Bl}_C(X) \to X$$

of the base locus $C \subset X$ of $\mathcal{S}$ and denote by

$$p \colon \widetilde{X} \to \mathbb{P}^1$$

the induced morphism. The exceptional divisor E of π has the form

$$E \cong C \times \mathbb{P}^1,$$

the restriction of π to E is the projection to C, the restriction of p to E is the projection to $\mathbb{P}^1$, and the normal bundle of E is

$$\mathcal{O}_E(E) \cong \mathcal{O}_C(K_C) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-1). \tag{40}$$

We denote by $\iota\colon E \hookrightarrow \widetilde{X}$ the embedding.

Since C is BNP-general (by Lemma 5.4(a)), Lemma 2.3 shows that C has a Mukai pair (ξ, η) of type (r, s). Consider the relative Lazarsfeld bundle $\mathbf{L}_{\widetilde{X}/\mathbb{P}^1}(\xi)$ on $\widetilde{X}$ defined by the exact sequence

$$0 \to \mathbf{L}_{\widetilde{X}/\mathbb{P}^1}(\xi) \longrightarrow \mathrm{H}^0(C, \xi) \otimes \mathcal{O}_{\widetilde{X}} \xrightarrow{\mathrm{ev}} \iota_*(\xi \boxtimes \mathcal{O}_{\mathbb{P}^1}) \to 0. \tag{41}$$

Below we investigate the pullbacks of $\mathbf{L}_{\widetilde{X}/\mathbb{P}^1}(\xi)$ to the fibers of p and their minimal resolutions.

Note that the fiber $S_t := p^{-1}(t)$ of p over a point $t \in \mathbb{P}^1$ is a polarized K3 surface with du Val singularities and its minimal resolution $\sigma_t\colon \widetilde{S}_t \to S_t$ is a smooth quasipolarized K3 surface (where the quasipolarization $H_{\widetilde{S}_t}$ is the pullback of H_{S_t}); sometimes we will think of σ_t as a map $\widetilde{S}_t \to \widetilde{X}$. Note also that the curve $S_t \cap E = C$ is contained in the smooth locus of S_t, hence the embedding $C \hookrightarrow S_t$ lifts to an embedding $j_t\colon C \hookrightarrow \widetilde{S}_t$, so that $\sigma_t(j_t(C)) = E \cap \widetilde{S}_t \subset \widetilde{X}$.

Recall that special Mukai classes on quasipolarized K3 surfaces were introduced in Definition 2.12. The crucial observation is the following.

Proposition 5.5. *Let $\mathcal{S} \in |H|$ be a pencil as in Lemma 5.4 and let $\mathbf{L}_{\widetilde{X}/\mathbb{P}^1}(\xi)$ be the relative Lazarsfeld bundle associated with a Mukai pair (ξ, η) of type (r, s) on the base locus C of $\mathcal{S}$. For each point $t \in \mathbb{P}^1$ the pullback $\sigma_t^*\mathbf{L}_{\widetilde{X}/\mathbb{P}^1}(\xi)$ is the Lazarsfeld bundle on the surface $\widetilde{S}_t$ associated with the curve $j_t(C)$ and the Mukai pair (ξ, η) on it, i.e.,*

$$\sigma_t^*\mathbf{L}_{\widetilde{X}/\mathbb{P}^1}(\xi) \cong \mathbf{L}_{\widetilde{S}_t}(\xi). \tag{42}$$

Moreover, there is a point $t_0 \in \mathbb{P}^1$ such that the surface $\widetilde{S}_{t_0}$ has a special Mukai class $\Xi \in \mathrm{Pic}(\widetilde{S}_{t_0})$ of type (r, s).

Proof. Pulling back the sequence (41) to $\widetilde{S}_t$ we obtain an exact sequence

$$0 \to \sigma_t^*\mathbf{L}_{\widetilde{X}/\mathbb{P}^1}(\xi) \longrightarrow \mathrm{H}^0(C, \xi) \otimes \mathcal{O}_{\widetilde{S}_t} \xrightarrow{\mathrm{ev}} j_{t*}\xi \to 0,$$

which coincides with the defining sequence of $\mathbf{L}_{\widetilde{S}_t}(\xi)$, hence the isomorphism (42).

Now we restrict the sequence (41) to E. Using (40) we obtain an exact sequence

$$0 \to \xi(-K_C) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow \iota^*\mathbf{L}_{\widetilde{X}/\mathbb{P}^1}(\xi) \longrightarrow \mathrm{H}^0(C, \xi) \otimes (\mathcal{O}_C \boxtimes \mathcal{O}_{\mathbb{P}^1}) \xrightarrow{\mathrm{ev}} \xi \boxtimes \mathcal{O}_{\mathbb{P}^1} \to 0.$$

Its last morphism is the pullback of $\mathrm{ev}\colon \mathrm{H}^0(C, \xi) \otimes \mathcal{O}_C \to \xi$ along p, therefore its kernel is isomorphic to $\mathbf{R}_C(\xi^{-1})^\vee \boxtimes \mathcal{O}_{\mathbb{P}^1}$. Dualizing, we obtain an exact sequence

$$0 \to \mathbf{R}_C(\xi^{-1}) \boxtimes \mathcal{O}_{\mathbb{P}^1} \to \iota^*\mathbf{L}_{\widetilde{X}/\mathbb{P}^1}(\xi)^\vee \to \eta \boxtimes \mathcal{O}_{\mathbb{P}^1}(-1) \to 0.$$

Consider its extension class

$$\epsilon \in \mathrm{Ext}^1(\eta \boxtimes \mathcal{O}_{\mathbb{P}^1}(-1), \mathbf{R}_C(\xi^{-1}) \boxtimes \mathcal{O}_{\mathbb{P}^1}) \cong \mathrm{Ext}^1(\eta, \mathbf{R}_C(\xi^{-1})) \otimes \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)).$$

For each $t \in \mathbb{P}^1$ we can evaluate ϵ at t and obtain an extension class $\epsilon(t) \in \mathrm{Ext}^1(\eta, \mathbf{R}_C(\xi^{-1}))$. Using (42) we obtain an isomorphism

$$(\iota^*\mathbf{L}_{\widetilde{X}/\mathbb{P}^1}(\xi))|_{C \times \{t\}} \cong j_t^*\sigma_t^*\mathbf{L}_{\widetilde{X}/\mathbb{P}^1}(\xi) \cong j_t^*\mathbf{L}_{\widetilde{S}_t}(\xi).$$

It follows that the extension of η by $\mathbf{R}_C(\xi^{-1})$ corresponding to the class $\epsilon(t)$ is isomorphic to $j_t^*\mathbf{L}_{\widetilde{S}_t}(\xi)$, which means that $\epsilon(t)$ is the Lazarsfeld extension class for all t, see Definition 4.6.

We claim that there is a unique point $t_0 \in \mathbb{P}^1$ such that $\epsilon(t_0) = 0$. Indeed, for a very general t we have $\mathrm{Pic}(\widetilde{S}_t) = \mathbb{Z} \cdot H_{\widetilde{S}_t}$ (by Lemma 5.4(b), (c)), in particular $\widetilde{S}_t$ does not have special Mukai classes. Therefore, Proposition 4.8 shows that $\epsilon(t)$ is a Mukai extension class as in Definition 4.9, and then Theorem 4.13 proves that the classes $\epsilon(t)$ for different very general t are all proportional. Thus, we can write

$$\epsilon = \epsilon_0 \otimes f_0,$$

where $\epsilon_0 \in \mathrm{Ext}^1(\eta, \mathbf{R}_C(\xi^{-1}))$ and $f_0 \in \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ is a linear function on $\mathbb{P}^1$. Hence there is a unique point $t_0 \in \mathbb{P}^1$ such that $f_0(t_0) = 0$, i.e., the Lazarsfeld extension class $\epsilon(t_0)$ vanishes. Applying Proposition 4.8 we conclude that $\widetilde{S}_{t_0}$ must have a special Mukai class Ξ of type (r, s). ☐

In Proposition 5.5 we constructed an anticanonical divisor $S_0 \subset X$ such that its minimal resolution $\sigma\colon \widetilde{S}_0 \to S_0$ has a special Mukai class Ξ. Recall from Corollary 2.15 that $\sigma_*\mathcal{O}_{\widetilde{S}_0}(\Xi)$ is a maximal Cohen–Macaulay globally generated sheaf on S_0.

Now we define a coherent sheaf $\mathbf{L}_X(\Xi)$ on X from the exact sequence

$$0 \to \mathbf{L}_X(\Xi) \longrightarrow \mathrm{H}^0(\widetilde{S}_0, \mathcal{O}_{\widetilde{S}_0}(\Xi)) \otimes \mathcal{O}_X \xrightarrow{\mathrm{ev}} \sigma_*\mathcal{O}_{\widetilde{S}_0}(\Xi) \to 0. \tag{43}$$

This is an incarnation of the Lazarsfeld bundle construction in dimension three rather than two.

Corollary 5.6. *If $S_0 \subset X$ is an anticanonical divisor with du Val singularities such that its minimal resolution $\sigma\colon \widetilde{S}_0 \to S_0$ has a special Mukai class Ξ of type (r, s), then $\mathbf{L}_X(\Xi)$ is a weak Mukai sheaf on X. Moreover, if $S \subset X$ is a smooth anticanonical divisor such that* $\mathrm{Pic}(S) = \mathbb{Z}\cdot H_S$, *$C = S \cap S_0$ is a smooth BNP-general curve, and $\xi \coloneqq \sigma_*\mathcal{O}_{\widetilde{S}_0}(\Xi)|_C$ then*

$$\mathbf{L}_X(\Xi)|_S \cong \mathbf{L}_S(\xi).$$

In particular, $\mathbf{L}_X(\Xi)|_S$ is a Mukai bundle on S.

Proof. Properties (a) and (b) of Definition 5.1 follow immediately from the defining exact sequence (43) and Corollary 2.15. Since the last term of (43) is a maximal Cohen–Macaulay sheaf on a Cartier divisor in X, it follows that $\mathbf{L}_X(\Xi)$ is maximal Cohen–Macaulay on X.

Moreover, Proposition 2.14 shows that $(\xi, \xi^{-1}(K_C))$ is a Mukai pair of type (r, s), and the restriction of (43) to S coincides with the defining exact sequence (19) of the Lazarsfeld bundle $\mathbf{L}_S(\xi)$, hence $\mathbf{L}_X(\Xi)|_S \cong \mathbf{L}_S(\xi)$. Since $\mathrm{Pic}(S) = \mathbb{Z} \cdot H_S$, it follows from Theorem 3.4 that $\mathbf{L}_S(\xi)$ is a Mukai bundle; in particular it is stable. Therefore, $\mathbf{L}_X(\Xi)$ is stable by the argument of Lemma 5.2. ☐

Proof of Theorem 5.3. Since $s \geq r \geq 2$ and $g \leq 12$, we have $r \in \{2, 3\}$. Therefore, Corollary 5.6 applies, proving that $\mathcal{U}_X \coloneqq \mathbf{L}_X(\Xi)$ is a weak Mukai sheaf on X. Note that, in particular, $\mathbf{L}_X(\Xi)$ is reflexive. Hence, we can dualize (43) to obtain an exact sequence

$$0 \to \mathrm{H}^0(\widetilde{S}_0, \mathcal{O}_{\widetilde{S}_0}(\Xi))^\vee \otimes \mathcal{O}_X \to \mathbf{L}_X(\Xi)^\vee \to \sigma_*\mathcal{O}_{\widetilde{S}_0}(H - \Xi) \to 0,$$

where the last term is identified in Corollary 2.15, which also proves its global generation and computes its cohomology. Since $h^1(\mathcal{O}_X) = 0$, it follows that the sheaf $\mathcal{U}_X^\vee \cong \mathbf{L}_X(\Xi)^\vee$ is globally generated, $h^0(\mathcal{U}_X^\vee) = r + s$ and $h^{>0}(\mathcal{U}_X^\vee) = 0$. ☐

5.2. **The Mukai bundle.** In this section we prove Theorem 1.1. So, from now on we assume that X is smooth. By [Mo, Theorem 7.5] a very general anticanonical divisor $S \subset X$ satisfies $\mathrm{Pic}(S) = \mathbb{Z} \cdot H_S$, hence Theorem 5.3 implies the existence of a weak Mukai sheaf $\mathbf{L}_X(\Xi)$ on X. Since a maximal Cohen–Macaulay sheaf on a smooth variety is locally free, we see that $\mathbf{L}_X(\Xi)$ is a vector bundle. It remains to prove its uniqueness and exceptionality.

We will need the following simple observation.

Lemma 5.7. *If $\mathcal{F}$ is a vector bundle on X then $\chi(\mathcal{F}, \mathcal{F}) = \frac{1}{2}\chi(\mathcal{F}|_S, \mathcal{F}|_S)$. In particular, if $\mathcal{F}|_S$ is numerically spherical, i.e., $\chi(\mathcal{F}|_S, \mathcal{F}|_S) = 2$, then $\mathcal{F}$ is numerically exceptional, i.e., $\chi(\mathcal{F}, \mathcal{F}) = 1$.*

Proof. Let $i\colon S \hookrightarrow X$ be the embedding. The short exact sequence $0 \to \mathcal{F}(-H) \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0$ combined with Serre duality and adjunction gives

$$\chi(\mathcal{F}, \mathcal{F}) = \chi(\mathcal{F}, \mathcal{F}(-H)) + \chi(\mathcal{F}, i_*i^*\mathcal{F}) = -\chi(\mathcal{F}, \mathcal{F}) + \chi(\mathcal{F}|_S, \mathcal{F}|_S),$$

which shows our claim. □

The following lemma reduces verification of some Ext-vanishing on X to Ext-vanishing on S.

Lemma 5.8. *Let $\mathcal{F}_1, \mathcal{F}_2$ be vector bundles on X and let $S \subset X$ be a divisor in $|H|$ such that*

$$\mathrm{Hom}(\mathcal{F}_1|_S, \mathcal{F}_2|_S) = \Bbbk \qquad \text{and} \qquad \mathrm{Ext}^1(\mathcal{F}_1|_S, \mathcal{F}_2(-H)|_S) = 0.$$

Then $\mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2) = \Bbbk$ and $\mathrm{Ext}^1(\mathcal{F}_1, \mathcal{F}_2(-kH)) = 0$ for all $k \ge 1$.

Proof. Consider the Koszul complex for three general global sections of $\mathcal{O}_S(H_S)$:

$$0 \to \mathcal{O}_S(-3H_S) \to \mathcal{O}_S(-2H_S)^{\oplus 3} \to \mathcal{O}_S(-H_S)^{\oplus 3} \to \mathcal{O}_S \to 0.$$

Tensoring it by $\mathcal{F}_2(H)|_S$ we obtain an exact sequence

$$0 \to \mathcal{F}_2(-2H)|_S \to \mathcal{F}_2(-H)|_S^{\oplus 3} \to \mathcal{F}_2|_S^{\oplus 3} \to \mathcal{F}_2(H)|_S \to 0$$

Since $\mathrm{Ext}^1(\mathcal{F}_1|_S, \mathcal{F}_2(-H)|_S) = 0$, a simple spectral sequence implies that $\mathrm{Ext}^1(\mathcal{F}_1|_S, \mathcal{F}_2(-2H)|_S)$ is a quotient of the space

$$\mathrm{Ker}\left(\mathrm{Hom}(\mathcal{F}_1|_S, \mathcal{F}_2|_S)^{\oplus 3} \to \mathrm{Hom}(\mathcal{F}_1|_S, \mathcal{F}_2(H)|_S)\right).$$

Since $\mathrm{Hom}(\mathcal{F}_1|_S, \mathcal{F}_2|_S) = \Bbbk$, the kernel is nonzero only if $\mathrm{Hom}(\mathcal{F}_1|_S, \mathcal{F}_2|_S)$ is annihilated by a section of $\mathcal{O}_S(H_S)$, which is impossible because $\mathcal{F}_i$ are locally free. Thus, $\mathrm{Ext}^1(\mathcal{F}_1|_S, \mathcal{F}_2(-2H)|_S) = 0$. Twisting the same Koszul complex by $-H$, $-2H$, and so on, and repeating the same argument, we see that $\mathrm{Ext}^1(\mathcal{F}_1|_S, \mathcal{F}_2(-kH)|_S) = 0$ for all $k \ge 1$.

Now consider the restriction exact sequence

$$\mathrm{Ext}^1(\mathcal{F}_1, \mathcal{F}_2(-(k+1)H)) \to \mathrm{Ext}^1(\mathcal{F}_1, \mathcal{F}_2(-kH)) \to \mathrm{Ext}^1(\mathcal{F}_1|_S, \mathcal{F}_2(-kH)|_S).$$

Its right term is zero for $k \ge 1$ as we just showed, hence its first arrow must be surjective. Thus, if $\mathrm{Ext}^1(\mathcal{F}_1, \mathcal{F}_2(-kH)) \ne 0$ for some $k \ge 1$, the same nonvanishing holds for all sufficiently large k, which contradicts Serre vanishing because $\mathcal{F}_i$ are locally free. This proves the vanishing of Ext^1.

Similarly, the restriction exact sequence for $k = 0$ gives

$$0 \to \mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2(-H)) \to \mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2) \to \mathrm{Hom}(\mathcal{F}_1|_S, \mathcal{F}_2|_S) \to \mathrm{Ext}^1(\mathcal{F}_1, \mathcal{F}_2(-H)),$$

and since the right term vanishes and the next to it term is $\Bbbk$, we have $\mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2) \ne 0$. Finally, if the dimension of $\mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2)$ is greater than 1, then $\mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2(-H)) \ne 0$, and then by induction $\mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2(-kH)) \ne 0$ for all $k \ge 0$, again contradicting Serre vanishing. □

Corollary 5.9. *If X is a smooth prime Fano threefold of genus $g \geq 6$ then the weak Mukai sheaf $\mathbf{L}_X(\Xi)$ constructed in Theorem 5.3 is a Mukai bundle on X. Moreover, any Mukai bundle on X is isomorphic to $\mathbf{L}_X(\Xi)$; in particular, the Mukai bundle on X is unique.*

Proof. By [Mo, Theorem 7.5] a very general anticanonical divisor $S \subset X$ satisfies $\operatorname{Pic}(S) = \mathbb{Z} \cdot H_S$. Since a maximal Cohen–Macaulay sheaf on a smooth variety is locally free, we see that $\mathbf{L}_X(\Xi)$ is a vector bundle. By Corollary 5.6 the restriction $\mathbf{L}_X(\Xi)|_S \cong \mathbf{L}_S(\xi)$ is a Mukai bundle. Moreover, Proposition 4.2 shows that

$$\operatorname{Hom}(\mathbf{L}_S(\xi), \mathbf{L}_S(\xi)) = \Bbbk \qquad \text{and} \qquad \operatorname{Ext}^1(\mathbf{L}_S(\xi), \mathbf{L}_S(\xi) \otimes \mathcal{O}_S(-H)) = 0.$$

Applying Lemma 5.8 to $\mathcal{F}_1 = \mathcal{F}_2 = \mathbf{L}_X(\Xi)$, we obtain

$$\operatorname{Hom}(\mathbf{L}_X(\Xi), \mathbf{L}_X(\Xi)) = \Bbbk \qquad \text{and} \qquad \operatorname{Ext}^1(\mathbf{L}_X(\Xi), \mathbf{L}_X(\Xi) \otimes \mathcal{O}_X(-H)) = 0.$$

By Serre duality, the second equality implies that $\operatorname{Ext}^2(\mathbf{L}_X(\Xi), \mathbf{L}_X(\Xi)) = 0$. On the other hand, we have $\chi(\mathbf{L}_X(\Xi), \mathbf{L}_X(\Xi)) = \frac{1}{2}\chi(\mathbf{L}_S(\xi), \mathbf{L}_S(\xi)) = 1$ by Lemma 5.7. Therefore,

$$\operatorname{Ext}^1(\mathbf{L}_X(\Xi), \mathbf{L}_X(\Xi)) = \operatorname{Ext}^3(\mathbf{L}_X(\Xi), \mathbf{L}_X(\Xi)) = 0,$$

hence $\mathbf{L}_X(\Xi)$ is exceptional. Combining with Corollary 5.6, we see that $\mathbf{L}_X(\Xi)$ is a Mukai bundle.

Let $\mathcal{U}_X$ be another Mukai bundle on X. By Lemma 5.2 and Theorem 3.4 we have $\mathcal{U}_X|_S \cong \mathbf{L}_S(\xi)$. Therefore, applying Lemma 5.8 to $\mathcal{F}_1 = \mathcal{U}_X$, $\mathcal{F}_2 = \mathbf{L}_X(\Xi)$, we obtain $\operatorname{Hom}(\mathcal{U}_X, \mathbf{L}_X(\Xi)) = \Bbbk$. But both $\mathcal{U}_X$ and $\mathbf{L}_X(\Xi)$ are H-Gieseker stable by Lemma 5.2, hence $\mathcal{U}_X \cong \mathbf{L}_X(\Xi)$. □

Proof of Theorem 1.1. If $s \geq r$, we apply Theorem 5.3 (the assumption $\operatorname{Pic}(S) = \mathbb{Z} \cdot H_S$ is satisfied by [Mo, Theorem 7.5]) and Corollary 5.9. If $s < r$, we apply Theorem 5.3 and Corollary 5.9 to construct the Mukai bundle $\mathcal{U}_s$ of type (s, r) and define

$$\mathcal{U}_r = \mathcal{U}_s^{\perp} \coloneqq \operatorname{Ker}\left(\mathrm{H}^0(X, \mathcal{U}_s^{\vee}) \otimes \mathcal{O}_X \to \mathcal{U}_s^{\vee}\right).$$

Then it is immediate to see that $\mathcal{U}_r$ is a Mukai bundle of type (r, s) on X and the dual of the defining exact sequence of $\mathcal{U}_r$

$$0 \to \mathcal{U}_s \to \mathrm{H}^0(X, \mathcal{U}_s^{\vee})^{\vee} \otimes \mathcal{O}_X \to \mathcal{U}_r^{\vee} \to 0$$

shows that $\mathcal{U}_r^{\vee}$ is globally generated and computes its cohomology. □

Remark 5.10. If X is a smooth prime Fano threefold of genus $g = 4$ then $X = Q \cap R$ is a complete intersection of a quadric and cubic hypersurfaces in $\mathbb{P}^5$; moreover, Q is smooth or is a cone over a smooth 3-dimensional quadric $\bar{Q}$, and if Q is a cone then R does not contain its vertex. In the first case the restriction to X of either of the two spinor bundles on Q is a Mukai bundle on X, but they are not isomorphic and thus a Mukai bundle is not unique. In the second case the pullback of the unique spinor bundle from $\bar{Q}$ is the unique weak Mukai sheaf on X, see [KS, Proposition 4.2]; however, the sheaf is not exceptional, hence a Mukai bundle does not exist on such X.

Furthermore, consider a complete intersection $X = Q \cap R$ of a quadric cone Q over a smooth quadric threefold $\bar{Q}$ and a cubic hypersurface R containing the vertex, where R is very general among such cubics; such X is a factorial 1-nodal Fano threefold of genus $g = 4$ admitting a smooth anticanonical divisor with Picard number 1. Let $\mathcal{U}_X$ be the reflexive extension of the pullback of the spinor sheaf on $\bar{Q}$ to the complement of the node in X. Then $\mathcal{U}_X$ is the unique weak Mukai sheaf on X, but it is neither locally free, nor exceptional.

We do not know an example of a weak Mukai sheaf that is not a vector bundle on a Fano threefold satisfying the assumptions of Theorem 5.3 for $g \geq 6$.

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