A TOUR TO STABILITY CONDITIONS ON DERIVED CATEGORIES

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ABSTRACT. These lecture notes are an introduction to the space of Bridgeland stability conditions on derived categories, based on lecture series by the author at various summer schools. They are geared towards graduate students interested in working on stability conditions. Besides many pictures and exercises we also include a simpler proof of Bridgeland's deformation result.

Preliminary version with many omissions, errors, and almost no motivation. Comments are very welcome.

1. INTRODUCTION

Stability conditions on derived categories were introduced by Bridgeland in [Bri07] in order to understand the work of Douglas on π-stability in superconformal field theories [Dou02]. A SCFT as considered by Douglas depends both on the complex structure on a Calabi-Yau threefold X *and* the choice of a Kähler parameter. D-Branes in this theory can be understood as objects in the derived category $\mathrm{D^b}(X)$ of X, and in fact the SCFT knows $\mathrm{D^b}(X)$ completely. A stability condition in the sense of Bridgeland captures the additional structure on $\mathrm{D^b}(X)$ given by the choice of a Kähler parameter.

Despite what the previous paragraph might suggest, these notes give a purely mathematical introduction to the space of Bridgeland stability conditions.

1.1. **On these notes.** These notes are intended for readers with some familiarity of basic algebraic geometry[1] and perhaps some previous exposure to derived categories.[2] They are probably better described as trying to prepare the reader to work with stability conditions, rather than an expository overview for bedtime reading.

Exercises marked with (*) require additional knowledge beyond basic algebraic geometry knowledge or the material presented here (most often related to Fourier-Mukai transforms).

The only part of our notes that might have a small claim of originality is perhaps our proof of Bridgeland's deformation result in section 5.5. I would like to thank Dan Grayson for bringing his quite beautiful Harder-Narasimhan filtration to my attention, on which our deformation proof is based. Both that proof and the entire notes are based on the viewpoint that in order to understand stability conditions

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[1] smooth projective curves, coherent sheaves, degree of a line bundle on a curve

[2] We do give a review of the aspects that are most relevant for stability conditions.

on derived categories, it is enough to understand stability conditions on abelian categories, and to understand *tilting* of an abelian category at a torsion pair.[3]

2. STABILITY IN ABELIAN CATEGORIES

2.1. **Stable vector bundles on algebraic curves.** Stability in algebraic geometry is a very classical concept, in the two (closely related) contexts of geometric invariant theory, and stability of vector bundles and coherent sheaves. We will say nothing about the former, and take a short through the latter.

Let X be a smooth, projective curve over $\mathbb{C}$ (a compact Riemann surface). If E is a vector bundle, it has two numerical invariants: the rank $\mathrm{rk}(E)$, and the degree[4] $\deg(E)$. We define its slope to be the quotient

$$\mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)}$$

its slope. In fact, this definition also makes sense for any coherent sheaf $\mathcal{F}$ on X, if we set $\mu(\mathcal{F}) = +\infty$ for any torsion-sheaf $\mathcal{F}$.

The following lemma, called the *see-saw property*, is extremely crucial:

Lemma 2.1.1. *Let $0 \to A \to E \to B \to 0$ be a short exact sequence coherent sheaves on X. Then*

$$\begin{aligned} \mu(A) < \mu(E) \quad &\Leftrightarrow \quad \mu(E) < \mu(B) \\ \mu(A) > \mu(E) \quad &\Leftrightarrow \quad \mu(E) > \mu(B) \end{aligned}$$

This follows by simple algebra, but even more convincingly from the picture in figure 1, where we have set $Z(\_\_) = i\,\mathrm{rk}(\_\_) - \deg(\_\_)$: The ordering of slopes is equivalent to the ordering of the complex numbers $Z(\_\_)$ by their arguments. By additivity of degree and rank on short exact sequence, we have $Z(E) = Z(A) + Z(B)$. The two cases correspond to the two possible orientations of the parallelogram $0, Z(A), Z(E), Z(B)$ (and equality to the case of a degenerate parallelogram).

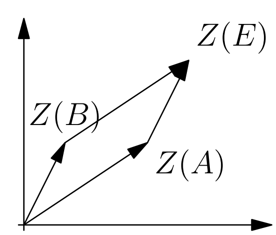


FIGURE 1. See-saw property

[3]The author was partially supported by the NSF grant DMS-0801356/DMS-1001056.

[4]The degree of a vector bundle can be characterized by as follows: for a line bundle $\mathcal{L} \cong \mathcal{O}_X(\sum a_i P_i)$, the degree is $d(\mathcal{L}) = \sum_i a_i$, and for higher-rank bundles it is determined by additivity on short exact sequences.

Remark 2.1.2. What we used in the above "proof by picture" are just two properties of the function Z:

(1) Z is additive on short exact sequences; in other words, Z is a group homomorphism $Z\colon K(\mathcal{A}) \to \mathbb{C}$ from the K-group[5] to $\mathbb{C}$.
(2) The image of Z is contained in a half-plane in $\mathbb{C}$, so that we can meaningfully compare the slopes of objects.

Definition 2.1.3. A vector bundle is slope-(semi-)stable if for all subbundles $A \hookrightarrow E$ we have $\mu(A) < (\leq)\mu(E)$.

Due to the see-saw property, we could ask equivalently that for all quotients $E \twoheadrightarrow B$ we have $\mu(E) > (\geq)\mu(B)$.

Example 2.1.4. (1) Any line bundle is stable.
(2) Let L_1 be a line bundle of degree one. An extension $0 \to \mathcal{O}_X \to E \to L_1 \to 0$ of L_1 by $\mathcal{O}_X$ is stable if and only if the extension does not split. Such extensions always exist when $g > 1$.

Lemma 2.1.5. *If E, E' are semistable and $\mu(E) > \mu(E')$, then* $\mathrm{Hom}(E, E') = 0$.

Proof. Given a non-zero morphism $\phi\colon E \to E'$, consider the image $\mathrm{im}\,\phi$ fitting into the maps $E \twoheadrightarrow \mathrm{im}\,\phi \hookrightarrow E'$. By the semistability of E and E', we get a contradiction from $\mu(E) \leq \mu(\mathrm{im}\,\phi) \leq \mu(E') < \mu(E)$. □

Classically, a lot of interest in stable vector bundles is due to the fact that stability allows the study of moduli of vector bundles via nicely behaved (finite type, separated) moduli spaces. In particular, the moduli of *stable* vector bundles of fixed Chern class is "bounded", which is not true for the moduli of arbitrary vector bundles: the bundles $\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$ form an unbounded set of vector bundles or rank 2 and degree 0 on $\mathbb{P}^1$, i.e. there can never be a family of vector bundles parameterized by a scheme of finite type such that each of the above bundles appear in the family. Further, one can construct a moduli space of (semi-)stable vector bundles by geometric invariant theory.

However, for our purposes the existence of Harder-Narasimhan filtration is the most interesting aspect:

Theorem 2.1.6. *For any coherent sheaf $\mathcal{F}$ there is a unique increasing filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}$$

such that the filtration quotients $\mathcal{F}_i/\mathcal{F}_{i-1}$ are semistable of slope μ_i, with $\mu_1 > \mu_2 > \cdots > \mu_n$.

The beautiful proof we reproduce here is due to Dan Grayson, [Gra84] and will apply in many similar situations. We will only prove the existence of the HN-filtration—the uniqueness follows by purely formal arguments from Lemma 2.1.5.

Proof. To visualize the slope function, we again use the complex plane by setting $Z(\mathcal{F}) = i\,\mathrm{rk}(\mathcal{F}) - \deg(\mathcal{F})$.

[5] The K-group of an abelian category is the quotient of the free abelian group generated by its objects by the relation $[B] = [A] + [C]$ for any short exact sequence $A \hookrightarrow B \twoheadrightarrow C$.

Consider the subset $\{Z(A) \mid A \subset \mathcal{F}\}$ of the complex plane, and let $\mathcal{H}_Z(\mathcal{F})$ be its convex hull. By Lemma 2.1.7, it is bounded from the left. As the image of Z is contained in the discrete subset $\mathbb{Z} \oplus i\mathbb{Z}$, it follows that there is a finite set of extremal points $0 = v_0, v_1, \dots, v_n = Z(\mathcal{F})$ of the set $\mathcal{H}_Z(\mathcal{F})$ that lie on or to the left the straight line through the origin containing $Z(\mathcal{F})$ (i.e, the intersection of $\mathcal{H}_Z(\mathcal{F})$ with the half-plane lying on or to the left of the line is the convex hull of $v_0, \dots, v_n$). Let $\mathcal{F}_i \subset \mathcal{F}$ be an arbitrary subobject of $\mathcal{F}$ with $Z(\mathcal{F}_i) = v_i$. We make the following claims:

(1) $\mathcal{F}_i \subset \mathcal{F}_{i+1}$
(2) The slopes $\mu(\mathcal{F}_{i+1}/\mathcal{F}_i)$ are decreasing.
(3) $\mathcal{F}_{i+1}/\mathcal{F}_i$ is semistable.

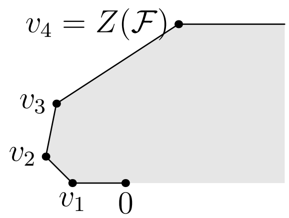


FIGURE 2. The set $\mathcal{H}_Z(\mathcal{F})$ and its extremal points

To prove claim (1), consider the intersection $\mathcal{F}_i \cap \mathcal{F}_{i+1} \subset \mathcal{F}$ and the span $\mathcal{F}_i + \mathcal{F}_{i+1} \subset \mathcal{F}$ of $\mathcal{F}_i, \mathcal{F}_{i+1}$ inside $\mathcal{F}$. Since $Z(\mathcal{F}_i)$ and $\mathcal{Z}(\mathcal{F}_{i+1})$ are adjacent extremal points of the set $\mathcal{H}_Z(\mathcal{F})$, both $Z(\mathcal{F}_i \cap \mathcal{F}_{i+1})$ and $Z(\mathcal{F}_i + \mathcal{F}_{i+1})$ lie on the line segment $v_i v_{i+1}$ or to the right of the straight line $(v_i v_{i+1})$. On the other hand, as $\mathcal{F}_i \cap \mathcal{F}_{i+1} \subset \mathcal{F}_i$, the imaginary part of $Z(\mathcal{F}_i \cap \mathcal{F}_{i+1})$ is bounded by $\Im Z(\mathcal{F}_i \cap \mathcal{F}_{i+1}) \leq \Im v_i$; similarly $\Im Z(\mathcal{F}_i + \mathcal{F}_{i+1}) \geq \Im v_{i+1}$; see fig. 3.

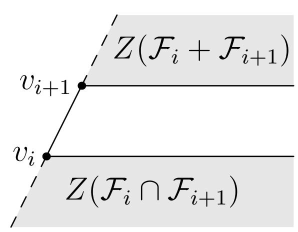


FIGURE 3. The location of $Z(\mathcal{F}_i \cap \mathcal{F}_{i+1})$, $Z(\mathcal{F}_i + \mathcal{F}_{i+1})$ relative to v_i, v_{i+1}

The short exact sequence

$$0 \to \mathcal{F}_i \cap \mathcal{F}_{i+1} \hookrightarrow \mathcal{F}_i \oplus \mathcal{F}_{i+1} \twoheadrightarrow \mathcal{F}_i + \mathcal{F}_{i+1} \to 0$$

implies

$$Z(\mathcal{F}_i \cap \mathcal{F}_{i+1}) + Z(\mathcal{F}_i + \mathcal{F}_{i+1}) = v_i + v_{i+1}.$$

As can be seen from the picture, this is only possible if $Z(\mathcal{F}_i \cap \mathcal{F}_{i+1}) = v_i$, i.e. if $\mathcal{F}_i \cap \mathcal{F}_{i+1} = \mathcal{F}_i$, i.e. if $\mathcal{F}_i$ is a subobject of $\mathcal{F}_{i+1}$.

The slope $\mu(\mathcal{F}_{i+1}/\mathcal{F}_i)$ is determined by the slope of the line segment $v_i v_{i+1}$, and claim (2) holds by convexity.

Finally, if $\overline{A} \subset \mathcal{F}_{i+1}/\mathcal{F}_i$ is a destabilizing subobject, consider its preimage $A \subset \mathcal{F}_{i+1}$. By additivity we have $Z(A) = Z(\overline{A}) + v_i$. As $Z(\overline{A})$ has bigger slope than $Z(\mathcal{F}_{i+1}/\mathcal{F}_i) = v_{i+1} - v_i$, this means that $Z(A)$ lies to the left of the straight line $(v_i v_{i+1})$, in contradiction to the convexity of $\mathcal{H}_Z(\mathcal{F})$. $\Box$ As the proof shows, the existence of an HN-filtration of $\mathcal{F}$ is equivalent to $\mathcal{H}_Z(\mathcal{F})$ having only finitely many extremal points on the left.

Lemma 2.1.7. *Let $\mathcal{F}$ be a coherent sheaf on X. Then there exists an integer $d \in \mathbb{Z}$ such that for any subsheaf $\mathcal{F}' \subset \mathcal{F}$ we have*

$$\deg(\mathcal{F}') \leq d$$

2.2. **Stability for quiver representations.** In the previous section we used extremely few ingredients of the specific situation of the category $\operatorname{Coh} X$ of coherent sheaves on a curve:

(1) We have invariants $\operatorname{rk}(\mathcal{F}), \deg(\mathcal{F})$ that are additive on short exact sequences.
(2) They satisfy a positivity property given by $\operatorname{rk}(\mathcal{F}) \geq 0$ and $\operatorname{rk}(\mathcal{F}) = 0 \Rightarrow \deg(\mathcal{F}) > 0$.
(3) Finally, we used the boundedness property of Lemma 2.1.7 and the discreteness of $\operatorname{rk}, \deg$ in the proof of the existence of HN-filtration.

It is easy to generalize this to other situations:

Definition 2.2.1. Given an abelian category $\mathcal{A}$, we say Z is a stability function for $\mathcal{A}$ if $Z\colon K(\mathcal{A}) \to \mathbb{C}$ is a group homomorphism from the K-group of $\mathcal{A}$ to $\mathbb{C}$ such that for any $0 \neq E \in \mathcal{A}$ we have

$$Z(E) \in \mathbb{H} = \left\{ z = m \cdot e^{i\pi\phi} \,\middle|\, m > 0, \phi \in (0,1]) \right\}$$

The positivity property of $\operatorname{rk}, \deg$ is replaced by the semi-closed half-plane $\mathbb{H}$. Then we can define the phase $\phi(E)$ of a non-zero object by $\phi(E) = \frac{1}{\pi}\arg(Z(E)) \in (0,1]$, and we say an object E is Z-semistable if the inequality $\phi(A) \leq \phi(E)$ holds for all subobjects $A \subset E$.

Now consider a finite quiver $Q = (Q_0, Q_1)$, i.e. a directed graph with vertices $Q_0 = \{0, \ldots, n\}$ and arrows Q_1. We may also allow relations R; a relation is a linear equation between directed paths starting and ending at the same vertex. Suppressing R from the notation, let $\operatorname{Rep} Q$ be the abelian category of its representations.

Example 2.2.2. The Kronecker quiver P_2 is given by two vertices $\{0, 1\}$ and two arrows $x, y\colon 0 \to 1$. A representation $\underline{V}$ of P_2 is a pair of vector spaces (V_0, V_1) together with morphisms $\phi_x, \phi_y\colon V_0 \to V_1$.

Example 2.2.3. The A_1-quiver has two vertices $\{0, 1\}$ with two morphisms $x_0, y_0 \colon 0 \to 1$ going from 0 to 1 and two morphisms $x_1, y_1 \colon 1 \to 0$ going in the other direction; it has relations $x_1 y_0 = y_1 x_0$ and $x_0 y_1 = y_0 x_1$. A representation of A_1 is a pair of vector spaces V_0, V_1 together with morphisms $\phi_0, \psi_0 \colon V_0 \to V_1$ and $\phi_1, \psi_1 \colon V_1 \to V_0$ satisfying the relations $\phi_1 \circ \psi_0 = \psi_1 \circ \phi_0$ and $\phi_0 \circ \psi_1 = \psi_0 \circ \phi_1$.

Pick a complex numbers $z_0, \dots, z_n \in \mathbb{H}$ in the semi-closed upper half-plane $\mathbb{H} = \left\{ \mathbb{R}_{>0} \cdot e^{i\pi\phi} \,\middle|\, \phi \in (0, 1] \right\}$. Then we can define a stability function Z by

$$Z(\underline{V}) = \sum_{i=0}^{n} \dim V_i \cdot z_i$$

The positivity properties of rank and degree are replaced by $Z(\underline{V}) \in \mathbb{H}$, and the boundedness property is evident from the fact that subrepresentations $\underline{W}$ of a given representation $\underline{V}$ can only have a finite number of possible dimension vectors $(\dim W_i)$.

2.3. **Exercises.**

Exercise 1. (Schur's Lemma) Let $\mathcal{A}$ be an abelian category and Z a stability function. Assume that E is Z-stable; show that any non-zero endomorphism $\phi \in \operatorname{End}(E)$ is an automorphism.

When $\mathcal{A}$ is a linear category over an algebraically closed field $k = \overline{k}$, it follows that $\operatorname{End} E$ consists only of scalars $k \cdot \mathrm{Id}$.

Exercise 2.
1. Consider the Kronecker quiver P_2 of example 2.2.2, and stability conditions given by $z_0, z_1 \in \mathbb{H}$.
   - (a) Show that if the phase of z_1 is bigger than the phase of z_0, then the only stable objects are the two simple representations, i.e. the representations with dimension vectors $(1, 0)$ and $(0, 1)$.
   - (b) Now assume that the phase of z_0 is bigger than the phase of z_1. Show that the there is a 1:1-correspondence between isomorphism classes of stable objects of dimension vector $(1, 1)$ and points on $\mathbb{P}^1$.[6]
2. Consider the same problem with P_{n+1}, the Kronecker quiver with $n + 1$ arrows and two vertices, and $\mathbb{P}^n$.
3. (*) For the second part of the above problem, consider the same situation for the A_1-quiver with relations of Example 2.2.3, and the blow-up X of the affine plane $A^2_{\mathbb{C}}$ at the origin.

Exercise 3. Let L_1 be a line bundle of degree 1 on a smooth projective curve X. Show that an extension $0 \to \mathcal{O}_X \to E \to L_1$ is stable if and only if the extension is non-split. (Note that such extensions exist for any L_1 whenever the genus of X is at least 2.)

[6] If you are familiar with the notion of moduli spaces in algebraic geometry, try to formulate the precise moduli problem and prove that $\mathbb{P}^1$ is a moduli space of stable objects with dimension vector $(1, 1)$.

Exercise 4. Show that the tangent bundle on $\mathbb{P}^2$ is slope-stable. (Note that to define slope-stability for a torsion-free sheaf on a variety of dimension two or higher, one only allows "saturated" subsheaves to test stability, i.e. subsheaves such that the quotient is also torsion-free.)

Exercise 5. Show that in the construction of the proof of Theorem 2.1.6, the choice of $\mathcal{F}_i$ with $Z(\mathcal{F}_i) = v_i$ is actually unique.

3. DERIVED CATEGORIES AND T-STRUCTURES

3.1. **Basics of derived categories.** It is probably an exaggeration to call this section even a crash-course on derived categories, but we do review some aspects of its construction that are particularly important in the context of stability conditions. For an expository overview we recommend [Căl05] instead, for details on derived categories in general [Wei94], and for a summary of the construction in case of $\mathrm{D^b}(X)$ the first three chapters of [Huy06].

We start with an abelian category $\mathcal{A}$, which could be

- the category $\mathrm{Coh}\, X$ of coherent sheaves on a projective algebraic variety X,
- the category $\mathrm{Rep}\, Q$ of finite-dimensional representations of a quiver Q.

Although this isn't actually true,[7] we will always pretend the category $\mathcal{A}$ has "enough injectives" (i.e., any objects A can be embedded $A \hookrightarrow I$ into an injective object I).

Its bounded derived category $\mathrm{D^b}(\mathcal{A})$ is constructed in three steps:

(1) We let $C^b(\mathcal{A})$ be the category of bounded complexes: Objects are complexes
$$E^\bullet = \dots \to E^i \to^{d^i} E^{i+1} \to^{d^{i+1}} E^{i+2} \to \dots$$
with $d^{i+1} \circ d^i = 0$ and $H^i(E) = 0$ for all but finitely many i, and morphisms $f^\bullet \colon E^\bullet \to F^\bullet$ are morphism $f_i \colon E^i \to F^i$ that are commute with the differential.

(2) The homotopy category $K^b(\mathcal{A})$ has the same objects $C^b(\mathcal{A})$, except that two morphisms $f^\bullet, g^\bullet \colon E^\bullet \to F^\bullet$ are considered identical if the difference $f - g$ is homotopic to zero: i.e. there exist maps $h^i \colon E^i \to F^{i-1}$ s.th. $f_i - g_i = d \circ h^i - h^{i+1} \circ d$.

(3) $\mathrm{D^b}(\mathcal{A})$ is the category obtained by inverting quasi-isomorphisms: a morphism $f^\bullet$ is considered a quasi-isomorphism if $f_* \colon H^i(E^\bullet) \to H^i(F^\bullet)$ is an isomorphism for all i. Then morphisms in $\mathrm{D^b}(\mathcal{A})$ are formal compositions $f^{-1} \circ g$ where f is a quasi-isomorphism.

However, when actually working with derived categories one almost never has to refer back to these definitions. Partly, this is due to the following fact:

[7] It is true for quasi-coherent sheaves on a projective variety, not for coherent sheaves. To remedy this problem, it is better to define $\mathrm{D^b}(X)$ by complexes of quasi-coherent sheaves for which every cohomology sheaf is coherent. As this category is equivalent to $\mathrm{D^b}(\mathrm{Coh}\, X)$, we can ignore this distinction.

Proposition 3.1.1. *(1) If $A^\bullet$ is any complex in $\mathrm{D}^\mathrm{b}(\mathcal{A})$ and $I^\bullet$ is a complex consisting of injectives, then*

$$\mathrm{Hom}_{\mathrm{D}^\mathrm{b}(\mathcal{A})}(A^\bullet, I^\bullet) = \mathrm{Hom}_{K^b(\mathcal{A})}(A^\bullet, I^\bullet)$$

(2) If $P^\bullet$ is a complex in $\mathrm{D}^\mathrm{b}(\mathcal{A})$ consisting of projective objects, and $B^\bullet$ is any complex, then

$$\mathrm{Hom}_{\mathrm{D}^\mathrm{b}(\mathcal{A})}(P^\bullet, B^\bullet) = \mathrm{Hom}_{K^b(\mathcal{A})}(P^\bullet, B^\bullet)$$

This fact is so useful because in the situation where $\mathcal{A}$ has enough injectives, any complex is quasi-isomorphic to a complex consisting of injectives. As an example, one can easily deduce the following fact:

Remark 3.1.2. Given an object $F \in \mathcal{A}$, we write $F[i]$ for the complex that is equal to F in degree $-i$, and 0 otherwise. Assuming that $\mathcal{A}$ has enough injectives, then

$$\mathrm{Hom}_{\mathrm{D}^\mathrm{b}(\mathcal{A})}(F[p], G[q]) = \begin{cases} 0 & \text{if } p > q \\ \mathrm{Ext}^{q-p}(F, G) & \text{if } p \leq q \end{cases}$$

Here is a first example how the derived category helps to organize homological algebra:

Remark 3.1.3. Let $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ be a short exact sequence of complexes in $\mathbb{C}^b(\mathcal{A})$. Then there exists a map $C^\bullet \to A^\bullet[1]$ in $\mathrm{D}^\mathrm{b}(\mathcal{A})$.

So the boundary maps familiar from derived functors between abelian category is induced by an actual morphism once we pass to the derived category. In the case where the complexes are all concentrated in degree zero, i.e. we have a short exact sequence $0 \to A[0] \to B[0] \to C[0] \to 0$ of objects in $\mathcal{A}$, the morphism is easy to understand: the complex $C[0]$ is quasi-isomorphic to the complex $A \hookrightarrow B$ (with A in degree -1 and B in degree 0), which has a natural map to $A[1]$.

In the general case, one has to show that $C^\bullet$ can be replaced with the *cone* of the morphism $A^\bullet \to B^\bullet$. We omit the definition of the mapping cone here; see e.g. [Wei94, section 1.8] instead. We will just note that it is an extension of $A^\bullet[1]$ by $B^\bullet$, and the extension is determined by the map $f \in \mathrm{Hom}(A^\bullet, B^\bullet) = \mathrm{Ext}^1(A^\bullet[1], B^\bullet)$. In particular, the cone decomposes as $\mathrm{cone}(f) = A^\bullet[1] \oplus B$ if and only if the map f is zero.

This notion of a triple of maps $A \to B \to C \to A[1]$ coming from short exact sequences is so useful that it is axiomatized in the notion of "exact triangles" in triangulated category: an exact triangle is a sequence of maps $A\colon B\colon C \to A[1]$ that is quasi-isomorphic to a triangle $A \to^g B \to \mathrm{cone}(g) \to A[1]$ coming from the cone construction.

In some sense exact triangles are the replacement of short exact sequences in abelian categories; in another sense this analogy is rather misleading as any notion of injectivity or surjectivity is lost in the derived category. Instead, it seems better to think of B as an extension of C by A. For example, the long exact Hom-sequence

$$\dots \to \mathrm{Hom}(F, A) \to \mathrm{Hom}(F, B) \to \mathrm{Hom}(F, C) \to \mathrm{Hom}(F, A[1]) \to \dots$$

tells us that if $\mathrm{Hom}(F, A) = \mathrm{Hom}(F, C) = 0$, then also $\mathrm{Hom}(F, B) = 0$, exactly what the intuition would tell us by analogy from extensions in abelian categories.

3.2. **Octahedral axiom.** The octahedral axiom is often glossed over in introductions to derived categories, which might give the impressions that it is too complicated to understand (given that it is already so difficult to draw on 2-dimensional paper), and not very useful. The latter definitely ceases to be true when dealing with stability conditions and t-structures, but fortunately the former is also a misconception.

The octahedral axiom answers a simple question: Given a composition

$$A \to^f B \to^g C,$$

is there any way to relate the three cones $\mathrm{cone}(f)$, $\mathrm{cone}(g)$ and $\mathrm{cone}(g \circ f)$? Phrased this way, the axiom is easy to guess: they form an exact triangle. More precisely, there is the following commutative diagram, where all the (almost) straight lines are part of an exact triangle:

$$\begin{tikzcd}
 & & D \arrow[dr] & & \\
 & B \arrow[ur] \arrow[dr, "g"] & & E \arrow[ddr] & \\
 & & C \arrow[ur] \arrow[drr] & & \\
A \arrow[uur, "f"] \arrow[urr, "g\circ f"] & & & & F
\end{tikzcd}$$

In the special case where $\mathcal{D} = \mathrm{D}^{\mathrm{b}}(\mathcal{A})$ is the derived category of an abelian category $\mathcal{A}$, and A, B, C are objects concentrated in degree zero (identified with objects in $\mathcal{A}$, and f, g are inclusions, then the octahedral axiom specializes to a very familiar statement:

$$(C/A)/(B/A) = C/B$$

More generally, in the same spirit in which exact triangles are a replacement of exact sequences, the use of the octahedral axiom replaces abelian category proofs based on diagram chasing.

3.3. **Filtration by cohomology.** It is very crucial that a complex $E^\bullet \in \mathrm{D}^{\mathrm{b}}(\mathcal{A})$ still has more information than its cohomology. The formal reason is that if $E^\bullet$ and $F^\bullet$ have isomorphic cohomology groups $H^i(E^\bullet) \cong H^i(F^\bullet)$, then there may not exist a morphism of complexes $f^\bullet \colon E^\bullet \to F^\bullet$ that induces these isomorphism. However, the following proposition gives a more satisfactory way to think about the additional information:

Proposition 3.3.1. *Given a complex $E \in \mathrm{D^b}(\mathcal{A})$, there exists a sequence of maps*

$$\begin{tikzcd}
0 \arrow[rr] & & E_k \arrow[rr] \arrow[dl] & & E_{k-1} \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{j+1} \arrow[rr] & & E_j = E \arrow[dl] \\
& H^{-k}(E)[k] \arrow[ul, dotted] & & H^{-k+1}(E)[k-1] \arrow[ul, dotted] & & & & H^{-j}(E)[j] \arrow[ul, dotted] &
\end{tikzcd}$$

where all triangles are exact.

We will refer to this as the filtration of E by its cohomology objects. In other words, E may be a non-trivial extension, rather than the direct sum of its cohomology objects $H^{-i}(E)[i]$.

This cohomology filtration is extremely useful and arguably not applied as often it should. For example, most proofs that rely on an elementary spectral sequence argument can be an argument based on the cohomology filtration.

3.4. **Bounded t-structures.** The notion of a t-structure can be motivated by the following question: Assuming we have an equivalence of derived categories $\mathrm{D^b}(\mathcal{A}) \cong \mathrm{D^b}(\mathcal{B})$, can we understand the image of $\mathcal{A} = \mathcal{A}[0]$ in $\mathrm{D^b}(\mathcal{B})$? In interesting examples, typically $\mathcal{A}$ does not get mapped to $\mathcal{B}$, so we would like to understand what structure the image of $\mathcal{A}$ in $\mathrm{D^b}(\mathcal{B})$ satisfies. It turns out that one can get a satisfactory and interesting concept by considering subcategories satisfying the Hom-vanishing of Remark 3.1.2 and the filtration of Proposition 3.3.1:

Definition 3.4.1. The heart of a bounded t-structure in a triangulated category $\mathcal{D}$ is a full additive subcategory $\mathcal{A}^\sharp \subset \mathcal{D}$ such that

(1) For $k_1 > k_2$, we have $\mathrm{Hom}(\mathcal{A}^\sharp[k_1], \mathcal{A}^\sharp[k_2]) = 0$.
(2) For every object E in $\mathcal{D}$ there are integers $k_1 > k_2 > \cdots > k_n$ and a sequence of exact triangles

$$\begin{tikzcd}
0 = E^0 \arrow[rr] & & E^1 \arrow[rr] \arrow[dl] & & E^2 \arrow[r] \arrow[dl] & \cdots & E^{n-1} \arrow[rr] & & E^n = E \arrow[dl] \\
& A_1 \arrow[ul, dotted] & & A_2 \arrow[ul, dotted] & & & & A_n \arrow[ul, dotted] &
\end{tikzcd}$$

with $A_i \in \mathcal{A}^\sharp[k_i]$.

The concept of t-structures was introduced in [BBD82], which is required reading for anyone interested in details about t-structures.

3.5. **Remarks.**

(1) A bounded t-structure is uniquely determined by its heart, which allows us to omit the definition bounded t-structure in this note.
(2) By Remark 3.1.2 and Proposition 3.3.1, the subcategory $\mathcal{A}[0] \subset \mathrm{D^b}(\mathcal{A})$ is the heart of a t-structure.
(3) The heart $\mathcal{A}^\sharp$ is automatically abelian: A morphism $A \to B$ between two objects in $\mathcal{A}^\sharp$ is defined to be an inclusion if its cone is also in $\mathcal{A}^\sharp$, and it is defined to be a surjection if the cone is in $\mathcal{A}^\sharp[1]$.
(4) The objects A_i are called the cohomology objects $H^i_\sharp(E)$ of E with respect to $\mathcal{A}^\sharp$. They are functorial and induce a long exact cohomology sequence for any exact triangle $A \to B \to C \to A[1]$ (see Exercise 10).

One can construct many non-trivial t-structures are given by *tilting at a torsion pair*.

Definition 3.5.1. A torsion pair in an abelian category $\mathcal{A}$ is a pair $(\mathcal{T}, \mathcal{F})$ of full additive subcategories with

(1) $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$.
(2) For all $E \in \mathcal{A}$ there exists a short exact sequence
$$0 \to T \to E \to F \to 0$$
with $T \in \mathcal{T}, F \in \mathcal{F}$.

Property (1) implies that that the filtration in (2) is automatically unique and functorial.

3.6. **Examples.**

(1) The canonical example of a torsion pair is $\mathcal{A} = \mathrm{Coh}\, X$, where we define $\mathcal{T}$ to be the torsion sheaves and $\mathcal{F}$ the torsion-free sheaves.
(2) Consider a finite quiver $Q = (Q_0, Q_1)$ with relations R, and assume that the vertex n is a sink, i.e. it has no outgoing arrows. Then let $\mathcal{T}$ be the subcategory of representations $\underline{V}$ concentrated at vertex n, i.e. with $V_i = \mathbb{C}^0$ for $i \neq n$, and $\mathcal{F}$ be the subcategory of representations $\underline{V}$ with $V_n = \mathbb{C}^0$. As n is assumed to be a sink, any representation $\underline{V}$ has a subrepresentation $(\mathbb{C}^0, \dots, \mathbb{C}^0, V_n)$, inducing the short exact sequence (2).
(3) There are two ways to generalize the previous construction to the case where n is allowed to have outgoing arrows; for simplicity we will assume that there are no loops going from the vertex n to itself:
   (a) Let $\mathcal{T}$ consist of representation generated by V_n, and $\mathcal{F}$ of representations with $V_n = \mathbb{C}^0$ as before.
   (b) Let $\mathcal{T}$ consist of representations concentrated at the vertex n, i.e. it consists of the direct sums $S_n^{\oplus k}$ where S_n is the simple one-dimensional representation concentrated at vertex n. Let $\mathcal{F}$ be the subcategory of representations with $\mathrm{Hom}(\underline{S}_n, \underline{V}) = 0$ for the equivalently, we can characterize $\mathcal{F}$ as the set of representations for which the intersection of the kernels of all maps $\phi_j \colon V_n \to V_i$ going out of V_n is trivial.
(4) Again consider a finite quiver $Q = (Q_0, Q_1)$, possibly with relations R, and again assume that the vertex n has no loops. This time we let $\mathcal{F}$ consist of $S_n^{\oplus k}$, and $\mathcal{T}$ consist of representations with $\mathrm{Hom}(\underline{V}, S_n) = 0$; more explicitly, $\underline{V}$ belongs to $\mathcal{F}$ if the images of the ingoing maps $\phi_j \colon V_i \to V_n$ span V_n.
(5) Let $\mathcal{A} = \mathrm{Coh}\, X$ be the category of coherent sheaves on a smooth projective curve X, and $\mu \in \mathbb{R}$ a real number. Let $\mathcal{A}_{\geq \mu}$ be the subcategory generated by torsion sheaves and vector bundles all of whose HN-filtration quotients have slope $\geq \mu$, and $\mathcal{A}_{<\mu}$ the category of vector bundles all of whose filtration quotients have slope $< \mu$. Then $(\mathcal{A}_{\geq \mu}, \mathcal{A}_{<\mu})$ is a torsion pair: property (1) follows from Lemma 2.1.5, and (2) is obtained by collapsing the HN-filtration into two parts: we let $T = E_i$ for i maximal such that $\mu_i \geq \mu$.

Proposition 3.6.1. *Given a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$, the following defines the heart of a bounded t-structure in $\mathrm{D^b}(\mathcal{A})$:*

$$\mathcal{A}^\sharp := \left\{ E \in \mathrm{D^b}(\mathcal{A}) \,\middle|\, H^0(E) \in \mathcal{T}, H^{-1}(E) \in \mathcal{F}, H^i(E) = 0 \quad \textit{for } i \neq 0, -1 \right\}$$

Objects in $\mathcal{A}$ can be thought of as an extension of F by T, with $T \in \mathcal{T}, F \in \mathcal{F}$, determined by an element in $\mathrm{Ext}^1(F, T)$. Objects in $\mathcal{A}^\sharp$ are instead an extension of some T by some $F[1]$, determined by an element in $\mathrm{Ext}^1(T, F[1]) = \mathrm{Ext}^2(T, F)$. More concretely, every object in $\mathcal{A}^\sharp$ can be represented by a two-term complex $E^{-1} \to^d E^0$ with $\ker d \in \mathcal{F}$ and $\operatorname{cok} d \in \mathcal{T}$. It is worth to keep the picture in fig. 4 in mind:

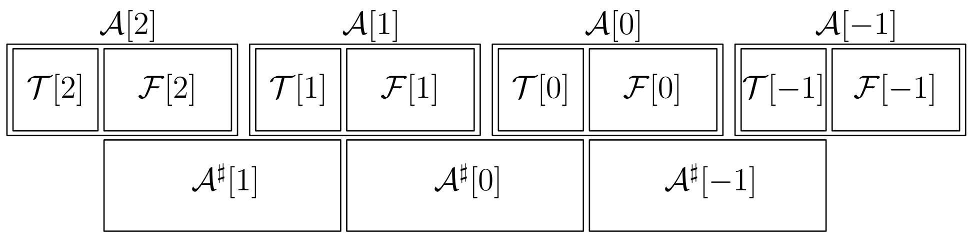


FIGURE 4. Schematic relation between $\mathcal{A}$ and its tilt $\mathcal{A}^\sharp$

In this picture, there are no morphisms going from the left to the right, and any object can be written as a successive extension of objects contained in one of the building blocks, starting with its right-most building block and extending it by objects further and further to the left.

The picture also suggests how to prove the proposition: the Hom-vanishing follows by extending known Hom-vanishings to extensions, and the filtration step $E_k^\sharp$ of E with respect to $\mathcal{A}^\sharp$ is given by an extension of the filtration step E^{k+1} with respect to $\mathcal{A}$ and the torsion part of $H^k(E)$.

In fact, the proposition holds in a more general situation, with the same proof: instead of working with $\mathrm{D^b}(\mathcal{A})$, we could have just assumed that $\mathcal{A}$ is the heart of a bounded t-structure inside a triangulated category $\mathcal{D}$, and used the same definition with H^i standing for the cohomology objects with respect to this t-structure. In particular, starting with the standard t-structure we can repeat this process multiple times. It seems that any bounded t-structure of interest in $\mathrm{D^b}(\mathcal{A})$ can be constructed by means of this process; in particular, in Bridgeland's space of stability conditions it will turn out that if we know the heart at one point in the space, then all other hearts appearing in the same component can be obtained by a sequence of tilts. The following Lemma, which we give without proof, gives an idea how such statements could be proved:

Lemma 3.6.2 ([Pol07, Lemma 1.1.2]). *If $\mathcal{A}, \mathcal{A}^\sharp \subset \mathcal{D}$ are the hearts of bounded t-structures in a triangulated category $\mathcal{D}$ such that $\mathcal{A}^\sharp$ is contained in the extension closure $\langle \mathcal{A}, \mathcal{A}[1] \rangle$ of $\mathcal{A}$ and its shift $\mathcal{A}[1]$, then $\mathcal{A}^\sharp$ is obtained from $\mathcal{A}$ by a tilt.*

The torsion pair is given by $\mathcal{T} = \mathcal{A} \cap \mathcal{A}^\sharp$ and $\mathcal{F} = \mathcal{A} \cap \mathcal{A}^\sharp[-1]$.

3.7. **Exercises.**

Exercise 6. Proof Remark 3.1.2.

Exercise 7. Translate the lemma of 9 to a triangulated category, and prove it!

Exercise 8. Pick a spectral sequence proof in a derived categories textbook and replace it by an argument using the filtration by cohomology.

Exercise 9. Use the filtration by cohomology to show that for a smooth projective curve X, every object in $\mathrm{D}^{\mathrm{b}}(X)$ is the direct sum of its cohomology sheaves. (The same statement holds for the category $\operatorname{Rep} Q$ of representations of a quiver Q without relations.)

Exercise 10.
(1) Prove that if $\mathcal{A}^\sharp \subset \mathcal{D}$ the heart of a bounded t-structure, $A \to B \to C$ is an exact triangle in $\mathcal{D}$ with $A, B \in \mathcal{A}$, then the cohomology objects $H^i_\sharp(C)$ C with respect to $\mathcal{A}^\sharp$ can be non-zero only for $i = -1, 0$.
(2) Show that $\mathcal{A}^\sharp$ is an abelian category if we define the kernel of $f\colon A \to B$ to be $H^{-1}_\sharp(\operatorname{cone} f)$ and the cokernel to be $H^0_\sharp(\operatorname{cone} f)$.
(3) Show that with this definition, any exact triangle $A \to B \to C$ induces a long exact cohomology sequence among the cohomology objects $H^i_\sharp(\_\_)$.

Exercise 11. Consider one of the examples (3a), (3b) or (4) of a torsion pair in the category $\operatorname{Rep} Q$ of representations of a quiver Q. Prove that it is indeed a torsion pair, and verify the explicit descriptions of $\mathcal{F}, \mathcal{T}$.

Exercise 12. (*) Let $X = \mathbb{P}^1, \mathcal{A} = \operatorname{Coh} X$, and let $\mathcal{A}^\sharp$ be the tilted heart for the torsion pair $(\mathcal{A}_{\geq 0}, \mathcal{A}_{<0})$. Let Q be the Kronecker quiver (the directed quiver with tow vertices and two arrows), and let $\Phi_T\colon \mathrm{D}^{\mathrm{b}}(\mathbb{P}^1) \to \mathrm{D}^{\mathrm{b}}(\operatorname{rep}_{\mathbb{C}}(Q))$ be the equivalence induced by the tilting bundle $T = \mathcal{O} \oplus \mathcal{O}(1)$. Show that $\mathcal{A}^\sharp$ is the inverse image of the heart of the standard t-structure.

Exercise 13. (*) Consider an elliptic curve E, and its auto-equivalence $\Phi\colon \mathrm{D}^{\mathrm{b}}(E) \to \mathrm{D}^{\mathrm{b}}(E)$ given by the Fourier-Mukai transform of the Poincaré line bundle. Determine the image $\Phi(\operatorname{Coh} E)$ of the heart of the standard t-structure.

4. Stability conditions on a triangulated category

Given a stability function $\mathcal{A}, Z$ on an abelian category for which Harder-Narasimhan filtrations exist, we can define a stability condition on its derived category $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$ by defining an object to be stable if and only if it is the shift $E[n]$ of a Z-stable object E, and by defining its phase to be $\phi(E[n]) = \phi(E) + n$; it will satisfy properties very similar to the existence of HN-filtrations. In the following section, we will make this notion precise.

4.1. **Definition of Stability conditions.** Before putting things together again, it is worth separating out the properties of phases of stable objects along with the existence of Harder-Narasimhan filtrations. The idea is that they refine the filtrations of an object given by a bounded t-structure:

Definition 4.1.1. A slicing $\mathcal{P}$ of a triangulated category $\mathcal{D}$ is a collection of full additive subcategories $\mathcal{P}(\phi)$ for each $\phi \in \mathbb{R}$ satisfying

(1) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$
(2) For all $\phi_1 > \phi_2$ we have $\mathrm{Hom}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) = 0$.
(3) For each $0 \neq E \in \mathcal{D}$ there is a sequence $\phi_1 > \phi_2 > \cdots > \phi_n$ of real numbers and a sequence of exact triangles

$$\begin{tikzcd}[column sep=small] 0 = E^0 \arrow[rr] & & E^1 \arrow[dl] \arrow[rr] & & E^2 \arrow[dl] \arrow[r] & \cdots & E^{n-1} \arrow[rr] & & E^n = E \arrow[dl] \\ & A_1 \arrow[ul, dotted] & & A_2 \arrow[ul, dotted] & & & & A_n \arrow[ul, dotted] & \end{tikzcd} \tag{1}$$

with $A_i \in \mathcal{P}(\phi_i)$ (which we call the Harder-Narasimhan filtration of E).

Remark 4.1.2. (1) We call the objects in $\mathcal{P}(\phi)$ semistable of phase ϕ.
(2) Given the slicing $\mathcal{P}$, the sequence of ϕ_i and the Harder-Narasimhan filtration are automatically unique. We set $\phi^+_{\mathcal{P}}(E) = \phi_1$ and $\phi^-_{\mathcal{P}}(E) = \phi_n$ (where we sometimes omit the subscript $\mathcal{P}$).
(3) If $\phi^-(A) > \phi^+(B)$, the $\mathrm{Hom}(A, B) = 0$.
(4) If $\mathcal{P}(\phi) \neq 0$ only for $\phi \in \mathbb{Z}$, then the slicing is equivalent to the datum of a bounded t-structure, with heart $\mathcal{A} = \mathcal{P}(0)$.
(5) More generally, given a slicing $\mathcal{P}$, let $\mathcal{A} = \mathcal{P}((0, 1])$ be the full extension-closed subcategory generated by all $\mathcal{P}(\phi)$ for $\phi \in (0, 1]$; equivalently, $\mathcal{A}$ is the subcategory of objects E with $\phi^+_{\mathcal{P}}(E) \leq 1$ and $\phi^-_{\mathcal{P}}(E) > 0$. Then $\mathcal{A}$ is the heart of a bounded t-structure. In other words, a slicing is always a refinement of a bounded t-structure.

While this gives a notion of semistable objects and successfully generalizes Harder-Narasimhan filtrations, it is rather unsatisfactory that we have to specify the semistable objects explicitly (instead of defining them implicitly by a slope function as in the case of vector bundles). The remedy for this lies in the following definition that brings back the stability functions Z used in section 2.

Definition 4.1.3. A stability condition on a triangulated category $\mathcal{D}$ is a pair $(Z, \mathcal{P})$ where $Z\colon K(\mathcal{D}) \to \mathbb{C}$ is a group homomorphism (called central charge) and $\mathcal{P}$ is a slicing, so that for every $0 \neq E \in \mathcal{P}(\phi)$ we have

$$Z(E) = m(E) \cdot e^{i\pi\phi}$$

for some $m(E) \in \mathbb{R}_{>0}$.

Indeed, the following proposition shows that to some extent (once we identify a t-structure), stability it intrinsically defined. It also describes how stability conditions are actually constructed:

Proposition 4.1.4 ([Bri07, Proposition 5.3])**.** *To give a stability condition $(Z, \mathcal{P})$ on $\mathcal{D}$ is equivalent to giving a heart $\mathcal{A}$ of a bounded t-structure with a stability function $Z_{\mathcal{A}}\colon K(\mathcal{A}) \to \mathbb{C}$ (see Definition 2.2.1) such that $(\mathcal{A}, Z_{\mathcal{A}})$ have the "Harder-Narasimhan property", i.e. any object in $\mathcal{A}$ has a HN-filtration by $Z_{\mathcal{A}}$-stable objects.*

We will focus on how to obtain a stability condition from the datum $(\mathcal{A}, Z_{\mathcal{A}})$, as this is how stability conditions are actually constructed:

Proof. If $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$, then[8] we have $K(\mathcal{D}) = K(\mathcal{A})$, so clearly how to go from Z and $Z_{\mathcal{A}}$ determine each other.

Given $(\mathcal{A}, Z_{\mathcal{A}})$, we define $\mathcal{P}(\phi)$ for $\phi \in (0,1]$ to be the $Z_{\mathcal{A}}$-semistable objects in $\mathcal{A}$ of phase $\phi(E) = \phi$. This is extended to all real numbers by $\mathcal{P}(\phi + n) = \mathcal{P}(\phi)[n] \subset \mathcal{A}[n]$ for $\phi \in (0,1]$ and $0 \neq n \in \mathbb{Z}$ (as forced by condition 1 of Definition 4.1.1). The compatibility condition (4.1.3) is satisfied by construction, so we just need show that $\mathcal{P}$ satisfies the remaining properties in Definition 4.1.1. The Hom-vanishing in condition no. 2 follows from Definition 3.4.1, part (1) in case $\lfloor \phi_1 \rfloor > \lfloor \phi_2 \rfloor$, and from Lemma 2.1.5 in case $\lfloor \phi_1 \rfloor = \lfloor \phi_2 \rfloor$. Finally, given $E \in \mathcal{D}$, its filtration by cohomology objects $A_i \in \mathcal{A}[k_i]$ according to 3.4.1, and the HN-filtrations $0 \hookrightarrow A_{i1} \hookrightarrow A_{i2} \hookrightarrow \ldots \hookrightarrow A_{im_i} = A_i$ given by the HN-property inside $\mathcal{A}$ can be combined into a HN-filtration of E: it begins with as

$$0 \to F_1 = A_{11}[k_1] \to F_2 = A_{12}[k_1] \to \cdots \to F_{m_1} = A_1[k_1] = E_1,$$

i.e. with the HN-filtration of A_1. Then the following filtration steps F_{m_1+i} are an extensions of $A_{2i}[k_2]$ by E_1 that can be constructed as the cone of the composition $A_{2i}[k_2] \to A_2[k_2] \to^{[1]} E_1$—the octahedral axiom shows that these have the same filtration quotients as $0 \to A_{21}[k_2] \to A_{22}[k_2] \ldots$; continuing this we obtain a filtration of E as desired.

Conversely, given the stability condition, we set $\mathcal{A} = \mathcal{P}((0,1])$ as before; by the compatibility condition in equation (4.1.3), the central charge $Z(E)$ of any $\mathcal{P}$-semistable object E lies in $\mathbb{H}$; since any object in $\mathcal{A}$ is an extension of semistable ones, this follows for all objects in $\mathcal{A}$ by the additivity. Finally, it is fairly straightforward to show that Z-semistable objects in $\mathcal{A}$ are exactly the semistable objects with respect to $\mathcal{P}$. □

4.2. **Examples.** If X is a smooth projective curve and $\mathcal{D} = \mathrm{D^b}(X)$, let $\mathcal{A} = \operatorname{Coh} X$ be the heart of the standard t-structure, and $Z(E) = -\deg(E) + i \cdot \operatorname{rk}(E)$. As we have seen in section 2.1, Z is a stability function with the Harder-Narasimhan property, and thus induces a stability condition on $\mathrm{D^b}(X)$. Its semistable objects are the shifts of slope-semistable vector bundles, and the shifts of 0-dimensional torsion sheaves.

Similarly, the stability conditions on a category of Quiver representations considered in section 2.2 leads to stability conditions on its derived category.

Remark 4.2.1. It is often highly non-trivial to satisfy the condition that Z sends objects of $\mathcal{A}$ to the semi-closed upper half plane $\mathbb{H}$. Already for a projective surface S and its category of coherent sheaves $\operatorname{Coh} S$, it is impossible to satisfy, and so far no one has succeeded in constructing a heart with a stability function in the derived category of a projective Calabi-Yau threefold.

[8] even though $\mathcal{D}$ might not be equivalent to $\mathrm{D^b}(\mathcal{A})$

5. SPACE OF STABILITY CONDITIONS

5.1. **The deformation result.** So far, extending a stability condition from an abelian category to its derived category hasn't really gained us anything. However, as will see we will have a lot more freedom to deform stability conditions in the derived category, leading to an interesting space of such stability conditions.

We will now restrict our attention to stability conditions $\sigma = (Z, \mathcal{P})$ satisfying two additional assumptions (in the formulation introduced in [KS08, Section 3.4]):

(1) We fix a finite-dimensional lattice Λ with a map $\lambda\colon K(\mathcal{D}) \to \Lambda$, and restrict our attention to stability conditions for which Z factors via Λ. Obviously, this is no restriction in case $K(\mathcal{D})$ is finite-dimension; when it is infinite-dimensional, a typical choice for Λ might be the numerical Grothendieck group $K_{\mathrm{num}}(\mathcal{D})$.[9]
(2) Let $\|\cdot\|$ be an arbitrarily fixed norm on $\Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}$. We assume that σ satisfies the "support property":[10]
$$\inf \left\{ \frac{|Z(E)|}{\|[E]\|} \,\middle|\, E \text{ is } \sigma\text{-semistable} \right\} > 0$$

Note that we committed abuse of notation by writing $[E]$ for $\lambda([E]) \in \Lambda$. Similarly, we denote by $\mathrm{Stab}(\mathcal{D})$ for the set of stability conditions satisfying these two additional properties (omitting Λ from the notation).

We can define a generalized metric[11] on the set of slicings by
$$d_S(\mathcal{P}, \mathcal{Q}) \quad = \sup\nolimits_{0 \neq E \in \mathcal{D}} \left\{ |\phi^-_{\sigma^2}(E) - \phi^-_{\sigma^1}(E)|, |\phi^+_{\sigma^2}(E) - \phi^+_{\sigma^1}(E)| \right\} \in [0, +\infty]$$
Combining this with the metric on $\Lambda^\vee := \mathrm{Hom}(\Lambda, \mathbb{C})$ induced by $\|\cdot\|$ we obtain a generalized metric on $\mathrm{Stab}(\mathcal{D})$ by
$$d(\sigma, \tau) = \sup\{d_S(\mathcal{P}, \mathcal{Q}), \|Z - W\|\}.$$
In particular, we have defined a topology on $\mathrm{Stab}(\mathcal{D})$. The main theorem of Bridgeland, motivated by Douglas' work on π-stability, is the following deformation result:

Theorem 5.1.1. [Bri07, Theorem 7.1] *The space* $\mathrm{Stab}(\mathcal{D})$ *of stability conditions is a smooth finite-dimensional complex manifold such that the map*
$$\mathcal{Z}\colon \mathrm{Stab}(\mathcal{D}) \to \mathrm{Hom}(\Lambda, \mathbb{C}) \quad \sigma = (Z, \mathcal{P}) \to Z$$
is a local homeomorphism at every point of $\mathrm{Stab}(\mathcal{D})$.

In other words, we can deform a stability condition $(Z, \mathcal{P})$ (uniquely) by deforming Z.[12]

---

[9] The numerical Grothendieck group is the quotient of $K(\mathcal{D})$ by the null-space of the Euler form $\chi(E, F) = \sum_i (-1)^i \mathrm{Hom}(E, F[i])$.

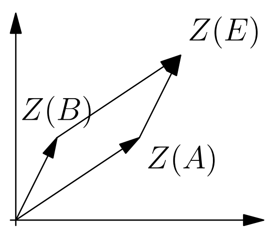
[10] Stability conditions satisfying this condition were called "full" in [Bri08].

[11] It satisfies the triangle inequality and is non-degenerate; however, its value might be infinite.

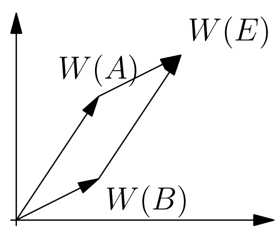
[12] A posteriori, this result also tells us how to characterize the complex manifold structure on $\mathrm{Stab}(\mathcal{D})$: it is the unique complex structure such that for every $E \in \mathcal{D}$ the function

We have already seen some of these deformations in the stability conditions for a quiver Q in section 2.2: our stability conditions depended on the choice of some $z_i \in \mathbb{H}$ for every vertex of Q. Bridgeland's theorem then says that we can deform these stability conditions further when z_i leave the upper half-plane, if we are willing to pay the price of working with the derived category $\mathrm{D}^{\mathrm{b}}(\operatorname{Rep} Q)$, and changing our abelian category $\operatorname{Rep} Q \subset \mathrm{D}^{\mathrm{b}}(\operatorname{Rep} Q)$ into the heart of a non-standard t-structure.

5.2. **Example 1: Wall-crossing within an abelian category.** Assume for a moment that we deform Z to W in such a way that we don't have to change the heart $\mathcal{A}$ of our stability condition, i.e. W satisfies condition ??? in ???. Assume now that there is a short exact sequence $0 \to A \to E \to B \to 0$ in $\mathcal{A}$, such that the classes $[A], [B]$ have no non-trivial subobjects (in particular, A is the only non-trivial subobject for Z, W). We say that there is a "wall" between Z, W if the orientation of the parallelogram of the central charges of A, E, B is different between Z, W (the wall consisting of central charges for which the parallelogram is degenerate):



(A) E is stable



(B) E is unstable

FIGURE 5. A simple wall-crossing

Then either E is stable (as in the case of Z in the figure), or E is unstable and $0 \hookrightarrow A \hookrightarrow E$ is the HN-filtration of E, with filtration quotients A, B. This is of course the most trivial case of a wall-crossing, but it is still the basic reason why we can at all hope to get a new stability condition after deforming Z to W.

5.3. **Example 2: Stability and tilting for quivers.** Now consider the case where $\mathcal{A} = \operatorname{Rep} Q$ is the category of representations of a quiver Q (possibly with relations), and stability conditions on $\mathrm{D}^{\mathrm{b}}(\operatorname{Rep} Q)$ constructed from the stability conditions on $\operatorname{Rep} Q$ considered in section 2.2. We want to find out what happens to $\mathcal{A} = \mathcal{P}((0,1])$ when one of the complex numbers, say z_n, leaves the semi-closed upper half plane $\mathbb{H}$. For simplicity we assume that all other $z_j, j \neq n$ are contained in the open half plane $\mathcal{H}$. Let S_n be the simple one-dimensional representation supported at the vertex n.

(1) If z_n crosses the negative real line, from $-x + i\epsilon$ to $-x - i\epsilon$, then $\mathcal{A} = \mathcal{P}((0,1])$ gets replaced by $\mathcal{Q}((0,1]) = \mathcal{A}^\sharp[-1]$, where $\mathcal{A}^\sharp$ is the tilt at the torsion pair $TT = \{S_n^{\oplus k}\}$, $\mathcal{F} = \{\underline{V} \mid \operatorname{Hom}(S_n, \underline{V}) = 0\}$ considered as

$\mathcal{Z}_E \colon \operatorname{Stab}(\mathcal{D}) \to \mathbb{C}, \mathcal{Z}_E(Z, \mathcal{P}) = Z(E)$ is holomorphic - but of course without the deformation result we would have no reason to assume that such a complex structure exists, or that it is unique.

example (3b) of section 3.6. It is easy to see that for ϵ sufficiently small, the new central charge W satisfies the positivity property $W(E) \in \mathbb{H}$ for $E \in \mathcal{Q}((0,1])$. It is also easy to see that this tilt is the only reasonable choice for $\mathcal{Q}((0,1])$: the object S_n is Z-stable, but its phase increases beyond 1; thus we should instead include $S_n[-1]$. Similarly, any other Z-stable object of $\mathcal{A}$ will remain stable, and thus should also be included in $\mathcal{Q}((0,1])$; the extension closure of these objects is given by $\mathcal{F}$.

(2) IF z_n crosses the positive real line, we instead let the new heart $\mathcal{Q}((0,1])$ be the tilt $\mathcal{A}^\sharp$ at the torsion pair determined by $\mathcal{F} = \{S_n^{\oplus k}\}$; see Example (4). This time, we are replacing S_n by $S_n[1]$, while keeping all other stable objects of $\mathcal{A}$.

Typically, the tilted heart $\mathcal{A}^\sharp$ is again the category of representations of a quiver; thus the space of stability condition on $\mathbb{D}^b(\operatorname{Rep} Q)$ knows about the rich theory of tilting of quivers.

5.4. **Example 3: Stability conditions for $\mathbb{P}^1$.** We can easily change our stability conditions of section 2.1 on the category of coherent sheaves $\operatorname{Coh} X$ on a curve X to depend on a parameter: We set

$$Z(\mathcal{F}) = -\deg \mathcal{F} + z \cdot \operatorname{rk} \mathcal{F}$$

Note that this just a reparametrization of the central charges and of the phases of stable objects—we do not change the ordering of sheaves by their phases, nor do we change which objects are stable. We will now study what happens when z crosses the real line in the case of $X = \mathbb{P}^1$.

First, we picture the case where z is close to the real line: by Grothendieck's theorem, every sheaf on $\mathbb{P}^1$ is the direct sum of line bundles and skyscraper sheaves of points; thus the only stable objects are line bundles and the skyscraper sheaves $\mathcal{O}_x$ (as well as their shifts); their central charges are picture in fig. 6 (where the doted arrows denote morphisms between the stable objects).

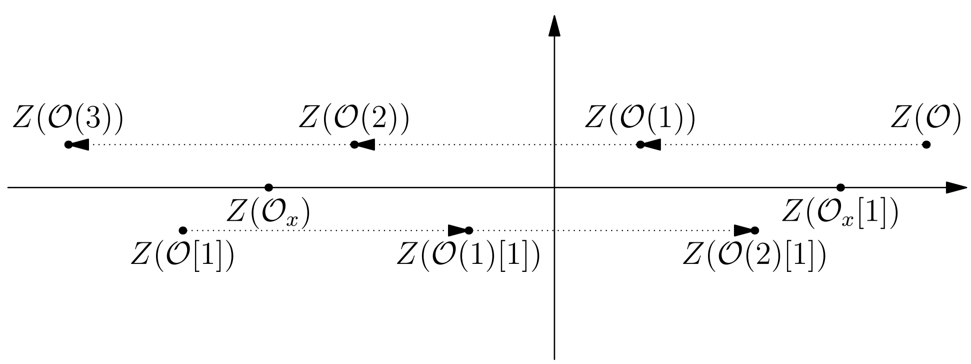


FIGURE 6. Stability condition for $\mathbb{P}^1$ with $z \approx 1.3 + i\epsilon$

Now consider what happens when z moves on a straight line from $x + i\epsilon$ to $x - i\epsilon$. The first thing to note is that the central charge of a stable object can never become zero; hence z can only reach the real line when $x \in \mathbb{R} \setminus \mathbb{Z}$, as otherwise there will be a line bundle with $Z(\mathcal{O}(n)) = 0$. To be specific, let us assume that $1 < x < 2$. Now, instead of trying to understand what will happen to the heart $\mathcal{P}((0, 1])$, it is easier to analyze everything inside the heart $\mathcal{A}^\sharp = \mathcal{P}((\frac{1}{2}, \frac{3}{2}])$, as this heart will remain constant along the path we are considering; it is the extension closure of $\{\mathcal{O}(2), \mathcal{O}(3), \dots\} \cup \{\mathcal{O}(1)[1], \mathcal{O}[1], \mathcal{O}(-1)[1], \dots\}$.

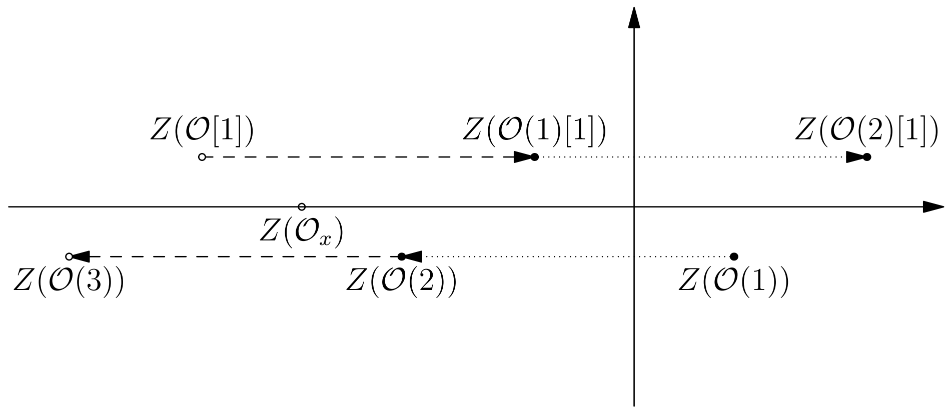


FIGURE 7. Stability condition for $\mathbb{P}^1$ with $z \approx 1.3 - i\epsilon$

This category contains short exact sequences $\mathcal{O}(n) \hookrightarrow \mathcal{O}(n+1) \twoheadrightarrow \mathcal{O}_x$ for $n \geq 2$, and $\mathcal{O}(3)$, $\mathcal{O}(4)$ etc. become unstable for $z = x - i\epsilon$; see fig. 7, where we have drawn destabilizing morphisms by dashed arrows. Similarly, the short exact sequences $\mathcal{O}_x \hookrightarrow \mathcal{O}(n)[1] \twoheadrightarrow \mathcal{O}(n+1)[1]$ for $n \leq 0$ destabilize $\mathcal{O}[1], \mathcal{O}(-1)[1], \dots$. Also, $\mathcal{O}(2) \hookrightarrow \mathcal{O}_x \twoheadrightarrow \mathcal{O}(1)[1]$ destabilizes $\mathcal{O}_x$. Thus the only remaining stable objects are $\mathcal{O}(2), \mathcal{O}(1)[1]$. One can also show that these are the only two stable objects in $\mathcal{P}((\frac{1}{2}, \frac{3}{2}])$. In fact, the heart $\mathcal{A}^\sharp$ is isomorphic to the category of representations $\operatorname{Rep} P_2$ of the Kronecker quiver (Example 2.2.2) with two arrows, with $\mathcal{O}(2), \mathcal{O}(1)[1]$ corresponding to the simple objects. One might say that we moved from a geometric chamber (with heart $\operatorname{Coh} \mathbb{P}^1$, and with the skyscraper sheaves $\mathcal{O}_x$ being stable) to an chamber of algebraic stability conditions.

Now we can also go back and describe the new heart $\mathcal{A}^{\sharp\sharp} = \mathcal{Q}((0, 1])$: the only stable objects with phase between 0 and 1 are $\mathcal{O}(1)[1]$ and $\mathcal{O}(2)[-1]$. By definition, $\mathcal{A}^{\sharp\sharp}$ is the extension closure of these two objects; however, as there are no non-trivial extensions between these two objects (nor self-extensions of one of the objects), the category $\mathcal{A}^{\sharp\sharp}$ is equivalent to category of pairs of vector spaces V_0, V_1. This is the easiest example I am aware of where the derived category $\mathrm{D}^{\mathrm{b}}(\mathcal{A}^{\sharp\sharp})$ of the heart is not isomorphic to the original derived category.

For a complete study of the space of stability conditions on $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^1)$, see [Oka06].

5.5. **Proof of the deformation result.** This section contains a sketch of a proof of Theorem 5.1.1. We first prove that the map $\mathcal{Z}$ is locally injective:[13]

[13]Note that $\frac{1}{4}$ in the Lemma could be replaced by 1, see [Bri07, Lemma 6.4].

Lemma 5.5.1. *If two stability conditions σ, τ satisfy $d_S(\mathcal{P}, \mathcal{Q}) < \frac{1}{4}$ and $Z = W$ then $\sigma = \tau$.*

Proof. Consider a σ-semistable object $E \in \mathcal{P}(\phi)$. It follows directly from the definition of d_S that we have chain of subcategories

$$\mathcal{P}(\phi) \subset \mathcal{Q}((\phi - \frac{1}{4}, \phi + \frac{1}{4})) \subset \mathcal{P}((\phi - \frac{1}{2}, \phi + \frac{1}{2}]) =: \mathcal{A}$$

of the abelian category $\mathcal{A}$. Any semistable HN-factor of E with respect to τ will have central charge lying in the sector of angle $\frac{\pi}{2}$ centered on the ray $\mathbb{R}_{>0} \cdot E^{i\pi\phi}$. Thus, if E is not τ-semistable, and if $A \to E$ is first object of its HN-filtration with respect to τ, then

(1) the morphism $A \to E$ is an inclusion in $\mathcal{A}$, and
(2) the phase of $Z(A)$ is bigger than the phase of E.

(Both statement follow from the observation that the cone of $A \to E$ has as HN-filtration factors exactly the remaining HN-factors of E.) This contradicts Proposition 4.1.4, as E is not Z-stable in $\mathcal{A}$. □

So it remains to prove the local surjectivity. To prove surjectivity onto an open ball in $\Lambda^\vee$ we may have to make this ball arbitrarily small; its size will be determined by the following quantity (which is positive by our assumption that σ satisfies the support property):

$$S(\sigma) := \inf \left\{ \frac{|Z(E)|}{\|[E]\|} \,\middle|\, E \text{ is } \sigma\text{-semistable} \right\} > 0$$

The quantity $S(\sigma)$ controls how fast the central charges $Z(E)$ of a stable object can vary, relative to the absolute value of the original central charge:

Remark 5.5.2. If E is σ-stable and W is a central charge close by Z satisfying $\|W - Z\| < S(\sigma) \cdot \epsilon$, then $W(E)$ is contained in a ball $B_{\epsilon \cdot |Z(E)|}(Z(E))$ of radius $\epsilon \cdot |Z(E)|$ around $Z(E)$.

We will need one more definition:

Definition 5.5.3. Given a stability condition $\sigma = (Z, \mathcal{P})$ and an object E, the mass $m_\sigma(E)$ is defined by

$$m_\sigma(E) = \sum_i |Z(A_i)|,$$

where A_i are the HN-factors of E.

Note that in the picture used in the proof of Theorem 2.1.6, the mass of $\mathcal{F}$ equals the length of the path $0 \to v_1 \to v_2 \to \cdots \to v_n$ bounding $\mathcal{H}_Z(\mathcal{F})$ from the left.

Lemma 5.5.4. *The function $S(\sigma)\colon \operatorname{Stab}(\mathcal{D}) \to \mathbb{R}_{>0}$ is continuous.*

Proof. By the symmetry of the situation, it is enough to prove that given δ, then for any $\tau = (W, \mathcal{Q})$ sufficiently close by $\sigma = (Z, \mathcal{P})$, we have $S(\tau) > S(\sigma) - \delta$.

So let us assume that $d_S(\mathcal{P}, \mathcal{Q}) < \epsilon$ and $\|W - Z\| < \epsilon$. Assume that E is τ-stable of phase $\psi(E)$. By the definition of the norm $\|\cdot\|$ applied to $W - Z$, we have

$$\frac{|W(E)|}{\|E\|} \geq \frac{|Z(E)|}{\|E\|} - \frac{|W(E) - Z(E)|}{\|E\|} \geq \frac{|Z(E)|}{\|E\|} - \epsilon$$

So it suffices to prove that $\frac{|Z(E)|}{\|E\|}$ can be bound from below arbitrarily close to $S(\sigma)$. Let $A_1, \ldots, A_n$ be the HN-factors of E respect to σ. By definition of d_S we have $\psi + \epsilon > \phi(A_i) > \psi - \epsilon$ for all i; hence the complex numbers $Z(A_i)$ and $Z(E)$ all lie in a sector of the complex plane of angle $2\pi\epsilon$. Hence

$$\begin{aligned} \frac{|Z(E)|}{\|E\|} \quad &> \quad \frac{\sum_i |Z(A_i)| \cdot \cos(2\pi\epsilon)}{\|E\|} > \cos(2\pi\epsilon) \frac{\sum_i |Z(A_i)|}{\sum_i \|A_i\|} \\ &> \quad \cos(2\pi\epsilon) \frac{\sum_i S(\sigma) \|A_i\|}{\sum_i \|A_i\|} = \cos(2\pi\epsilon) \cdot S(\sigma) \end{aligned}$$

□

It follows that it suffices to prove the following:

Lemma 5.5.5. *There exists* $\epsilon > 0$ *such that for any stability condition* $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(\mathcal{D})$ *and any group homomorphism* $W \colon \Lambda \to \mathbb{C}$ *with* $\|W - Z\| < \epsilon \cdot S(\sigma)$ *and either*

(1) $\Im W = \Im Z$ *or*
(2) $\Re W = \Re Z$*,*

then there exists a stability condition $\tau = (W, \mathcal{Q})$ *with* $d_S(\mathcal{P}, \mathcal{Q}) < \epsilon$*.*

In fact, the proof below works for any $\epsilon < 1$, but it is psychologically easier to imagine ϵ being small.

The continuity of $S(\sigma)$ is crucial for this reduction: we obtain a ball of stability conditions $\tau = (W, \mathcal{Q})$ nearby σ mapping isomorphically to an $\epsilon \cdot S(\sigma)$-sized neighborhood of $Z + \mathrm{Hom}(\Lambda, \mathbb{R})$. Then, using the second case of the Lemma, the unions of the neighborhoods of τ inside $W + i \,\mathrm{Hom}(\Lambda, \mathbb{R})$ will form an open neighborhood of the original σ.

Proof. We consider the first case (the proof of the second case works in exactly the same way by replacing $\mathcal{P}((0, 1])$ with $\mathcal{P}((\frac{1}{2}, \frac{3}{2}])$ and swapping real and imaginary parts of complex numbers in every argument below.) The convenience of this assumption $\Im W = \Im Z$ lies in the fact that W is still a stability function for $\mathcal{A}^\sharp = \mathcal{P}((0, 1])$: this is evident for objects with $\Im Z(E) > 0$, and follows from $\|W - Z\| < S(\sigma)$ for objects in $\mathcal{P}(1)$ (which are automatically semistable). So it remains to prove that under the assumptions, the pair $(\mathcal{A}^\sharp, W)$ has the HN-property:

We proceed as in the proof of 2.1.6: let $\mathcal{H}_W(E)$ be the convex hull of the set $W(A)$ for all subobjects $A \hookrightarrow E$. We have to prove that $\mathcal{H}_W(E)$ is bounded to the left, and that it has a finite number of extremal points to the left of the line segment connection 0 and $W(E)$.

Given $A \hookrightarrow E$, consider the set $\mathcal{H}_Z(E)$, its subset $\mathcal{H}_Z(A)$, and the piecewise-linear path $0 \to v_1 \to v_2 \to \cdots \to v_n = Z(A)$ bounding $\mathcal{H}_Z(A)$ which corresponds to the the HN-filtration of A.

Its length $m_\sigma(A)$ is bounded by the length of the path that follows the boundary of $\mathcal{H}_Z(E)$ up to the height $\Im(A)$ and then moves horizontally to $Z(A)$; see fig. 8.

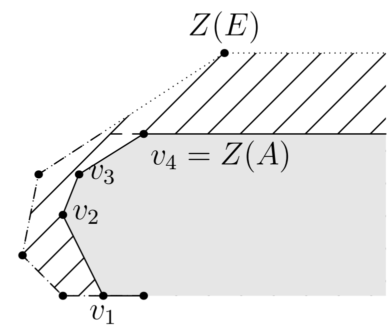


FIGURE 8. The convex sets $\mathcal{H}_Z(A)$ and $\mathcal{H}_Z(E)$

It follows that there is a constant $C(E)$ such that any subobject $A \hookrightarrow E$ satisfies

$$\Re Z(A) \geq C(E) + m_\sigma(A)$$

Since the real parts of $W(A_i)$ and $Z(A_i)$ for any HN-filtration quotient A_i of A can differ by at most $\epsilon \cdot |Z(A_i)|$ (see Remark 5.5.2), we obtain

$$\Re W(A) \geq \Re Z(A) - \epsilon m_\sigma(A) \geq C(E) + (1-\epsilon) m_\sigma(A) > C(E)$$

Also note that if $A \hookrightarrow E$ is an extremal point of the set $\mathcal{H}_W(A)$, then in particular $\max(0, \Re W(E)) > \Re W(A)$, which means the mass $m_\sigma(A)$ is bounded from above by a constant $D(E) = \frac{\max(0, \Re W(E)) - C(E)}{(1-\epsilon)}$, a constant depending only on E. In particular, the central charge of every Harder-Narasimhan filtration factor A_i is bounded from above by $|Z(A_i)| < D(E)$; by the support property, the norm $\|A_i\| < \frac{D(E)}{S(\sigma)}$ is also bounded, and hence there are only finitely many classes $[A_i] \in \Lambda$ that can appear as the class of a HN-factor of such an object. It follows that there are only finitely many extremal points of the set $\mathcal{H}_W(\mathcal{A})$, and we can conclude as in the proof of Theorem 2.1.6.

Finally, it is easy enough to verify the support property for the stability condition $(W, \mathcal{Q})$ that we obtained; it also follows from Lemma 5.5.4 for small enough ϵ. $\square$

5.6. **Exercises.**

Exercise 14. Prove claim (5) in Remark 4.1.2.

Exercise 15. Giving a slicing $\mathcal{P}$ with heart $\mathcal{A} = \mathcal{P}((0,1])$, and $\phi \in (0,1)$, prove that $\mathcal{T} = \mathcal{P}((\phi, 1]), \mathcal{F} = \mathcal{P}((0,\phi])$ define a torsion pair in $\mathcal{A}$. Show that the heart $\mathcal{A}^\sharp$ obtained by tilting at $\mathcal{T}, \mathcal{F}$ is equal to $\mathcal{P}((\phi, \phi+1])$.

Exercise 16. Find the Harder-Narasimhan filtrations of all $\mathcal{O}_{\mathbb{P}^1}(n)$ in the algebraic stability condition considered in section 5.4.

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