

THE SPACE OF STABILITY CONDITIONS ON THE LOCAL PROJECTIVE PLANE

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ABSTRACT. We study the space of stability conditions on the total space of the canonical bundle over the projective plane. We explicitly describe a chamber of geometric stability conditions, and show that its translates via autoequivalences cover a whole connected component. We prove that this connected component is simply-connected. We determine the group of autoequivalences preserving this connected component.

Finally, we show that there is a submanifold isomorphic to the universal covering of a moduli space of elliptic curves with level structure, with the morphism given by solutions of Picard-Fuchs equations. This result is motivated by the notion of Π -stability and by mirror symmetry.

1. INTRODUCTION

In this paper, we study the space of stability conditions on the derived category of the local \mathbb{P}^2 . Our approach is based on the chamber decomposition given by the wall-crossing for stable objects of the class of skyscraper sheaves of points.

1.1. Motivation. Consider a projective Calabi-Yau threefold Y containing a projective plane $\mathbb{P}^2 \subset Y$. Ideally, one would like to study the space of Bridgeland stability conditions on its derived category $D^b(Y)$. Understanding the geometry of this space would give insights on the group of autoequivalences of $D^b(Y)$ and give a global picture of mirror symmetry. Understanding wall-crossing for counting invariants of semi-stable objects would have many implications for Donaldson-Thomas type invariants on Y .

However, no single example of stability condition on a projective Calabi-Yau threefold has been constructed. Instead, in this article we focus on the full subcategory $D_{\mathbb{P}^2}^b(Y)$ of complexes concentrated on \mathbb{P}^2 . The local model for this situation is the total space $X = \text{Tot } \mathcal{O}_{\mathbb{P}^2}(-3)$ of the canonical bundle of \mathbb{P}^2 , called the “local \mathbb{P}^2 ”: $D_{\mathbb{P}^2}^b(Y)$ is then equivalent to the derived category $\mathcal{D}_0 := D_0^b(X)$ of coherent sheaves supported on the zero-section.

Denote by $\text{Stab}(\mathcal{D}_0)$ the space of stability conditions (Z, \mathcal{P}) on \mathcal{D}_0 (see Appendix B for a quick introduction to stability conditions). It is a three-dimensional complex manifold coming with a local homeomorphism to $\mathcal{Z}: \text{Stab}(\mathcal{D}_0) \rightarrow \text{Hom}(K(\mathcal{D}_0), \mathbb{C}) \cong \mathbb{C}^3$. The goal of this article is to study the space $\text{Stab}(\mathcal{D}_0)$ as a test case for the properties we would expect in the case of Y .

This space was first studied in [Bri06], where it was suggested that the space is closely related to the Frobenius manifold of the quantum cohomology of \mathbb{P}^2 . Further, understanding how Donaldson-Thomas type counting invariants of semi-stable objects depend on the stability conditions $(Z, \mathcal{P}) \in \text{Stab}(\mathcal{D}_0)$ (i.e., wall-crossing phenomena) would be highly interesting. For example, due to the derived equivalence $\mathcal{D}_0 \cong \mathbb{D}^b([\mathbb{C}^3/\mathbb{Z}_3])$ of [BKR01] it would give a new explanation for the relation between the Gromov-Witten potentials of X and of $\mathbb{C}^3/\mathbb{Z}_3$ (“crepant resolution conjecture”, see [Coa09, CCIT09]). It could also explain the modularity properties of the Gromov-Witten potential of X observed in [ABK08].

While these questions remain open, our results give a good description of a connected component of $\text{Stab}(\mathcal{D}_0)$, explain its relation to autoequivalences of \mathcal{D}_0 , and do give a global mirror symmetry picture.

1.2. Geometric stability conditions. In order to study $\text{Stab}(\mathcal{D}_0)$, we use one of its chamber decompositions. We consider a chamber $U \subset \text{Stab}(\mathcal{D}_0)$ consisting of “geometric” stability conditions, which have the property that all skyscraper sheaves $k(x)$, $x \in \mathbb{P}^2$, are stable of the same phase (see Definition 2.1 for the precise definition).

Our first result is a complete description of the geometric chamber (see Theorem 2.5): U is an open, connected, simply-connected, 3-dimensional subset of $\text{Stab}(\mathcal{D}_0)$. Up to shifts, a stability condition $(Z, \mathcal{P}) \in U$ is determined by its central charge Z , and we give explicit inequalities cutting out the set $\mathcal{Z}(U) \subset \text{Hom}(K(\mathcal{D}_0), \mathbb{C}) \cong \mathbb{C}^3$ of central charges Z for $(Z, \mathcal{P}) \in U$. The most interesting part of the boundary of U has a fractal-like structure; its shape is determined by the set of Chern classes of semi-stable vector bundles on \mathbb{P}^2 .

Let $\text{Stab}^\dagger(\mathcal{D}_0)$ be the connected component of $\text{Stab}(\mathcal{D}_0)$ containing U and let \bar{U} be the closure of U in $\text{Stab}^\dagger(\mathcal{D}_0)$. We can directly construct every wall of U , i.e. the components of the boundary $\partial U = \bar{U} \setminus U$ of U (see Theorem 5.1). We use this to prove the following result (see Corollary 5.2):

Theorem 1. *The translates of \bar{U} under the group of autoequivalences generated by spherical twists at spherical sheaves in \mathcal{D}_0 cover the whole connected component $\text{Stab}^\dagger(\mathcal{D}_0)$.*

The translates of U are disjoint, and each translate is a chamber on which the moduli space of stable objects of class $[k(x)]$ is constant.

1.3. Topology of $\text{Stab}^\dagger(\mathcal{D}_0)$. In [Bri06], Bridgeland described an open connected subset Stab_a of $\text{Stab}^\dagger(\mathcal{D}_0)$ consisting of “algebraic” stability conditions that can be described in terms of quivers. We will see that the subset Stab_a is not dense (in particular, it does not contain the “large volume limit” point: see Proposition 6.11). Nevertheless, by combining Bridgeland’s description of Stab_a with Theorem 1, we prove the following result:

Theorem 2. *The connected component $\text{Stab}^\dagger(\mathcal{D}_0)$ is simply-connected.*

1.4. Autoequivalences. In our situation, the local homeomorphism $\mathcal{Z} : \text{Stab}^\dagger(\mathcal{D}_0) \rightarrow \text{Hom}(K(\mathcal{D}_0), \mathbb{C})$ is not a covering of its image. This is a fundamental difference to the case of Calabi-Yau 2-categories (as studied in [Bri08, Tho08, Bri05a, IUU06, HMS08]). Further, there is no non-trivial subgroup of autoequivalences of \mathcal{D}_0 that acts as a group of deck transformation of the map \mathcal{Z} . But, in any case, using Theorem 1 we can classify all autoequivalences $\text{Aut}^\dagger(\mathcal{D}_0)$ which preserve the connected component $\text{Stab}^\dagger(\mathcal{D}_0)$:

Theorem 3. *The group $\text{Aut}^\dagger(\mathcal{D}_0)$ is isomorphic to a product $\mathbb{Z} \times \Gamma_1(3) \times \text{Aut}(X)$.*

Recall that $\Gamma_1(3)$ is the group on two generators α and β subject to the relation $(\alpha\beta)^3 = 1$. It is isomorphic to the subgroup generated by the spherical twist at the structure sheaf $\mathcal{O}_{\mathbb{P}^2}$ of the zero-section $\mathbb{P}^2 \hookrightarrow X$, and by the tensor product with $\mathcal{O}_X(1)$. The group \mathbb{Z} is identified with the subgroup generated by the shift by 1 functor $[1]$ and $\text{Aut}(X)$ denotes the group of automorphisms of X .

1.5. Π -stability and mirror symmetry. Stability conditions on a derived category were originally introduced by Bridgeland in [Bri07] to give a mathematical foundation for the notion of Π -stability in string theory, in particular in Douglas’ work, see e.g. [Dou02, AD02] and references therein. However, it has been understood that only a subset of Bridgeland stability conditions is physically meaningful, i.e. there is a submanifold M of the space of stability conditions Y that parametrizes Π -stability conditions, and that is isomorphic to the (universal covering) of the complex Kähler moduli space. In fact, M is (the universal covering of) a slice of the moduli space \mathcal{M} of SCFTs containing the sigma model associated to Y ; in the physics literature, it is often referred to as the “Teichmüller space”.

By mirror symmetry, M is also isomorphic to the universal covering of the moduli space of mirror partners \widehat{Y} of Y . As explained in [Bri09], this

leads to a purely purely algebro-geometric mirror symmetry statement; we prove such a result in Section 9:

The mirror partner for the local \mathbb{P}^2 is the universal family over the moduli space $\mathcal{M}_{\Gamma_1(3)}$ of elliptic curves with $\Gamma_1(3)$ level structures. Its fundamental group is $\Gamma_1(3)$. Let $\widetilde{\mathcal{M}}_{\Gamma_1(3)}$ be the universal cover, with $\Gamma_1(3)$ acting as the group of deck transformations.

Theorem 4. *There is an embedding $I: \widetilde{\mathcal{M}}_{\Gamma_1(3)} \hookrightarrow \text{Stab}^\dagger(\mathcal{D}_0)$ which is equivariant with respect to the action by $\Gamma_1(3)$ on both sides.*

Here the $\Gamma_1(3)$ -action on $\text{Stab}^\dagger(\mathcal{D}_0)$ is induced by the subgroup $\Gamma_1(3) \subset \text{Aut}^\dagger(\mathcal{D}_0)$ identified in Theorem 3.

On the level of central charges, the embedding is given in terms of a Picard-Fuchs differential equation: for a fixed $E \in \mathcal{D}_0$, the function $(\mathcal{Z} \circ I)(z)(E): \widetilde{\mathcal{M}}_{\Gamma_1(3)} \rightarrow \mathbb{C}$ is a solution of the Picard-Fuchs equation. In particular, while classical enumerative mirror symmetry gives an interpretation of their formal expansions at special points of M in terms of genus-zero Gromov-Witten invariants on Y , the space of stability conditions allows us to interpret solutions of Picard-Fuchs equations globally.

1.6. Relation to existing work. Various examples of stability conditions in local Calabi-Yau situations have been studied in the literature. In particular, the local derived category of curves inside surfaces has been studied in [Tho08, Bri05a, IU05, IUU06, Oka06, MMS09, BT09], and results similar to Theorem 1, Theorem 2, and Theorem 3 have been obtained. Some examples of stability conditions on projective spaces were studied in [Mac07, ABL07, Ohk08]. Other local Calabi-Yau threefold cases were studied in [Tod08b, Tod09], and, as already mentioned, an open subset of $\text{Stab}^\dagger(\mathcal{D}_0)$ has been described in [Bri06].

However, our approach follows the ideas in [Bri08] more closely than most of the above mentioned articles, as we describe stability conditions in terms of stability of sheaves on \mathbb{P}^2 , rather than in terms of exceptional collections and quivers. Applying this approach in our situation is possible due to the classical results of Drezet and Le Potier [DLP85]; in particular, the fractal boundary of $\mathcal{Z}(U)$ discussed in Section 1.2 is directly due to their results. At the same time, Sections 6 and 7 rely heavily on the work in [GR87] on exceptional collection and mutations.

Stability conditions around the orbifold point can be understood in terms of stability of quiver representations as studied in [CI04]; in particular our Theorem 1 could be understood as a derived version of [CI04, Theorem 1.2] applied to our situation.

There does not seem to be an equivalent of Theorem 4 in the literature for a Calabi-Yau 3-category; however, it is motivated by the conjectural picture described in [Bri09, Section 7].

There are many articles in the mathematical physics literature related to Π -stability and mirror symmetry for $\mathbb{C}^3/\mathbb{Z}_3$ and the local \mathbb{P}^2 (as well as other local del Pezzo surfaces), and our presentation in Section 9 is very much guided by [Asp05] and [ABK08]. In particular, Theorem 4 is based on the computations of analytic continuations and monodromy for solutions of the Picard-Fuchs equation of the mirror of the local \mathbb{P}^2 in [AGM94, Asp05, ABK08]; in some sense, we are just lifting their results from the level of central charges to the level of stability conditions.

In order for this to work, the “central charges predicted by physicists” had to survive a non-trivial test: they had to satisfy the inequalities of Definition 2.4 (see Observation (a), page 42). The fact that they survived this test is somewhat reassuring for the case of compact Calabi-Yau threefolds: identifying similar inequalities (which would be based on inequalities for Chern classes of stable objects), and checking that the central charges satisfy them, is the major obstacle towards constructing stability conditions on compact Calabi-Yau threefolds.

1.7. Open questions. Bridgeland’s conjecture [Bri06, Conj. 1.3] remains open; it would identify $\text{Stab}^\dagger(\mathcal{D}_0)$ with an open subset of the extended Frobenius manifold of the quantum cohomology of \mathbb{P}^2 . Theorems 1 and 2 of this paper essentially complete the study of $\text{Stab}(\mathcal{D}_0)$ as started in *loc. cit.*; and Theorem 4 clarifies the discussion in *loc. cit.* about the “small quantum cohomology locus”, as this locus corresponds to the image of $\widehat{\mathcal{M}}_{\Gamma_1(3)}$. What is missing from a proof of the whole conjecture, as pointed out in *loc. cit.*, is still a better understanding of the Frobenius manifold side.

It seems natural to conjecture that the full group $\text{Aut}(\mathcal{D}_0)$ of autoequivalences of \mathcal{D}_0 preserves the connected component $\text{Stab}^\dagger(\mathcal{D}_0)$; in fact, this last one may be the only three-dimensional component of $\text{Stab}(\mathcal{D}_0)$. In this case, Theorem 3 would give a complete description of $\text{Aut}(\mathcal{D}_0)$.

Maybe the most intriguing question about $\text{Stab}^\dagger(\mathcal{D}_0)$ related to our results is whether there is an intrinsic characterization of the image of the map I of Theorem 4, a question raised in other contexts in [Bri09]. To this end, note that the central charge on the image can also be given in terms of an analytic continuation of the genus zero Gromov-Witten potential of X (see [ABK08, Iri08]; that this agrees with our description using the mirror is classical enumerative mirror symmetry). But the genus-zero Gromov-Witten potential is in turn determined by counting invariants of one-dimensional torsion sheaves ([PT07, Tod08a]), i.e. counting invariants of stable objects close to the large-volume limit.

It would also be interesting to generalize some of the results of this paper to other “local del Pezzo surfaces”. In such a case, the starting point would be a generalization of the result in [DLP85] on the description of the Chern classes of stable sheaves. Already for $\mathbb{P}^1 \times \mathbb{P}^1$ the situation is more complicated: see [Rud94, Rud96] for results in this direction.

1.8. Plan of the paper. The paper is organized as follows. In Section 2 we define geometric stability conditions and state Theorem 2.5, which classifies them. Sections 3 and 4 are devoted to the proof of Theorem 2.5. In Section 5 we describe the boundary ∂U of the geometric chamber and prove Theorem 1. Algebraic stability conditions are introduced in Section 6 in order to prove Theorem 2 (whose proof will take Section 7).

In Section 8 we study the group of autoequivalences and prove Theorem 3. Section 9 discusses how the previous results fit into expectations from mirror symmetry, and includes the proof of Theorem 4. Finally, four appendices complete the paper: In Appendix A, we review the results of Drezet and Le Potier as we need them in the proof of Theorem 2.5. Appendix B is a brief introduction to stability conditions and contains an improved criterion for the existence of Harder-Narasimhan filtrations. In Appendix C we prove a technical result on representability of trivial autoequivalences. In Appendix D we give a sketch of the proof that the central charges we define in Section 9 satisfy the inequalities in Definition 2.4.

1.9. Notation. We work over the complex numbers \mathbb{C} . We let X denote the total space of $\mathcal{O}_{\mathbb{P}^2}(-3)$, and $i: \mathbb{P}^2 \hookrightarrow X$ the inclusion of the zero-section. We let $\text{Coh}_0 := \text{Coh}_{\mathbb{P}^2}(X) \subseteq \text{Coh}(X)$ be the subcategory of coherent sheaves on X supported (set-theoretically) on the zero-section. We write $\mathcal{D}_0 = \text{D}_0^b(X)$ for the subcategory of $\text{D}^b(\text{Coh}(X))$ of complexes with bounded cohomology, such that all of its cohomology sheaves are in Coh_0 . (Note that $\mathcal{D}_0 \cong \text{D}^b(\text{Coh}_0)$ as observed in [IU05], Notation and Convention.) The space of stability conditions on \mathcal{D}_0 will be denoted by $\text{Stab}(\mathcal{D}_0)$, and its two subsets of geometric and algebraic stability conditions by U and Stab_a , respectively (see Definitions 2.1 and 6.3).

An object S in \mathcal{D}_0 is called *spherical* if $\text{Ext}^p(S, S) \cong \mathbb{C}$ for $p = 0, 3$ and is zero otherwise. For a spherical object S we denote by ST_S the spherical twist associated with S , defined by the exact triangle

$$\text{Hom}^*(S, M) \otimes S \xrightarrow{ev} M \longrightarrow \text{ST}_S(M),$$

for $M \in \mathcal{D}_0$ (see [ST01]).

By abuse of notation, we will write $\mathcal{O}_{\mathbb{P}^2}(n) \in \mathcal{D}_0$ for the spherical objects given by the push-forwards to X of line bundles on \mathbb{P}^2 . For $x \in \mathbb{P}^2$ we denote by $k(x)$ the skyscraper sheaf in X of length one concentrated at x .

The Grothendieck group of \mathcal{D}_0 is denoted by $K(\mathcal{D}_0)$. It is isomorphic to $\mathbb{Z}^{\oplus 3}$. For any $E \in \mathcal{D}_0$, we write $r(E), d(E), c(E)$ for the components of the Chern character of its push-forward to \mathbb{P}^2 ; more precisely, if $\pi: X \rightarrow \mathbb{P}^2$ is the projection, then we write

$$\mathrm{ch}(\pi_*(E)) = r(E) \cdot [\mathbb{P}^2] + d(E) \cdot [\mathrm{line}] + c(E) \cdot [\mathrm{pt}].$$

For a complex number $z \in \mathbb{C}$, we write $\Re z$ (resp. $\Im z$) for its real (resp. imaginary) part.

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2. GEOMETRIC STABILITY CONDITIONS

We assume familiarity with the notion of stability conditions on a derived category; see Appendix B for a short summary, and [Bri07], [KS08, Section 3.4] for a complete reference.

We begin by constructing and classifying “geometric” stability conditions on \mathcal{D}_0 . Loosely speaking, geometric stability conditions are those that are most closely connected to the geometry of sheaves on X ; in the definition below, we require that the simple objects of $\mathrm{Coh}_0 X$ remain stable, but it will also turn out that the semi-stable objects are at most two-term complexes of sheaves in $\mathrm{Coh}_0 X$.

Definition 2.1. *A stability condition σ on \mathcal{D}_0 is called geometric if the following two conditions are satisfied:*

- (a) *All skyscraper sheaves $k(x)$ of closed points $x \in \mathbb{P}^2$ are σ -stable of the same phase.*
- (b) *The connected component of $\mathrm{Stab}(\mathcal{D}_0)$ containing σ has maximal dimension (equal to 3).*

We write U for the set of geometric stability conditions, and refer to it as the “geometric chamber”; in fact, we will see that it is precisely one of the chambers with respect to the chamber decomposition given by the wall-crossing phenomenon for semi-stable objects of class $[k(x)]$.

Part (b) of the definition is a technical condition to ensure that the wall-crossing for semi-stable objects behaves nicely; stability conditions satisfying this condition were called *full* in [Bri08], and they satisfy the *support property* of [KS08, Sect. 3.4].

We recall that a Bridgeland stability condition can be constructed by giving the heart of a t-structure $\mathcal{A} \subset \mathcal{D}_0$, and a compatible central charge $Z: K(\mathcal{A}) = K(\mathcal{D}_0) \rightarrow \mathbb{C}$ that sends objects in \mathcal{A} to the semi-closed upper half plane (see Remark B.1). The t-structures appearing in geometric stability conditions are given by the now familiar notion of *tilting* (see [HRS96]):

For purely 2-dimensional sheaves $\mathcal{F} \in \text{Coh}_0$, the slope function $\mu(\mathcal{F}) = \frac{d(\mathcal{F})}{r(\mathcal{F})}$ gives a notion of slope-stability (as in Definition A.1). By the same arguments as in the case of a projective variety, Harder-Narasimhan filtrations exist. Thus for any $B \in \mathbb{R}$, we can form a torsion pair $(\text{Coh}_0^{>B}, \text{Coh}_0^{\leq B})$ in Coh_0 , where

- $\text{Coh}_0^{\leq B}$ is generated by stable objects of slope $\mu \leq B$ (and extensions), and
- $\text{Coh}_0^{>B}(X)$ is generated by stable object of slope $\mu > B$ and zero- or one-dimensional torsion sheaves.

Definition 2.2. Let $\text{Coh}_0^{\sharp(B)} \subset \mathcal{D}_0$ be the tilt of Coh_0 at the torsion pair $(\text{Coh}_0^{>B}, \text{Coh}_0^{\leq B})$, that is

$$\text{Coh}_0^{\sharp(B)} = \left\{ E \in \mathcal{D}_0 : \begin{array}{l} \bullet \mathcal{H}^i(E) = 0, \text{ for all } i \neq 0, -1 \\ \bullet \mathcal{H}^0(E) \in \text{Coh}_0^{>B} \\ \bullet \mathcal{H}^{-1}(E) \in \text{Coh}_0^{\leq B} \end{array} \right\}.$$

The structure of central charges compatible with $\text{Coh}_0^{\sharp(B)}$ (and thus, as we will see, the structure of the whole geometric chamber) depends tightly on the set of Chern classes for which there exist semistable torsion-free sheaves. In the case of \mathbb{P}^2 , Drezet and Le Potier have given a complete description of this set:

It is most naturally described in terms of the discriminant $\Delta(\mathcal{F})$, which is defined as

$$\Delta(\mathcal{F}) = \frac{d(\mathcal{F})^2}{2r(\mathcal{F})^2} - \frac{c(\mathcal{F})}{r(\mathcal{F})}.$$

For an exceptional vector bundle E_α of rank r_α and slope α , the discriminant is given by $\Delta_\alpha = \frac{1}{2} - \frac{1}{2r_\alpha^2}$. As the slopes of exceptional vector bundles can be constructed explicitly, it remains to describe the slopes and discriminants of non-exceptional stable torsion-free sheaves.

Slightly reformulating the results of [DLP85], one can construct a function $\delta_\infty^{DP} : \mathbb{R} \rightarrow [\frac{1}{2}, 1]$ (cf. [LP97]). It is periodic of period 1 and Lipschitz-continuous with Lipschitz constant $\frac{3}{2}$. We refer to Appendix A for the precise definition of δ_∞^{DP} ; its construction is motivated by the following observation: If F_β is a slope-stable sheaf with $\beta < \alpha$, then $\text{Hom}(E_\alpha, F_\beta) = 0$; if we additionally assume $\beta > \alpha - 3$, then, by Serre duality, also $\text{Ext}^2(E_\alpha, F_\beta) = 0$ and hence $\chi(E_\alpha, F_\beta) \leq 0$. Using Riemann-Roch this yields an inequality of the form $\Delta(F_\beta) \geq p_\alpha(\beta)$ for $\alpha - 3 < \beta < \alpha$, with $p_\alpha(x)$ being a quadratic polynomial. The function δ_∞^{DP} is the supremum of all the quadratic polynomials p_α restricted to the ranges where the inequality is valid.

The main result of [DLP85] is that $\Delta \geq \delta_\infty^{DP}(\mu)$ is not only a necessary, but also a sufficient condition for the existence of a stable torsion-free sheaf of slope μ and discriminant Δ . For later use, we paraphrase their result as follows:

Define $S_\infty \subset \mathbb{R}^2$ to be the closed subset lying above the hyperplane cut out by the graph of δ_∞^{DP} , i.e.

$$S_\infty = \{(\mu, \Delta) \in \mathbb{R}^2 : \Delta \geq \delta_\infty^{DP}(\mu)\}.$$

Theorem 2.3. [DLP85] *Let $S \subset \mathbb{Q}^2$ be the set of pairs $(\mu(\mathcal{F}), \Delta(\mathcal{F}))$ where \mathcal{F} is any slope-stable torsion-free sheaf \mathcal{F} on \mathbb{P}^2 . Similarly, let $S_E \subset \mathbb{Q}^2$ be the corresponding set for slopes and discriminants of exceptional vector bundles. Then S is the disjoint union*

$$S = S_E \cup (S_\infty \cap \mathbb{Q}^2)$$

The set S_E has no accumulation points in $\mathbb{R}^2 \setminus S_\infty$.

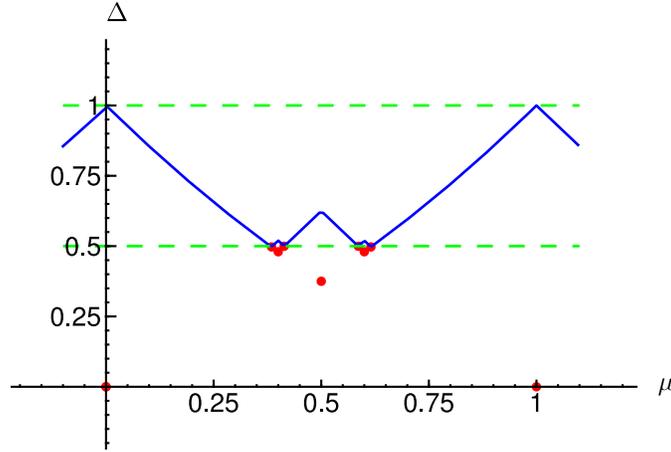
We explain this reformulation of Drezet and Le Potier's result in Appendix A, along with their explicit description of the set S_E ; see also Figure 1.

Definition 2.4. *We define the set $G \subset \mathbb{C}^2$ as the set of pairs $a, b \in \mathbb{C}$ satisfying the following three inequalities (where we set $B := -\frac{\Im b}{\Im a}$ if a satisfies the first inequality):*

- (1) $\Im a > 0$,
- (2) $\Re b > -B \cdot \Re a - \delta_\infty^{DP}(B) + \frac{1}{2}B^2$,

and, in case there exists an exceptional vector bundle of slope B and discriminant Δ_B ,

- (3) $\Re b > -B \cdot \Re a - \Delta_B + \frac{1}{2}B^2$.

Figure 1: δ_∞^{DP} and exceptional objects

Theorem 2.5. For $a, b \in \mathbb{C}$, denote by $Z_{a,b}: K(\mathcal{D}_0) \rightarrow \mathbb{C}$ the central charge given by

$$(4) \quad Z_{a,b}(E) = -c(E) + ad(e) + br(E)$$

for $E \in \mathcal{D}_0$.

Then there exists a geometric stability condition $\sigma_{a,b} = (Z_{a,b}, \mathcal{P}_{a,b})$ with $Z_{a,b}$ as above if and only if $(a, b) \in G \subset \mathbb{C}^2$. Its heart is, up to shifts, given by $\mathcal{P}_{a,b}((0, 1]) = \text{Coh}_0^{\sharp(B)}$. The shifts $k(x)[n]$ of skyscraper sheaves are the only stable objects of class $\pm[k(x)]$.

Any geometric stability condition is equivalent to a stability condition $\sigma_{a,b}$ up to the action of a unique element in \mathbb{C} .

The action of \mathbb{C} is given in Remark B.4: it is the lift to the space of stability conditions of the multiplication by $\exp(z)$ on $\text{Hom}(K(\mathcal{D}_0), \mathbb{C})$.

The theorem can be rephrased as stating that $U/\mathbb{C} \cong G$, with a section given by $(a, b) \mapsto \sigma_{a,b}$. Later, in Remark 7.2, we will see that this slice of the \mathbb{C} -action can be extended to a whole connected component of $\text{Stab}(\mathcal{D}_0)$. The theorem will be proved in the following two sections.

The best visualization of the set of allowed central charges is given by the following observation: as long as $\Im a > 0$, the central charge can be thought of as a surjective map $K(\mathcal{D}_0)_{\mathbb{R}} \cong \mathbb{R}^3 \rightarrow \mathbb{C} \cong \mathbb{R}^2$. Up to the action of $GL_2^+(\mathbb{R})$ on \mathbb{R}^2 (which does not affect the set of stable objects), this map is determined by its kernel, and by the orientation induced on $K(\mathcal{D}_0)_{\mathbb{R}}/\text{Ker } Z$. As long as $\Im a > 0$, the orientation does not change. The kernel intersects the affine hyperplane $r = 1$ of $K(\mathcal{D}_0)_{\mathbb{R}}$ in a single point.

The inequalities are equivalent to requiring that this point lies below the graph of $\Delta = \delta_\infty^{DP}(\mu)$, and not on any of the rays going up vertically from a point in S_E (see Figure 1).

However, for several reasons it is helpful to classify geometric stability conditions up to the action of \mathbb{C} , rather than the action of $\mathrm{GL}_2^+(\mathbb{R})$. The subgroup \mathbb{C} acts on $\mathrm{Stab}(\mathcal{D}_0)$ with closed orbits, it has no stabilizers, and it has a well-behaved quotient $\mathrm{Stab}(\mathcal{D}_0)/\mathbb{C}$. The stability conditions $\sigma_{a,b}$ constitute a slice of this action on $U \subset \mathrm{Stab}(\mathcal{D}_0)$, and the boundary of U can be identified, up to the \mathbb{C} -action, with the boundary of the set $\{\sigma_{a,b} : (a,b) \in G\}$. None of these statements would hold for the $\mathrm{GL}_2^+(\mathbb{R})$ -action, and the picture of the preceding paragraph only gives a partial picture of the boundary of U : One can see that every ray starting at a point of S_E going up vertically may give two walls in the boundary ∂U of the set U , but we cannot see that many of these walls intersect at points where the central charge lies in the real line.

3. CONSTRAINING GEOMETRIC STABILITY CONDITIONS

In this section we will show that geometric stability conditions can only be of the form given in Theorem 2.5. The general idea is the same as in [Bri08]: if we assume that the skyscraper sheaves $k(x)$ are stable of phase 1, then Hom-vanishing helps to constrain the form of objects in $\mathcal{P}((0, 1])$, and we can identify $\mathcal{P}((0, 1])$ with an explicit tilt of the standard t-structure. By the existence of a well-behaved chamber decomposition for the wall-crossing for stable objects of class $[k(x)]$, the set of geometric stability conditions is open, and we need to prove inequalities for the central charge only when it is defined over \mathbb{Q} .

The proof will be broken into several lemmata and propositions. The following observation shows that the bound of Theorem 2.3 translates into bounds for stable objects in Coh_0 :

Lemma 3.1. *A sheaf $\mathcal{F} \in \mathrm{Coh}_0(X)$ is a pure μ -stable sheaf if and only if it is the push-forward $\mathcal{F} = i_*\mathcal{F}_0$ of some μ -stable pure sheaf $\mathcal{F}_0 \in \mathrm{Coh}(\mathbb{P}^2)$.*

Proof. Since $i_*: \mathrm{Coh} \mathbb{P}^2 \hookrightarrow \mathrm{Coh}_0(X)$ is a full subcategory, closed under subobjects and quotients, and since i_* preserves the ordering by slopes, it follows $\mathcal{F}_0 \in \mathrm{Coh} \mathbb{P}^2$ is stable if and only if $i_*\mathcal{F}_0$ is stable.

Now assume that \mathcal{F} is stable. Then $\mathrm{End} \mathcal{F} = \mathbb{C} \cdot \mathrm{Id}$. Let Z be the scheme-theoretic support of \mathcal{F} . Its global sections act faithfully on \mathcal{F} , so $H^0(\mathcal{O}_Z) \cong \mathbb{C}$. Hence Z must be contained in the scheme-theoretic inverse image of the origin under the contraction $X \rightarrow \mathbb{C}^3/\mathbb{Z}_3$, i.e. $Z \subset \mathbb{P}^2$. \square

Now assume we are given a geometric stability condition. After a rescaling by \mathbb{C} , we may assume that all skyscraper sheaves $k(x)$ of closed points are stable with phase 1 and $Z(k(x)) = -1$.

Lemma 3.2 ([Bri08], Lemma 10.1). *Let (Z, \mathcal{P}) be a stability condition such that the skyscraper sheaves $k(x)$ are stable of phase 1, with $Z(k(x)) = -1$.*

- (a) *For any object $E \in \mathcal{P}((0, 1])$, its cohomology sheaves $\mathcal{H}^i(E)$ vanish unless $i = 0, -1$.*
- (b) *Further, for any such $E \in \mathcal{P}((0, 1])$ the cohomology sheaf $\mathcal{H}^{-1}(E)$ is pure of dimension 2.*
- (c) *If $E \in \mathcal{P}(1)$ is stable and $E \neq k(x)$ for all $x \in \mathbb{P}^2$, then there is a vector bundle \mathcal{F} on \mathbb{P}^2 such that $E \cong i_*\mathcal{F}[-1]$.*

Proof. If $E \in \mathcal{P}((0, 1))$, then $\mathrm{Hom}(E, k(x)[i]) = 0$ for $i < 0$, and $\mathrm{Hom}(k(x)[i], E) = \mathrm{Hom}(E, k(x)[3+i]) = 0$ for $i \geq 0$. By [BM02, Prop. 5.4], this implies that E is quasi-isomorphic to a 3-term complex of locally free sheaves $E^{-2} \xrightarrow{d^{-2}} E^{-1} \rightarrow E^0$. Hence $\mathcal{H}^{-2}(E)$ is locally free on X ; since $\mathcal{H}^{-2}(E) \in \mathrm{Coh}_0(X)$, it must vanish. This implies the first claim for such E .

This also shows that $\mathcal{H}^{-1}(E)$ is the cokernel of an injective map between locally free sheaves, which implies the second claim.

If E is stable of phase 1, then additionally $\mathrm{Hom}(E, k(x)) = 0$ (as they are both stable objects of the same phase). Hence E is isomorphic to a two-term complex of vector bundles $E^{-2} \xrightarrow{d^{-2}} E^{-1}$. By the same argument as in the previous case, the map d^{-2} must be injective, so that E is isomorphic to the shift of a sheaf: $E \cong \mathcal{F}'[1]$. Since E is stable, \mathcal{F}' can only have scalar endomorphisms, and thus \mathcal{F}' is the push-forward of a sheaf \mathcal{F} on \mathbb{P}^2 . Using $0 = \mathrm{Hom}(i_*\mathcal{F}[1], i_*k(x)) = \mathrm{Hom}(\mathcal{F}[1], k(x) \oplus k(x)[1])$, it follows that \mathcal{F} is a vector bundle.

Since all the assertions of the lemma are properties that are closed under extensions, this finishes its proof. \square

Proposition 3.3. *Let $\mathrm{Stab}^* \subset \mathrm{Stab}(\mathcal{D}_0)$ be a connected component of full dimension, and fix a primitive class $\alpha \in K(\mathcal{D}_0)$. Then there exists a collection of walls $W_i^\alpha, i \in I$ with the following properties:*

- *Every wall W_i^α is a smooth (but not necessarily closed) submanifold of real codimension one.*
- *The collection W_i^α is locally finite (i.e., every compact subset $K \subset \mathrm{Stab}^*$ intersects only a finite number of walls).*
- *If $C \subset \mathrm{Stab}^*$ is a connected component of the complement of $\bigcap_{i \in I} W_i^\alpha$ and $\sigma_1, \sigma_2 \in C$, then an object $E \in \mathcal{D}_0$ with $[E] = \alpha$ is σ_1 -stable if and only if it is σ_2 -stable.*

Proof. Every wall W_i^α can be described as the set of stability conditions where a given object E_i of class α is semistable, and has a given object F_i as a stable subobject of the same phase. We first show that the set of such

walls intersecting an open ball $B_{\frac{1}{8}}(\sigma)$ of radius $\frac{1}{8}$ around $\sigma = (Z, \mathcal{P})$ is finite:

Since the metric $\|\cdot\|_\sigma$ (defined by equation (23)) is finite, and since $K(\mathcal{D}_0)$ is a discrete subgroup of $K(\mathcal{D}_0) \otimes \mathbb{R} \cong \mathbb{R}^3$, there exist only finitely many classes $\beta \in K(\mathcal{D}_0)$ such that there exists a stable object F with $[F] = \beta$ and $|Z(F)| < |Z(\alpha)|$. If ϕ is an arbitrary choice with $Z(\alpha) \in \mathbb{R}_{>0} \cdot e^{i\pi\phi}$, it follows that there exist only finitely many classes $\beta \in K(\mathcal{D}_0)$ such that there exists a strict inclusion $F \hookrightarrow E$ in the quasi-abelian category $\mathcal{P}((\phi - \frac{1}{4}, \phi + \frac{1}{4}))$ with F of class β and E of class α . However, any wall W_i^α gives rise to such a pair $F_i \hookrightarrow E_i$, and it is completely determined by the class $[F_i]$.

All other claims follows exactly with the same arguments as given in [Bri08, Section 9]. \square

Remark 3.4. *As can be seen from the proof, the proposition follows from the fact that stability conditions in Stab^* satisfy the support property of [KS08, Sect. 3.4].*

Corollary 3.5. *The set U of geometric stability conditions is open in the space of stability conditions $\text{Stab}(\mathcal{D}_0)$. Its boundary $\partial U = \overline{U} \setminus U$ is given by a locally finite union of walls, and each wall is a real submanifold in $\text{Stab}(\mathcal{D}_0)$ of codimension one.*

Proof. From claim (c) of Lemma 3.2 it follows that the set U coincides with a union of some of the chambers given in Proposition 3.3, applied to the class $\alpha = [k(x)]$. \square

We proceed to show that any geometric stability condition is necessarily of the form given in Theorem 2.5.

By Lemma 3.2, we have $\mathcal{P}((0, 1]) \subset \langle \text{Coh}_0(X), \text{Coh}_0(X)[1] \rangle$. This implies that $\mathcal{P}((0, 1])$ is obtained from $\text{Coh}_0(X)$ by tilting at the torsion pair

$$\begin{aligned} \mathcal{T} &= \text{Coh}_0(X) \cap \mathcal{P}((0, 1]) \\ \mathcal{F} &= \text{Coh}_0(X) \cap \mathcal{P}((-1, 0]) \end{aligned}$$

(see e.g. [Pol07, Lemma 1.1.2]).

Since we assume $Z(k(x)) = -1$, the central charge can be written in the form of equation (4); in particular $\Im Z(E) = d(E) \cdot \Im a + r(E) \Im b$. By mimicking the proof of [Bri08, Prop. 10.3], it follows that $\Im a > 0$ and, after setting $B = -\frac{\Im b}{\Im a}$, $\mathcal{P}((0, 1]) = \text{Coh}_0^{\sharp(B)}$.

It remains to prove the inequality on $\Re b$. We first assume that $B \in \mathbb{Q}$. For any semi-stable torsion-free sheaf \mathcal{F} on \mathbb{P}^2 of slope B we have $\Im Z(i_* \mathcal{F}) =$

0 and $i_*\mathcal{F}[1] \in \text{Coh}_0^{\sharp(B)}$, hence we must have $\Re Z(i_*\mathcal{F}) > 0$. It follows that:

$$\begin{aligned} 0 &< \frac{\Re Z(i_*\mathcal{F})}{r} = \Re b + \Re aB - \frac{c}{r} \\ &= \Re b + \Re aB + \Delta(\mathcal{F}) - \frac{1}{2}B^2 \end{aligned}$$

Applying Theorem 2.3, we obtain the inequalities (2) and (3).

Finally, we need to treat the case $B \notin \mathbb{Q}$. By Corollary 3.5, there exists an open neighborhood $V \subset \mathbb{C}^2$ of (a, b) , such that any $(a', b') \in V$ with $\frac{\Im b'}{\Im a'} \in \mathbb{Q}$ satisfy inequality (2). Hence it holds for (a, b) , too.

4. CONSTRUCTING GEOMETRIC STABILITY CONDITIONS

We now come to the proof of existence of geometric stability conditions. The main problem is to prove the existence of Harder-Narasimhan filtrations for the stability function $Z_{a,b}$ on $\text{Coh}_0^{\sharp(B)}$. We prove this directly in the case where the image of $\Im Z$ is discrete, and then use Bridgeland's deformation result to extend it to the more general case. In order to make the extension effective, we have to bound the metric $\|\cdot\|_\sigma$ on $\text{Hom}(K(\mathcal{D}_0), \mathbb{C}) \cong \mathbb{C}^3$ defined by equation (23) from above. To do so, we in turn have to control $|Z(E)|$ for stable objects E from below.

Our arguments in this section build on [Bri08, ABL07].

Given $(a, b) \in G \subset \mathbb{C}^2$, let $\gamma_{a,b}: \mathbb{R} \rightarrow \mathbb{C}$ be the infinite path $\gamma_{a,b}(t) = x(t) + iy(t)$ defined by

$$(5) \quad \begin{aligned} x(t) &= \Re b + \frac{1}{2}\Re a^2 + \delta_\infty^{DP}(t) - \frac{1}{2}(t + \Re a)^2 \\ y(t) &= \Im a t + \Im b \end{aligned}$$

and let $S_{a,b} \subset \mathbb{C}$ be the closed subset cut out by $\gamma_{a,b}$ that lies on or to the right of $\gamma_{a,b}$, i.e.

$$S_{a,b} := \{x + iy : \exists t \text{ with } y = y(t), x \geq x(t)\}.$$

Since $\delta_\infty^{DP}(t) \in [\frac{1}{2}, 1]$, the path $\gamma_{a,b}$ is contained between the graphs of two parabolas with distance $\frac{1}{2}$, see Figure 2.

Lemma 4.1. *Let $\mathcal{F} \in \text{Coh}(\mathbb{P}^2)$ be a torsion-free μ -stable sheaf on \mathbb{P}^2 of rank r that is not an exceptional vector bundle. Then $\frac{Z_{a,b}(i_*\mathcal{F})}{r} \in S_{a,b}$.*

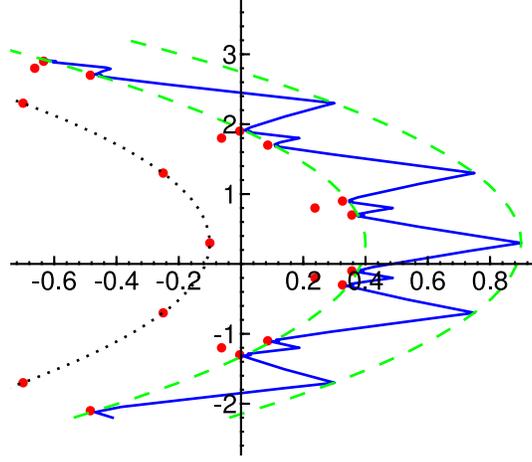


Figure 2: The path $\gamma(t)$ and the central charge of exceptional objects

Proof. We write $\frac{\text{ch}(\mathcal{F})}{r} = [\mathbb{P}^2] + \mu[l] + \frac{c}{r}[\text{pt}]$ and $\frac{Z_{a,b}(i_*\mathcal{F})}{r} = x + iy$. Using Theorem 2.3, we obtain:

$$\begin{aligned} y &= -\Im a\mu - \Im b \\ x &= -\frac{c}{r} - \Re a\mu - \Re b = \Delta(\mathcal{F}) - \frac{1}{2}\mu^2 - \Re a\mu - \Re b \\ &\geq \delta_\infty^{DP}(\mu) - \frac{1}{2}\mu^2 - \Re a\mu - \Re b \end{aligned}$$

Setting $t = \mu$ yields the claim. \square

Lemma 4.2. *For any $(a, b) \in G$, the central charge $Z_{a,b}$ is a stability function for $\text{Coh}_0^{\sharp(B)}$: if $0 \neq E \in \text{Coh}_0^{\sharp(B)}$, then $Z_{a,b}(E) \in \mathbb{H}$.*

Proof. It is sufficient to prove the claim for torsion sheaves, and for (shifts of) slope-stable torsion-free sheaves. The claim is only non-trivial for objects $\mathcal{F}[1]$, where \mathcal{F} is μ -stable of slope $\mu(\mathcal{F}) = B$. In such a case, $Z_{a,b}(\mathcal{F})$ is lying on the real line. By Lemma 3.1, \mathcal{F} is the push-forward of a μ -stable sheaf on \mathbb{P}^2 . If \mathcal{F} is not an exceptional vector bundle, the previous lemma shows $Z_{a,b}(i_*\mathcal{F}) \in \mathbb{R}_{>0}$. If it is exceptional, the same computation yields this statement from inequality (3). \square

In other words, inequality (2) is equivalent to $0 \notin S_{a,b}$, i.e. the path is passing through the real line with positive real part; and inequality (3) is equivalent to $Z(\mathcal{E}) \notin \mathbb{R}_{\leq 0}$ for any exceptional vector bundle \mathcal{E} on \mathbb{P}^2 . Together, they guarantee that the central charge of a slope-stable sheaf never lies on the negative real line.

Due to the above Lemma, as in Remark B.1 we can define a phase function $\phi(E) := (1/\pi) \arg(Z(E)) \in (0, 1]$, for all $0 \neq E \in \text{Coh}_0^{\sharp(B)}$.

Lemma 4.3. ([Bri08, Proposition 7.1]) *Let $a, b \in \mathbb{C}$ be such that $B = -\Im(b)/\Im(a) \in \mathbb{Q}$ and $Z_{a,b}(\text{Coh}_0^{\sharp(B)} \setminus \{0\}) \subseteq \mathbb{H}$. Then Harder-Narasimhan filtrations exist for $(Z_{a,b}, \text{Coh}_0^{\sharp(B)})$.*

Proof. We use Proposition B.2. Since $B \in \mathbb{Q}$, the image of $\Im(Z_{a,b})$ is discrete in \mathbb{R} .

Let $F \in \text{Coh}_0^{\sharp(B)}$ and let

$$(6) \quad 0 = A_0 \subseteq A_1 \subseteq \dots \subseteq A_j \subseteq A_{j+1} \subseteq \dots \subseteq F,$$

be a sequence of subobjects, with $A_j \in \mathcal{P}'_{a,b}(1)$. As in Proposition B.2, $\mathcal{P}'_{a,b}(1)$ is the full subcategory of $\text{Coh}_0^{\sharp(B)}$ whose objects have phase 1 with respect to $Z_{a,b}$. We need to show that (6) stabilizes.

To this end, we first observe that (6) induces a sequence of inclusions

$$(7) \quad 0 = \mathcal{H}^{-1}(A_0) \hookrightarrow \dots \hookrightarrow \mathcal{H}^{-1}(A_j) \hookrightarrow \dots \hookrightarrow \mathcal{H}^{-1}(F).$$

Since Coh_0 is Noetherian, (7) must terminate. We can therefore assume that $\mathcal{H}^{-1}(A_j) \cong \mathcal{F}_{-1}$ for all j and some $\mathcal{F}_{-1} \in \text{Coh}_0^{\leq B}$. Let \mathcal{F}_0 denote the cokernel of the inclusion $\mathcal{F}_{-1} \hookrightarrow \mathcal{H}^{-1}(F)$; then, by the long exact cohomology sequence, we have $\mathcal{F}_0 \in \text{Coh}_0^{\leq B}$ as well.

Now, observe that the simple objects of $\mathcal{P}'_{a,b}(1)$ are skyscraper sheaves $k(x)$ ($x \in \mathbb{P}^2$) and objects of the form $i_*\mathcal{G}[1]$, for $\mathcal{G} \in \text{Coh}(\mathbb{P}^2)$ a locally-free μ -stable sheaf on \mathbb{P}^2 with $\mu(\mathcal{G}) = B$. Indeed, this can be proved in precisely the same way as Lemma 3.2, (c). In particular, $\mathcal{H}^0(A_j)$ is a torsion sheaf of dimension zero.

Let B_j be the cokernel in $\text{Coh}_0^{\sharp(B)}$ of $A_j \hookrightarrow F$; then we have an exact sequence

$$0 \rightarrow \mathcal{F}_0 \xrightarrow{f_j} \mathcal{H}^{-1}(B_j) \rightarrow \mathcal{H}^0(A_j) \xrightarrow{g_j} \mathcal{H}^0(F).$$

The cokernel $\text{cok } f_j$ is zero-dimensional; since \mathcal{F}_0 is fixed $\mathcal{H}^{-1}(B_j)$ is pure of dimension 2, the length of $\text{cok } f_j$ is bounded. As the length of $\text{im } g_j$ is also bounded, we get a bound on the length of $\mathcal{H}^0(A_j)$. At the same time, if D_j denotes the cokernel in $\text{Coh}_0^{\sharp(B)}$ of $A_j \hookrightarrow A_{j+1}$, we must have $\mathcal{H}^{-1}(D_j) = 0$, and thus an inclusion $\mathcal{H}^0(A_j) \hookrightarrow \mathcal{H}^0(A_{j+1})$. Hence, for $j \gg 0$, $\mathcal{H}^0(A_j) \cong \mathcal{H}^0(A_{j+1})$ and so (6) stabilizes. \square

Notice that, in the assumptions of Lemma 4.3, the pair $(Z_{a,b}, \text{Coh}_0^{\sharp(B)})$ defines a locally-finite stability condition on \mathcal{D}_0 . Indeed this follows immediately from [Bri08, Lemma 4.4].

In the rest of this section, we will use Bridgeland's deformation result to extend the existence of Harder-Narasimhan filtrations to the case where

a, b are not rational. In order to make the deformation effective, we need to bound the metric $\|\cdot\|_{\sigma_{a,b}}$ on $\text{Hom}(K(\mathcal{D}_0), \mathbb{C}) \cong \mathbb{C}^3$ defined in equation (23) relative to an arbitrarily chosen metric, with the bound depending continuously on $a, b \in G$.

We define the following functions $G \rightarrow \mathbb{R}_{\geq 0}$:

$$\gamma_{\min}(a, b) := \inf \{ |\gamma_{a,b}(t)| : t \in \mathbb{R} \}$$

$$E_{\min}(a, b) := \inf \left\{ \left| \frac{Z_{a,b}(\mathcal{E})}{r(\mathcal{E})} + t \right| : t \in \mathbb{R}_{\geq 0}, \mathcal{E} \text{ exceptional v. bundle} \right\}$$

$$S_{\min}(a, b) := \min(\gamma_{\min}(a, b), E_{\min}(a, b)).$$

Lemma 4.4. *The function S_{\min} is continuous and satisfies*

$$0 < S_{\min}(a, b) \leq \inf \left\{ \left| \frac{Z_{a,b}(\mathcal{F})}{r(\mathcal{F})} + t \right| : t \in \mathbb{R}_{\geq 0}, \mathcal{F} \in \text{Coh}_0 \text{ slope-stable} \right\}$$

for all $(a, b) \in G$.

Proof. The path $\gamma_{a,b}(t)$ depends continuously on a, b, t , does not go through zero, and satisfies $\lim_{t \rightarrow \pm\infty} |\gamma_{a,b}(t)| = +\infty$. It follows that γ_{\min} is a positive continuous function.

Since $Z_{a,b}(\mathcal{E}) \notin \mathbb{R}_{\leq 0}$, the term $\inf \left\{ \left| \frac{Z_{a,b}(\mathcal{E})}{r(\mathcal{E})} + t \right| : t \in \mathbb{R}_{\geq 0} \right\}$ is positive for every exceptional vector bundle \mathcal{E} on \mathbb{P}^2 . Further, Theorem 2.3 together with the computation of Lemma 4.1 shows that every accumulation point of the set

$$\left\{ \frac{Z_{a,b}(\mathcal{E})}{r(\mathcal{E})} : \mathcal{E} \text{ exceptional v. bdl} \right\}$$

is contained in $S_{a,b}$. Hence E_{\min} is also a positive continuous function.

It remains to prove $S_{\min}(a, b) \leq \left| \frac{Z_{a,b}(\mathcal{F})}{r(\mathcal{F})} + t \right|$ for all \mathcal{F}, t . It holds by definition when \mathcal{F} is an exceptional vector bundle. Otherwise, the claim follows as $\frac{Z_{a,b}(\mathcal{F})}{r(\mathcal{F})} + t$ is contained in $S_{a,b}$. \square

Let $|\cdot|_{\infty}$ be the supremums-norm on $K(\mathcal{D}_0) \otimes \mathbb{R} \cong \mathbb{R}^3$ in the coordinates (r, d, c) . Let $M(a, b)$ be the matrix

$$M(a, b) := \begin{pmatrix} 1 & 0 & 0 \\ \Im b & \Im a & 0 \\ \Re b & \Re a & -1 \end{pmatrix},$$

and let $N(a, b) := \|M(a, b)^{-1}\|_{\infty}$ be the norm of its inverse, where $\|\cdot\|_{\infty}$ is the operator norm with respect to the supremums-norm on \mathbb{R}^3 .

Lemma 4.5. *If $E \in \text{Coh}_0^{\sharp(B)}$ is $Z_{a,b}$ -stable, then*

$$\frac{|Z_{a,b}(E)|}{|E|_{\infty}} \geq \frac{\min(S_{\min}(a, b), 1)}{N(a, b)}.$$

We first show how to conclude the proof of Theorem 2.5 from the lemma:

Corollary 4.6. *There exists a geometric stability condition $\sigma_{a,b}$ for arbitrary pairs $(a, b) \in G \subset \mathbb{C}^2$.*

Proof. Let $V \subset U$ be the subset of geometric stability conditions $\sigma = (Z, \mathcal{P})$ such that all skyscraper sheaves $k(x)$ are stable of phase 1 with $Z(k(x)) = -1$. For any such stability condition, the central charge is of the form $Z = Z_{a,b}$ of equation (4), and thus \mathcal{Z} induces a map $\mathcal{Z}_V: V \rightarrow \mathbb{C}^2, \sigma \mapsto (a, b)$.

By the results of Section 3, the map \mathcal{Z}_V is injective, so it is a homeomorphism onto its image. Using Corollary 3.5 and the deformation property, we see that the image of \mathcal{Z}_V is open in \mathbb{C}^2 . By Lemma 4.3, it contains the dense subset of $(a, b) \in G$ such that $B = -\frac{\Im b}{\Im a} \in \mathbb{Q}$ is rational. Hence it suffices to prove that the image is closed in G .

Assume the contrary, and that $(a, b) \in G$ are in the boundary of \mathcal{Z}_V . From Lemma 4.4, it follows that for all $(a', b') \in G$ sufficiently close to (a, b) , we have

$$\|Z_{a,b} - Z_{a',b'}\|_\infty < \frac{1 \min(S_{\min}(a', b'), 1)}{8 N(a', b')}.$$

From Lemma 4.5, it follows that

$$\|Z_{a,b} - Z_{a',b'}\|_{\sigma_{a',b'}} < \frac{1}{8}$$

for all such (a', b') for which a geometric stability condition $\sigma_{a',b'}$ exists.

By Bridgeland's effective deformation result (see Theorem B.3) there exists a stability condition $\sigma_{a,b} = (Z_{a,b}, \mathcal{P}_{a,b})$ in the neighborhood of $\sigma_{a',b'}$. By choosing (a', b') appropriately, we may assume that $\sigma_{a,b}$ is on one of the walls in the sense of Corollary 3.5; in particular, $k(x)$ is semistable, and there is an inclusion $E \hookrightarrow k(x)$ in $\mathcal{P}_{a,b}(1)$ with E being stable. In particular, $0 = \Im Z_{a,b}(E) = \Im a \cdot d(E) + \Im b \cdot r(E)$. Since $k(x)$ is stable with respect to $\sigma_{a',b'}$, we have $\Im Z_{a',b'}(E) \neq 0$, and thus we have $r(E) \neq 0$ or $d(E) \neq 0$. Since $\Im a > 0$, it follows that $r(E) \neq 0$; but then $B = -\frac{\Im b}{\Im a} = \frac{d(E)}{r(E)} \in \mathbb{Q}$, and so we already know that there exists a geometric stability condition $\sigma \in V$ with $\mathcal{Z}_V(\sigma) = (a, b)$. \square

Proof. (Lemma 4.5) Writing $r = r(E)$ etc., we have

$$\begin{aligned} |E|_\infty &= \max(r, d, c) \\ &= |M(a, b)^{-1} \cdot (r, \Im a \cdot d + \Re b \cdot r, -c + \Re a \cdot d + \Re b \cdot r)|_\infty \\ &\leq N(a, b) \cdot |(r, \Re Z_{a,b}(E), \Im Z_{a,b}(E))|_\infty \\ &\leq N(a, b) \cdot \max(|r|, |Z_{a,b}(E)|) \end{aligned}$$

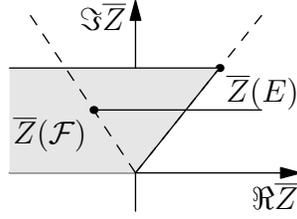


Figure 3: Location of $\bar{Z}(\mathcal{F})$ relative to $\bar{Z}(E)$

Thus the claim follows if we can show

$$|\bar{Z}(E)| > S_{\min(a,b)}$$

where $E \in \text{Coh}_0^{\sharp(B)}$ is any $Z_{a,b}$ -stable objects with non-zero rank, and where we wrote $\bar{Z}(E) = \frac{Z_{a,b}(E)}{|r(E)|}$.

Assume first that $H^0(E)$ has non-zero rank. We have

$$\Im Z_{a,b}(E) \geq \Im Z_{a,b}(H^0(E)),$$

as $E \twoheadrightarrow H^0(E)$ is a quotient in $\text{Coh}_0^{\sharp(B)}$ and $Z_{a,b}$ is a stability function, and we have $|r(E)| \leq r(H^0(E))$; together, they show

$$\Im \bar{Z}(E) \geq \Im \bar{Z}(H^0(E)).$$

Let $H^0(E) \twoheadrightarrow \mathcal{F}$ be a semi-stable quotient with $\mu(E) \geq \mu(\mathcal{F}) > B$ (such a quotient always exists due to the existence of HN-filtrations for slope stability). Then

$$\begin{aligned} \Im \bar{Z}(H^0(E)) &= \frac{\Im a \cdot d(H^0(E)) + \Im b \cdot r(E)}{r(H^0(E))} = \Im a \mu(H^0(E)) - B \\ &\geq \Im a \mu(\mathcal{F}) - B = \Im \bar{Z}(\mathcal{F}) \end{aligned}$$

(where we used the assumption $\Im a > 0$ in the inequality).

On the other hand, as E is $Z_{a,b}$ -semistable and $\mathcal{F} \in \text{Coh}_0^{\sharp(B)}$, and so the phase of $Z_{a,b}(\mathcal{F})$ is at least as big as the phase of $Z_{a,b}(E)$. Hence the line segment connecting 0 and $\bar{Z}(E)$ intersects the ray $\bar{Z}(\mathcal{F}) + t, t \geq 0$ (see Figure 3). By Lemma 4.4, this implies the claim.

A dual argument holds in case $H^{-1}(E)$ is non-zero, by considering a slope-stable sheaf $\mathcal{F} \hookrightarrow H^{-1}(E)$ with $B \geq \mu(\mathcal{F}) \geq \mu(H^{-1}(E))$. Finally, when $H^{-1}(E)$ is zero and $H^0(E)$ has rank zero, there is nothing to prove. \square

Remark 4.7. *The methods used in the last part of this section also apply in the situation [ABL07, Section 2]: the stability conditions constructed*

there for rational divisors D, F deform to produce stability condition for arbitrary \mathbb{R} -divisors D, F with F ample.

5. BOUNDARY OF THE GEOMETRIC CHAMBER

In this section we will show that the set of boundary walls of the geometric chamber U can be described explicitly using exceptional vector bundles \mathcal{E} on \mathbb{P}^2 . For any such \mathcal{E} , the push-forward $i_*\mathcal{E}$ is a spherical object in \mathcal{D}_0 . We denote by $\text{ST}_{\mathcal{E}}: \mathcal{D}_0 \rightarrow \mathcal{D}_0$ the spherical twist at $i_*\mathcal{E}$. If r is the rank of \mathcal{E} and $x \in \mathbb{P}^2$, we also write \mathcal{E}^x for the kernel of the natural map $i_*\mathcal{E}^{\oplus r} \rightarrow k(x)$.

The goal of this section is to prove the following theorem:

Theorem 5.1. *For every exceptional vector bundle \mathcal{E} there exist two codimension one walls $W_{\mathcal{E}}^+, W_{\mathcal{E}}^- \in \partial U$ with the following properties:*

- (a) *Stability conditions in $W_{\mathcal{E}}^+$ are characterized by the property that $i_*\mathcal{E}$ and all skyscraper sheaves $k(x)$ are semistable of the same phase ϕ , with $i_*\mathcal{E}$ being a subobject of $k(x)$ in $\mathcal{P}(\phi)$; at a general point of $W_{\mathcal{E}}^+$, the Jordan-Hölder filtration of any skyscraper sheaf $k(x)$ is given by*

$$(8) \quad i_*\mathcal{E}^{\oplus r} \rightarrow k(x) \rightarrow \mathcal{E}^x[1].$$

Similarly, we have $\sigma \in W_{\mathcal{E}}^-$ if $i_\mathcal{E}[2]$ is semistable of the same phase ϕ as $k(x)$, and $i_*\mathcal{E}[2]$ is a quotient of $k(x)$ in $\mathcal{P}(\phi)$. At a general point in $W_{\mathcal{E}}^-$, the Jordan-Hölder filtration of $k(x)$ is given by*

$$(9) \quad \text{ST}_{\mathcal{E}}^{-1}(\mathcal{E}^x[1]) \rightarrow k(x) \rightarrow i_*\mathcal{E}^{\oplus r}[2].$$

- (b) *We have $W_{\mathcal{E}}^+ = \bar{U} \cap \text{ST}_{\mathcal{E}}(\bar{U})$, i.e. $W_{\mathcal{E}}^+$ is the wall between U and $\text{ST}_{\mathcal{E}}(U)$; similarly, $W_{\mathcal{E}}^- = \bar{U} \cap \text{ST}_{\mathcal{E}}^{-1}(\bar{U})$.*

There are no other walls in ∂U .

Corollary 5.2. *The translates of \bar{U} under the group of autoequivalences generated by $\text{ST}_{\mathcal{E}}$ cover the whole connected component $\text{Stab}^{\dagger}(\mathcal{D}_0)$ of U in the space of stability conditions.*

Proof. Let $\alpha: [0, 1] \rightarrow \text{Stab}(\mathcal{D}_0)$ be a path of stability conditions with $\alpha(0) \in U$. By Proposition 3.3, there exists a finite set of walls $W_i^{[k(x)]}$ intersecting α , such that the set of stable objects of class $[k(x)]$ is constant in the complement of the intersection points. We may also assume that α intersects each wall transversely and in a generic point of the wall. Using the above theorem, it follows by induction that every open interval in the complement is contained in the translate of U under a sequence of spherical twists. \square

Most of the existing proofs of statements similar to the above claims are in the situation of a Calabi-Yau 2-category. In that situation, Lemma 5.2 of [Bri08] applies, which guarantees, via an Euler characteristic computation, that every non-trivial Harder-Narasimhan filtration of a skyscraper sheaf $k(x)$ contains a spherical object. In a Calabi-Yau 3-category, spherical objects cannot be characterized via their Euler characteristic among stable objects. Instead, our proof is obtained by a direct geometric analysis of the boundary of U .

Consider a stability condition (Z, \mathcal{P}) in the boundary ∂U of U . Since the skyscraper sheaves $k(x)$ are semi-stable, Z satisfies $Z(k(x)) \neq 0$ and, up to the action of \mathbb{C} , we can still assume $k(x) \in \mathcal{P}(1)$ and $Z = Z_{a,b}$ for $a, b \in \mathbb{C}$ as in equation (4). By Bridgeland's deformation result and Theorem 2.5, $a, b \in \mathbb{C}$ must satisfy one of the following conditions:

Case a : $\Im a = 0$

Case E : $\Im a > 0$, there exists an exceptional vector bundle of slope $B = -\frac{\Re a}{\Im b}$, and we have $\delta_\infty^{DP}(B) > -\Re b - B \cdot \Re a + \frac{1}{2}B^2 > \Delta_B$.

Case δ_∞^{DP} : $\Im a > 0$ and $-\Re b - B \cdot \Re a + \frac{1}{2}B^2 = \delta_\infty^{DP}(B)$.

We begin by showing that ‘‘Case δ_∞^{DP} ’’ cannot exist.

Lemma 5.3. *Let $\sigma_t = (\mathcal{P}_t, Z_t)$ for $t \in I \subset \mathbb{R}$ be a path in the space of stability conditions such that $\Im Z_t$ is constant. Then $\mathcal{P}_t((0, 1])$ and $\mathcal{P}_t(1)$ are constant, too.*

Proof. Let $t_1, t_2 \in I$ be such that σ_{t_1} and σ_{t_2} are close with respect to the metric on $\text{Stab}(\mathcal{D}_0)$; to be specific, we assume $d(\sigma_{t_1}, \sigma_{t_2}) < \frac{1}{8}$. Given $\phi \in (0, 1]$, the objects $E \in \mathcal{P}_{t_2}(\phi)$ can be characterized as the Z_{t_2} -stable objects in the quasi-abelian category $\mathcal{P}_{t_1}((\phi - \frac{1}{8}, \phi + \frac{1}{8}))$ (see [Bri07, Section 7]). We want to show $E \in \mathcal{P}_{t_1}((0, 1])$.

In case $\frac{1}{8} \leq \phi \leq \frac{7}{8}$ we are done. If $\phi \in (\frac{7}{8}, 1]$, then for any $A \in \mathcal{P}_{t_1}((1, \phi + \frac{1}{8}))$ we have $\Im Z_{t_2}(A) = \Im Z_{t_1}(A) < 0$, and thus the phase of A with respect to Z_{t_2} is bigger than one. Hence A cannot be a subobject of E . By considering the Harder-Narasimhan filtration of E with respect to \mathcal{P}_{t_1} , this implies that $E \in \mathcal{P}_{t_1}((\phi - \frac{1}{8}, 1])$. A similar argument applies for $0 < \phi < \frac{1}{8}$.

It follows that $\mathcal{P}_{t_2}((0, 1]) \subset \mathcal{P}_{t_1}((0, 1])$, and thus they must be equal. The claim about $\mathcal{P}_t(1)$ follows easily. \square

Lemma 5.4. *There are no stability conditions $\sigma = (Z_{a,b}, \mathcal{P})$ in ∂U such that $\Im a > 0$ and $-\Re b - B \cdot \Re a + \frac{1}{2}B^2 = \delta_\infty^{DP}(B)$.*

Proof. Due to Corollary 3.5, we may assume that B is irrational.

Consider the path $\sigma_t = (Z_{a(t), b(t)}, \mathcal{P}_t)$, $t \in [0, 1]$ in $\text{Stab}(\mathcal{D}_0)$ starting at σ induced by deforming Z as $Z_{a(t), b(t)}$, with $a(t) = a$ constant, and $b(t) =$

$b - \epsilon t$. Due to the shape of the boundary of $\mathcal{Z}(U)$ around a, b , the stability conditions σ_t for $t > 0$ are geometric.

By the previous lemma, it follows that $\mathcal{P}_0((0, 1]) = \mathcal{P}_1((0, 1]) = \text{Coh}_0^{\sharp(B)}$, and that $\mathcal{P}_0(1)$ is generated by the skyscraper sheaves $k(x)$. The sheaves $k(x)$ have no subobjects in this category, thus they are stable. By Corollary 3.5, this contradicts the assumption that σ is in the boundary of U . \square

The following lemma deals with “case a ”:

Lemma 5.5. *Let $(Z_{a,b}, \mathcal{P}) \in \partial U$ be a stability condition with $\Im a = 0$. Then $\Im b = 0$, i.e. the image of Z is contained in the real line.*

Proof. Writing inequality (2) without denominators we get

$$(\Im a)^2 \Re b > \Im a \cdot \Im b \cdot \Re a - (\Im a)^2 \delta_\infty^{DP}(B) + \frac{1}{2}(\Im b)^2$$

By continuity, this implies $(\Im b)^2 \leq 0$ on the boundary with $\Im a = 0$. \square

In particular, the part of the boundary with $\Im a = 0$ is a codimension two subset; by Corollary 3.5, it must be contained in the closure of the remaining part of the boundary.

We will now consider boundary stability conditions in “Case E ”. Let a, b, B be as in the assumption, and let \mathcal{E} be the exceptional vector bundle on \mathbb{P}^2 of slope B ; then $Z_{a,b}(i_*\mathcal{E}) \in (-1, 0)$ and $Z_{a,b}(i_*\mathcal{G}) \in \mathbb{R}_{>0}$ for any other slope-stable sheaf \mathcal{G} on \mathbb{P}^2 of slope B . This suggests that $i_*\mathcal{E}$ is semi-stable, of phase $\phi(i_*\mathcal{E}) = \pm 1$ (depending on whether $Z(i_*\mathcal{E})$ approaches the real line from above or below when we approach $Z_{a,b}$ by geometric stability conditions); and in the case $\phi(i_*\mathcal{E}) = +1$ the t -structure $\mathcal{P}((0, 1])$ should be given as in the following proposition.

We will prove this by constructing the stability conditions in the boundary directly, and prove that they deform to geometric stability conditions.

Proposition 5.6. *Let B be the slope of an exceptional vector bundle \mathcal{E} on \mathbb{P}^2 . Then there is a torsion pair $(\mathcal{T}^\mathcal{E}, \mathcal{F}^\mathcal{E})$ on Coh_0 where*

- $\mathcal{T}^\mathcal{E}$ is the extension-closed subcategory of Coh_0 generated by torsion sheaves, by slope-semistable sheaves of slope $\mu > B$, and by \mathcal{E} , and
- $\mathcal{F}^\mathcal{E}$ is generated by slope-semistable sheaves \mathcal{G} of slope $\mu \leq B$ that also satisfy $\text{Hom}(i_*\mathcal{E}, \mathcal{G}) = 0$.

Proof. From the construction, it is clear that $\text{Hom}(\mathcal{T}^\mathcal{E}, \mathcal{F}^\mathcal{E}) = 0$.

Now given any $\mathcal{G} \in \text{Coh}_0$, let $\mathcal{G}_{>B} \hookrightarrow \mathcal{G} \rightarrow \mathcal{G}_{\leq B}$ the unique short exact sequence with $\mathcal{G}_{>B} \in \text{Coh}_0^{>B}$ and $\mathcal{G}_{\leq B} \in \text{Coh}_0^{\leq B}$. Let $V = \text{Hom}(i_*\mathcal{E}, \mathcal{G}_{\leq B})$ and let \mathcal{F} be the cokernel such that the following sequence is exact on the right:

$$V \otimes i_*\mathcal{E} \xrightarrow{f} \mathcal{G}_{\leq B} \rightarrow \mathcal{F} \rightarrow 0$$

We claim that $\mathcal{F} \in \mathcal{F}^\mathcal{E}$, and that the kernel \mathcal{T} of the composition $\mathcal{G} \rightarrow \mathcal{G}_{<B} \rightarrow \mathcal{F}$ lies in $\mathcal{T}^\mathcal{E}$.

The image of the evaluation map f is slope-semistable of slope B ; hence so is the kernel of f . Since $i_*\mathcal{E}$ is stable, the kernel is of the form $i_*\mathcal{E} \otimes V'$ for some $V' \subset V$; by the definition of V , this forces $V' = 0$, i.e. the above sequence is exact. This shows that \mathcal{T} is an extension of $\mathcal{G}_{>B}$ and $i_*\mathcal{E}$.

Since $\mathrm{Hom}(i_*\mathcal{E}, i_*\mathcal{E}) = \mathbb{C}$ and $\mathrm{Ext}^1(i_*\mathcal{E}, i_*\mathcal{E}) = 0$, the long exact sequence associated to $\mathrm{Hom}(i_*\mathcal{E}, _)$ shows that $\mathrm{Hom}(i_*\mathcal{E}, \mathcal{F}) = 0$, and thus $\mathcal{F} \in \mathcal{F}^\mathcal{E}$ as desired. \square

A similar result as Proposition 5.6 is also in [Yos09, Prop. 2.7].

We continue to assume that $a, b \in \mathbb{C}$ satisfy $B = -\frac{\Im a}{\Im b}$ and the inequality of boundary ‘‘Case E ’’. Let $\mathrm{Coh}_0^\mathcal{E} = \langle \mathcal{T}^\mathcal{E}, \mathcal{F}^\mathcal{E}[1] \rangle$ be the t -structure given by tilting at the torsion pair of Proposition 5.6. From the previous discussion, it follows that $Z_{a,b}$ is a stability function for $\mathrm{Coh}_0^\mathcal{E}$.

Proposition 5.7. *Harder-Narasimhan filtrations exist for the stability function $Z_{a,b}$ on $\mathrm{Coh}_0^\mathcal{E}$.*

Proof. The proof is similar to Lemma 4.3. Indeed, since $B = -\frac{\Im b}{\Im a}$ is the slope of \mathcal{E} , it is rational; hence the imaginary part of $Z_{a,b}$ is discrete, and we can again apply Proposition B.2. The only difference with the case treated in Lemma 4.3 is that, for an object A in $\mathcal{P}'_{a,b}(1)$, $\mathcal{H}^0(A)$ is isomorphic to an extension of $i_*\mathcal{E}^a$ by a torsion sheaf of dimension zero. To deal with this difference, we replace all arguments using the length of a maximal zero-dimensional subsheaf by using the function

$$e: \mathrm{Coh}_0 \rightarrow \mathbb{Z}, \quad e(\cdot) := \dim(\mathrm{Hom}(i_*\mathcal{E}, \cdot)).$$

In many respects, it has the same formal properties needed (e.g. subadditivity on short exact sequences), and the proof goes through:

Assume that $F \in \mathrm{Coh}_0^\mathcal{E}$ has an infinite sequence

$$(10) \quad 0 = A_0 \subseteq A_1 \subseteq \dots \subseteq A_j \subseteq A_{j+1} \subseteq \dots \subseteq F,$$

of subobjects with $A_j \in \mathcal{P}'_{a,b}(1)$. Denote by B_j the cokernel in $\mathrm{Coh}_0^\mathcal{E}$ of $A_j \hookrightarrow F$.

By arguing as in Lemma 4.3, we can assume $\mathcal{H}^{-1}(A_j) = \mathcal{F}_{-1}$ and $\mathcal{H}^0(B_j) = \mathcal{G}_1$, for some $\mathcal{F}_{-1}, \mathcal{G}_1 \in \mathrm{Coh}_0$ and for all j . Let \mathcal{F}_0 be the cokernel in Coh_0 of $\mathcal{F}_{-1} \hookrightarrow \mathcal{H}^{-1}(F)$ and let \mathcal{G}_0 be the kernel of $\mathcal{H}^0(F) \rightarrow \mathcal{G}_1$. Then we have an exact sequence

$$0 \rightarrow \mathcal{F}_0 \xrightarrow{g} \mathcal{H}^{-1}(B_j) \rightarrow \mathcal{H}^0(A_j) \xrightarrow{f} \mathcal{G}_0 \rightarrow 0,$$

and we let $\mathcal{K}_j = \ker f = \mathrm{cok} g$. By the long exact Hom-sequences, we have $e(\mathcal{H}^0(A_j)) \leq e(\mathcal{K}_j) + e(\mathcal{G}_0) \leq e(\mathcal{H}^{-1}(B_j)) + \dim \mathrm{Ext}^1(i_*\mathcal{E}, \mathcal{F}_0) + e(\mathcal{G}_0)$.

By definition of $\mathcal{F}^\mathcal{E}$, we have $\mathrm{Hom}(i_*\mathcal{E}, \mathcal{H}^{-1}(B_j)) = 0$, and thus $e(\mathcal{H}^0(A_j))$ is bounded.

Now consider a filtration step of (10), and let D_j be the cokernel in $\mathrm{Coh}_0^\mathcal{E}$ of $A_j \hookrightarrow A_{j+1}$. From the exact sequence

$$0 \rightarrow \mathcal{H}^{-1}(D_j) \rightarrow \mathcal{H}^0(A_j) \rightarrow \mathcal{H}^0(A_{j+1}) \xrightarrow{\phi} \mathcal{H}^0(D_j) \rightarrow 0,$$

we argue similarly as before:

$$e(\ker \phi) = e(\mathcal{H}^{-1}(D_j)) + e(\ker \phi) \geq e(\mathcal{H}^0(A_j))$$

On the other hand, as $\mathcal{T}^\mathcal{E}$ is closed under quotients, we have $\ker \phi \in \mathcal{T}^\mathcal{E}$, and thus $\ker \phi \in \mathcal{P}'_{a,b}(1)$ and $\mathrm{Ext}^1(i_*\mathcal{E}, \ker \phi) = 0$. Thus we have

$$e(\mathcal{H}^0(A_{j+1})) = e(\ker \phi) + e(\mathcal{H}^0(D_j)) \geq e(\ker \phi) \geq e(\mathcal{H}^0(A_j))$$

with equality only if $\mathcal{H}^0(D_j) = 0$. By the boundedness established above, we do have $\mathcal{H}^0(D_j) = 0$ for $j \gg 0$; but then $\mathcal{H}^0(A_j) \twoheadrightarrow \mathcal{H}^0(A_{j+1})$ stabilizes as Coh_0 is Noetherian. \square

We denote by $\mathcal{W}_\mathcal{E}^+$ the set of stability conditions constructed in the above proposition.

Proposition 5.8. *For any $\sigma \in \mathcal{W}_\mathcal{E}^+$, the skyscraper sheaves $k(x)$ are σ -semistable with Jordan-Hölder filtration given as in Theorem 5.1. Their images $\mathrm{ST}_\mathcal{E}(k(x))$ under the spherical twist at $i_*\mathcal{E}$ are also σ -semistable, of the same phase as $k(x)$, with Jordan-Hölder filtration given by*

$$(11) \quad \mathcal{E}^x[1] \rightarrow \mathrm{ST}_\mathcal{E}(k(x)) \rightarrow i_*\mathcal{E}^{\oplus r}.$$

Proof. The sheaves \mathcal{E}^x are slope-semistable of the same slope as \mathcal{E} , and the long exact Hom-sequence shows $\mathrm{Hom}(\mathcal{E}, \mathcal{E}^x) = 0$. Hence $\mathcal{E}^x \in \mathcal{F}^E$, and $\mathcal{E}^x[1] \in \mathcal{P}(1)$, and we indeed have a short exact sequence as in (8) in $\mathcal{P}(1)$. We claim that $\mathcal{E}^x[1]$ is $Z_{a,b}$ -stable, i.e. we will show that there are no non-trivial short exact sequences $A \hookrightarrow \mathcal{E}^x[1] \twoheadrightarrow B$ in the abelian category $\mathcal{P}(1)$:

Let C denote the kernel of the composition $k(x) \twoheadrightarrow \mathcal{E}^x[1] \rightarrow B$ in $\mathcal{P}(1)$. By the long exact cohomology sequence, C is isomorphic to a sheaf \mathcal{C} . As observed in the proof of Proposition 5.7, we have $\mathcal{C} \cong \mathcal{E}^k \oplus \mathcal{T}$, where \mathcal{T} is a zero-dimensional torsion sheaf. Using the octahedral axiom (or the 9-lemma, according to the reader's taste) for the composition $\mathcal{E}^r \hookrightarrow \mathcal{C} \hookrightarrow k(x)$, we obtain an exact sequence $\mathcal{E}^r \hookrightarrow \mathcal{C} \twoheadrightarrow A$. On the other hand, by the long exact Hom-sequence applied to $A \hookrightarrow \mathcal{E}^x[1] \twoheadrightarrow B$, we have $\mathrm{Hom}(\mathcal{E}, A) = 0$, and thus $\mathrm{Hom}(\mathcal{E}, \mathcal{C}) \cong \mathbb{C}^r$. As r divides $\dim \mathrm{Hom}(\mathcal{E}, \mathcal{T})$, this is only possible if $\mathcal{C} \cong \mathcal{E}^r$, or $\mathcal{C} \cong k(x)$; in either case the short exact sequence is trivial as claimed.

By using adjunction one sees that $\mathrm{RHom}(i_*E, k(x)) = \mathbb{C}^r \oplus \mathbb{C}^r[-1]$. The long exact cohomology sequence shows $\mathcal{H}^0(\mathrm{ST}_\mathcal{E}(k(x))) \cong E^{\oplus r}$ and

$\mathcal{H}^{-1}(\mathrm{ST}_\varepsilon(k(x))) = E^x$, and so there is an exact triangle as in (11). This shows $\mathrm{ST}_\varepsilon(k(x)) \in \mathcal{P}(1)$, and that (11) is a Jordan-Hölder filtration. \square

We can deform Z such that $Z(k(x)) = -1$ remains constant, $Z(i_*E)$ moves to the upper half-plane, and $Z(E^x[1])$ moving to the lower-half plane; then by Lemma 5.9, all $k(x)$ become stable. It follows that the closure W_ε^+ of the orbit of $\mathcal{W}_\varepsilon^+$ under the action of \mathbb{C} is a wall of ∂U . The objects $k(x)$ and $\mathrm{ST}_\varepsilon(k(x))$ become stable on opposite sides of the wall, and thus $W_\varepsilon^+ \in \overline{U} \cap \mathrm{ST}_\varepsilon(\overline{U})$. If we apply $\mathrm{ST}_\varepsilon^{-1}$ to W_ε^+ , we obtain a wall where the Jordan-Hölder filtration of $k(x)$ is given by the image of (11) under $\mathrm{ST}_\varepsilon^{-1}$, which is indeed the exact triangle (9).

This finishes the proof of Theorem 5.1. Note that the proof also implies that two such walls can only intersect at points where the image of the central charge is contained in a line $e^{i\pi\phi} \cdot \mathbb{R} \subset \mathbb{C}$. In that case, the heart $\mathcal{P}(\phi)$ of the associated t-structure has finite length; in fact, it is one of the “quiver” stability conditions constructed in [Bri06], and used in the following section.

Lemma 5.9. *Let $E \in \mathcal{D}_0$ and $\sigma \in \mathrm{Stab}^\dagger(\mathcal{D}_0)$ be a stability condition such that E is σ -semistable, and assume that there is a Jordan-Hölder filtration $A^r \hookrightarrow E \twoheadrightarrow B$ of E such that A, B are σ -stable, $\mathrm{Hom}(E, A) = 0$, and $[E]$ and $[A]$ are linearly independent classes in $K(\mathcal{D}_0)$. Then σ is in the closure of the set of stability conditions where E is stable.*

Proof. Let ϕ be the phase of A, E, B with respect to $\sigma = (Z, \mathcal{P})$. By similar arguments as in the proof of Proposition 3.3, we can show that for any stability condition σ' sufficiently close to σ , E can only be destabilized by subobjects $F \hookrightarrow E$ in $\mathcal{P}(\phi)$. Now let $\sigma' = (Z', \mathcal{P}')$ be such a stability condition close by with A, B stable and $\phi'(A) < \phi'(E) < \phi'(B)$. Assume that $F \in \mathcal{P}(\phi)$ is a stable destabilizing subobject of E with respect to Z' . If the image of the composition $F \rightarrow B$ is zero, then F factors via A , hence $\phi'(F) < \phi'(A)$. So $F \rightarrow B$ must be surjective; its kernel $G \in \mathcal{P}(\phi)$ is a subobject of A^r , and thus of the form A^k for some $k < r$. Hence the quotient of $F \hookrightarrow E$ is isomorphic to A^{r-k} , in contradiction to $\mathrm{Hom}(E, A) = 0$. \square

6. ALGEBRAIC STABILITY CONDITIONS

In this section we study the open subset Stab_a , introduced by Bridgeland in [Bri06], consisting of *algebraic stability conditions*. We first introduce open subsets $\Theta_\mathfrak{S}$, associated to a collection of spherical objects \mathfrak{S} , and study their boundary in $\mathrm{Stab}^\dagger(\mathcal{D}_0)$. The subset of algebraic stability conditions will then be the union of all $\Theta_\mathfrak{S}$. Then we study in detail the relation between Stab_a and U . In particular, we show that Stab_a contains the

boundary of U (described in the previous section) and, vice versa, that the intersection of U with Stab_a is strictly contained in U . In the next section we will apply all of this to prove Theorem 2.

Let $\mathfrak{E} = \{\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2\}$ be an exceptional collection of vector bundles on \mathbb{P}^2 . Then the subcategory of \mathcal{D}_0 generated by extensions of $i_*\mathcal{E}_0[2]$, $i_*\mathcal{E}_1[1]$, and $i_*\mathcal{E}_2$

$$\mathcal{A}_{\mathfrak{E}} := \langle i_*\mathcal{E}_2, i_*\mathcal{E}_1[1], i_*\mathcal{E}_0[2] \rangle$$

is the heart of a bounded t -structure on \mathcal{D}_0 . By [Bri05b], the category $\mathcal{A}_{\mathfrak{E}}$ can also be described as the category of nilpotent modules over a certain algebra.

Definition 6.1. *A heart of a bounded t -structure on \mathcal{D}_0 is called quivery if it is of the form $\Phi(\mathcal{A}_{\mathfrak{E}})$, for some exceptional collection \mathfrak{E} of vector bundles on \mathbb{P}^2 and for some autoequivalence Φ of \mathcal{D}_0 given by composition of spherical twists associated to exceptional vector bundles.*

Notice that a quivery subcategory is of finite length, with simple objects $\Phi(i_*\mathcal{E}_2)$, $\Phi(i_*\mathcal{E}_1[1])$, and $\Phi(i_*\mathcal{E}_0[2])$, which are also spherical in \mathcal{D}_0 . A quivery subcategory is called *ordered* if it comes with an ordering of S_0, S_1, S_2 of its simple objects compatible with the requirement that $\text{Hom}^k(S_j, S_l) = 0$ unless $0 \leq k \leq 3$ and $j - l \equiv k \pmod{3}$. A collection $\mathfrak{S} = \{S_0, S_1, S_2\}$ of spherical objects of \mathcal{D}_0 is called an *ordered quivery collection* if it arises as an ordered collection of simple objects in an ordered quivery subcategory $\mathcal{A}_{\mathfrak{S}}$.

By [Bri05b, Thm. 4.11], we can define an action on the set of quivery ordered subcategories of \mathcal{D}_0 of the affine braid group B_3 , i.e., the group generated by elements τ_j ($j \in \mathbb{Z}_3$) and r subject to the relations

$$r\tau_j r^{-1} = \tau_{j+1}, \quad \tau_j \tau_{j+1} \tau_j = \tau_{j+1} \tau_j \tau_{j+1}, \quad r^3 = 1.$$

Indeed, to define such an action is sufficient to set how the generators of B_3 act on the simple objects of a ordered quivery category:

$$\begin{aligned} \tau_1\{S_0, S_1, S_2\} &:= \{S_1[-1], \text{ST}_{S_1}(S_0), S_2\} \\ r\{S_0, S_1, S_2\} &:= \{S_2, S_0, S_1\}. \end{aligned}$$

By [Bri05b, Prop. 4.10], the image via τ_j of a quivery category is quivery as well, and thus the action is well-defined. Notice, in particular, that

$$\tau_2\{S_0, S_1, S_2\} = \{S_0, S_2[-1], \text{ST}_{S_2}(S_1)\}.$$

Remark 6.2. *Let $\mathfrak{E} = \{\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2\}$ be an exceptional collection of vector bundles on \mathbb{P}^2 . Then*

$$\tau_1\{i_*\mathcal{E}_2, i_*\mathcal{E}_1[1], i_*\mathcal{E}_0[2]\} = \{i_*\mathcal{F}_2, i_*\mathcal{F}_1[1], i_*\mathcal{F}_0[2]\},$$

where $\mathfrak{F} = \{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2\}$ is another exceptional collection of vector bundles on \mathbb{P}^2 , called the left mutation of \mathfrak{E} at \mathcal{E}_1 (see [GR87, Bon89]). Similarly, for τ_2 we have the left mutation of \mathfrak{E} at \mathcal{E}_0 , for τ_1^{-1} we have the right mutation of \mathfrak{E} at \mathcal{E}_2 , and for τ_2^{-1} we have the right mutation of \mathfrak{E} at \mathcal{E}_1 . Since all exceptional collections of vector bundles on \mathbb{P}^2 can be obtained by a sequence of mutations from $\mathfrak{E}_1 := \{i_*\mathcal{O}_{\mathbb{P}^2}(-1), i_*\Omega_{\mathbb{P}^2}(1), i_*\mathcal{O}_{\mathbb{P}^2}\}$, all ordered quivery subcategories can be obtained from

$$\mathcal{A}_1 := \mathcal{A}_{\mathfrak{E}_1} = \langle i_*\mathcal{O}_{\mathbb{P}^2}, i_*\Omega_{\mathbb{P}^2}(1)[1], i_*\mathcal{O}_{\mathbb{P}^2}(-1)[2] \rangle$$

by the action of B_3 .

Definition 6.3. A stability condition σ on \mathcal{D}_0 is called algebraic if there exists $M \in \widetilde{\text{GL}}_2(\mathbb{R})$ such that the heart of $\sigma \cdot M$ is quivery. Denote by Stab_a the subset of $\text{Stab}(\mathcal{D}_0)$ consisting of algebraic stability conditions.

Using [Bri05b, Prop. 4.10] and [Mac07, Cor. 3.20] it follows that Stab_a is an open connected 3-dimensional submanifold of $\text{Stab}(\mathcal{D}_0)$. Moreover, it is easy to construct stability conditions in Stab_a in which the skyscraper sheaves are all stable (for example, a stability condition with heart \mathcal{A}_1 in which $\phi(i_*\mathcal{O}_{\mathbb{P}^2}) < \phi(i_*\Omega_{\mathbb{P}^2}(1)[1]) < \phi(i_*\mathcal{O}_{\mathbb{P}^2}(-1)[2])$). Hence $\text{Stab}_a \subset \text{Stab}^\dagger(\mathcal{D}_0)$, but the inclusion is strict (this can be deduced from Proposition 6.11 and Remark 6.12). Finally, by its own definition, Stab_a is invariant under the subgroup of the autoequivalences of \mathcal{D}_0 which is generated by spherical twists $\text{ST}_{\mathcal{F}}$, with \mathcal{F} an exceptional bundle on \mathbb{P}^2 .

Definition 6.4. Let \mathfrak{S} be an ordered quivery collection. We denote by $\Theta_{\mathfrak{S}}$ the open subset of Stab_a consisting of stability conditions whose heart is, up to the action of $\widetilde{\text{GL}}_2(\mathbb{R})$, equivalent to $\mathcal{A}_{\mathfrak{S}}$. With a slight abuse of notation, when \mathfrak{E} is an exceptional collection of vector bundles on \mathbb{P}^2 , we denote by $\Theta_{\mathfrak{E}}$ the open subset of Stab_a consisting of stability conditions whose heart is, up to the action of $\widetilde{\text{GL}}_2(\mathbb{R})$, equivalent to $\mathcal{A}_{\mathfrak{E}}$.

Lemma 6.5. The region $\Theta_{\mathfrak{S}} \subset \text{Stab}(\mathcal{D}_0)$ is characterized as the subset where S_0, S_1, S_2 are stable, and where their phases $\phi_j := \phi(S_j)$ satisfy

$$(12) \quad |\phi_j - \phi_{j+1}| < 1 \quad \text{for } j = 0, 1, 2.$$

It is homeomorphic to

$$\mathcal{C}_{\mathfrak{S}} = \{(m_0, m_1, m_2, \phi_0, \phi_1, \phi_2) \in \mathbb{R}^6 : m_j > 0 \text{ and (12) holds for all } j\}$$

Proof. Evidently S_j are stable in $\Theta_{\mathfrak{S}}$, and satisfy equation (12). Conversely, if S_j are stable in (Z, \mathcal{P}) satisfying equation (12), then for $\phi := \min \phi_j$ we have $\mathcal{A}_{\mathfrak{S}} \subset \mathcal{P}((\phi, \phi + 1])$, thus $\mathcal{A}_{\mathfrak{S}} = \mathcal{P}((\phi, \phi + 1])$ and $(Z, \mathcal{P}) \in \Theta_{\mathfrak{S}}$. \square

Notice that, for later use, $\mathrm{ST}_{S_j}^{-1}(S_{j+1})$ is an extension of S_{j+1} by m copies of S_j , where $m = \dim \mathrm{Hom}^1(S_{j+1}, S_j)$. Hence, its class in $K(\mathcal{D}_0)$ is given by

$$[\mathrm{ST}_{S_{j+1}}(S_j)] = m[S_j] + [S_{j+1}].$$

Moreover, it belongs to $\mathcal{A}_{\mathfrak{S}}$ and, if $\phi(S_{j+1}) > \phi(S_j)$ then it is also σ -stable. A similar observation holds true for $\mathrm{ST}_{S_{j+1}}(S_j)$.

The next proposition generalizes [Bri06, Theorem 1.1].

Proposition 6.6. *Let $\mathfrak{S} = \{S_0, S_1, S_2\}$ be an ordered quivery collection. Then the closure $\overline{\Theta}_{\mathfrak{S}}$ of $\Theta_{\mathfrak{S}}$ in $\mathrm{Stab}^\dagger(\mathcal{D}_0)$ is contained in Stab_a .*

Proof. Let $\bar{\sigma} = (\overline{Z}, \overline{P}) \in \overline{\Theta}_{\mathfrak{S}} \setminus \Theta_{\mathfrak{S}}$ be the limit of a sequence $\{\sigma_s\}_{s \in \mathbb{N}}$, with $\sigma_s \in \Theta_{\mathfrak{S}}$. Then S_0, S_1 , and S_2 are $\bar{\sigma}$ -semistable; up to the action of \mathbb{C} , we have the following possibilities for their phases:

- (a) The image of \overline{Z} is a line in the plane.
- (b) The image of \overline{Z} is not a line in the plane and there exists j such that $0 = \overline{\phi}(S_j) = \overline{\phi}(S_{j+1}) - 1$ (here and in the sequel all the indices are taken modulo 3).
- (c) The image of \overline{Z} is not a line in the plane and there exists j such that $0 = \overline{\phi}(S_j) = \overline{\phi}(S_{j-1}) - 1$.

We begin with case (b). First of all notice that $0 < \overline{\phi}(S_{j-1}) < 1$. Moreover, up to the action of \mathbb{C} , we can assume every σ_s has heart $\mathcal{A}_{\mathfrak{S}}$.

Let P_m be the Kronecker quiver with $m = \dim \mathrm{Hom}^1(S_{j+1}, S_j) > 0$, i.e. the quiver with two vertices and m arrows from the first to the second vertex. Consider the faithful functor $I: \mathrm{D}^b(P_m) \rightarrow \mathrm{Tr}(S_j, S_{j+1}) \subseteq \mathcal{D}_0$, which maps the two simple quiver representations of P_m corresponding to the two vertices respectively to S_{j+1} and S_j . Here $\mathrm{Tr}(S_j, S_{j+1})$ denotes the triangulated subcategory of \mathcal{D}_0 generated by S_j and S_{j+1} . For $s \gg 0$, the stability condition σ_s induces a stability condition on $\mathrm{Tr}(S_j, S_{j+1})$, whose heart is the abelian category generated by extensions by S_j and S_{j+1} . Now the functor I restricted to $\mathrm{mod}\text{-}P_m$ is full and faithful. By [MMS09, Prop. 2.12], σ_s induces a stability condition $I^{-1}\sigma_s$ in $\mathrm{D}^b(P_m)$. Hence, by [MMS09, Lemma 2.9], $I^{-1}\bar{\sigma} \in \mathrm{Stab}(P_m)$. By [Mac07, Lemma 4.2], there exists an integer $k \in \mathbb{Z}$ such that $\bar{\sigma} \in \Theta_{\tau_{j+1}^k \mathfrak{S}}$. More explicitly, if $\tau_{j+1}^k \mathfrak{S} = \{R_0, R_1, R_2\}$, then what we proved is that R_j are stable with respect to $\bar{\sigma}$, and that we have Jordan-Hölder filtrations given by $S_{j-1} = R_{j-1}$ and

$$(13) \quad \begin{aligned} R_j^{\oplus u_j} &\rightarrow S_j[\epsilon] \rightarrow R_{j+1}^{\oplus v_j} \\ R_j^{\oplus u_{j+1}} &\rightarrow S_{j+1}[\epsilon - 1] \rightarrow R_{j+1}^{\oplus v_{j+1}}, \end{aligned}$$

where $\epsilon = 0, 1$ according to k .

For possibility (c), we have similarly $0 < \bar{\phi}(S_{j+1}) < 1$. Then $\bar{\sigma} \in \Theta_{\tau_j \mathfrak{E}}$ and S_0, S_1 , and S_2 remain stable in $\bar{\sigma}$.

Finally, if the image of Z lies in a line (case (a)), then we can deform $\bar{\sigma}$ in $\bar{\Theta}_{\mathfrak{E}}$ in such a way to reduce to the situation of case (b). We can apply the previous procedure and find $g_1 \in B_3$ such that $\bar{\sigma} \in \bar{\Theta}_{g_1 \mathfrak{E}}$. If $\bar{\sigma} \in \Theta_{g_1 \mathfrak{E}}$ we have finished the proof. Assume not. Then we continue and again deform $\bar{\sigma}$ in $\bar{\Theta}_{g_1 \mathfrak{E}}$ to reduce again to case (b). We produce a new element $g_2 \in B_3$ and so on. This procedure must eventually terminate at a step N : indeed at every step, by (13), we are constructing a filtration of S_0, S_1 , and S_2 into $\bar{\sigma}$ -semistable objects of the same phase. But $\bar{\sigma}$ is locally-finite. Hence at a certain point we produce a stable factor and so $\bar{\sigma} \in \Theta_{g_N \mathfrak{E}}$, as wanted. \square

We can now study the relation of Stab_a with U .

Lemma 6.7. *Let \mathfrak{E} be an exceptional collection of vector bundles on \mathbb{P}^2 . Then $\Theta_{\mathfrak{E}} \cap U \neq \emptyset$ and it is connected.*

Proof. First of all notice that all skyscraper shaves $k(x)$ for $x \in \mathbb{P}^2$ belong to $\mathcal{A}_{\mathfrak{E}}$. Consider the stability condition $\bar{\sigma} \in \Theta_{\mathfrak{E}}$ with heart $\mathcal{A}_{\mathfrak{E}}$, whose simple objects have phases

$$\bar{\phi}(i_* \mathcal{E}_2) = \bar{\phi}(i_* \mathcal{E}_1[1]) = \bar{\phi}(i_* \mathcal{E}_0[2]) = 1.$$

Let \mathcal{E}_2^x be the kernel of $\mathcal{E}_2^{\oplus r} \rightarrow k(x)$ as in Section 5; then $\mathcal{E}_2^x \in \mathcal{A}_{\mathfrak{E}}$, and in fact it is contained in the abelian category generated by extensions by $i_* \mathcal{E}_1[1]$ and $i_* \mathcal{E}_0[2]$. We can deform $\bar{\sigma}$ slightly to a stability condition $\sigma \in \Theta_{\mathfrak{E}}$ with

- $\phi(i_* \mathcal{E}_2) = \phi(k(x)) = \phi(\mathcal{E}_2^x)$,
- $\phi(i_* \mathcal{E}_1[1]) < \phi(i_* \mathcal{E}_0[2])$.

We claim that \mathcal{E}_2^x is σ -stable. Indeed, as in the proof of Proposition 6.6, we can consider the faithful functor $I: \text{D}^b(P_m) \rightarrow \text{Tr}(i_* \mathcal{E}_0, i_* \mathcal{E}_1)$, where $m = \dim \text{Hom}(i_* \mathcal{E}_0, i_* \mathcal{E}_1) > 0$. Then $\mathcal{E}_2^x \cong I(\tilde{\mathcal{E}}_2^x)$, and to prove that \mathcal{E}_2^x is σ -stable is equivalent to prove that $\tilde{\mathcal{E}}_2^x$ is $I^{-1}\sigma$ -stable in $\text{D}^b(P_m)$. But the stability of $\tilde{\mathcal{E}}_2^x$ follows immediately from [Kin94, Prop. 4.4].

Hence $k(x)$ is σ -semistable, and its two Jordan–Hölder factors are \mathcal{E}_2^x and $i_* \mathcal{E}_2^{\oplus r_2}$, where r_2 is the rank of \mathcal{E}_2 . By Lemma 5.9, $\sigma \in \Theta_{\mathfrak{E}} \cap \bar{U}$, and so $\Theta_{\mathfrak{E}} \cap U \neq \emptyset$ since $\Theta_{\mathfrak{E}}$ is open.

To prove connectedness, we may first use the action by \mathbb{C} to fix the phase of $k(x)$ to be 1 with $Z(k(x)) = -1$. Then every class of a subobject of $k(x)$ gives a linear inequality for the imaginary part of Z , and thus $\Theta_{\mathfrak{E}} \cap U$ is cut out by a finite number of half-spaces. \square

Corollary 6.8. *We have*

$$\text{Stab}^\dagger(\mathcal{D}_0) = \text{Stab}_a \cup \bigcup \Phi(U),$$

where the union is taken over all autoequivalences Φ of \mathcal{D}_0 which belongs to the subgroup generated by spherical twists $\text{ST}_{\mathcal{F}}$ (\mathcal{F} an exceptional vector bundle on \mathbb{P}^2).

Proof. By Corollary 5.2, we know that $\text{Stab}^\dagger(\mathcal{D}_0) = \bigcup \Phi(\overline{U})$. We only need to show that the boundary ∂U is contained in Stab_a .

Consider the boundary of type $W_{\mathcal{E}}^+$, for \mathcal{E} an exceptional vector bundle on \mathbb{P}^2 , as in Theorem 5.1. Complete \mathcal{E} to an exceptional collection $\mathfrak{E} = \{\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2 = \mathcal{E}\}$ of vector bundles on \mathbb{P}^2 . By Lemma 6.7, $\Theta_{\mathfrak{E}} \cap W_{\mathcal{E}}^+ \neq \emptyset$.

Since $\Theta_{\mathfrak{E}}$ is open, there exists a stability condition σ , whose heart $\text{Coh}_0^{\mathcal{E}}$ is given by Proposition 5.6, which is also in $\Theta_{\mathfrak{E}}$. This means that, since $i_*\mathcal{E}_0[1]$ and $i_*\mathcal{E}_1[1]$ belong to $\text{Coh}_0^{\mathcal{E}}$,

$$(14) \quad \phi(i_*\mathcal{E}_0[1]) < \phi(i_*\mathcal{E}_1[1]).$$

But this implies that (14) holds for all stability conditions in $W_{\mathcal{E}}^+$ for which the central charge is not contained in a line. Indeed, if the (14) holds, then the stability condition is in $\Theta_{\mathfrak{E}}$. If the two phases are equal, then, since $\mathcal{E}_2^x[1]$ is a quotient of $k(x)$ ($x \in \mathbb{P}^2$) and at the same time it lies in the abelian subcategory generated by $i_*\mathcal{E}_0[2]$ and $i_*\mathcal{E}_1[1]$, the central charge is contained in a line.

Hence $W_{\mathcal{E}}^+ \subseteq \Theta_{\mathfrak{E}}$ and, by Proposition 6.6, $W_{\mathcal{E}}^+ \subseteq \text{Stab}_a$. For the case of the boundary of type $W_{\mathcal{E}}^-$, simply observe that $W_{\mathcal{E}}^- = \text{ST}_{\mathcal{E}}^{-1}(W_{\mathcal{E}}^+) \subseteq \text{Stab}_a$. By Theorem 5.1, $\partial U \subseteq \text{Stab}_a$, as wanted. \square

Notice that, in the proof of Corollary 6.8, we actually showed that

$$(15) \quad \partial U \subseteq \bigcup \overline{\Theta}_{\mathfrak{E}},$$

where the union is taken over all exceptional collections of vector bundles on \mathbb{P}^2 . It follows that:

Remark 6.9. *There is a one-to-one correspondence between quivery subcategories and loci in $\text{Stab}^\dagger(\mathcal{D}_0)$ of codimension 2 where the image of the central charge is contained in a line.*

Indeed, such a degenerate stability condition must lie, up to translation by spherical twists, in the boundary ∂U . It has a unique heart (up to shifts), which must be $\mathcal{A}_{\mathfrak{E}}$ for some exceptional collection \mathfrak{E} .

Corollary 6.10. *$\text{Stab}_a \cap U$ is connected.*

Proof. Let $\mathfrak{E} = \{\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2\}$ be an exceptional collection of vector bundles on \mathbb{P}^2 . By Lemma 6.7, $\Theta_{\mathfrak{E}} \cap U$ is nonempty and connected. We first claim that we can connect in $\text{Stab}_a \cap U$ any stability condition in $\Theta_{\mathfrak{E}} \cap U$ to a stability condition in $\Theta_{\mathfrak{E}_1} \cap U$.

We proceed by induction on the length of a mutation from \mathfrak{E}_1 to \mathfrak{E} . By Remark 6.2, we need to show that a stability condition in $\Theta_{\tau_j^{\pm 1}\mathfrak{E}} \cap U$, for $j =$

1, 2, can be connected to a stability condition in $\Theta_{\mathfrak{E}} \cap U$. Let $\sigma \in \Theta_{\mathfrak{E}} \cap U$. Then there exists a continuous family $G(t) \in \widetilde{\mathrm{GL}}_2(\mathbb{R})$, $t \in \mathbb{R}$, such that $\sigma \cdot G(t) \rightarrow \bar{\sigma}$, for $t \rightarrow +\infty$, where $\bar{\sigma} \in \Theta_{\mathfrak{E}}$ is a stability condition having $\mathcal{A}_{\mathfrak{E}}$ as heart and $\bar{\phi}(i_*\mathcal{E}_2) = \bar{\phi}(i_*\mathcal{E}_1[1]) = \bar{\phi}(i_*\mathcal{E}_0[2]) = 1$. Hence $\bar{\sigma} \in \bar{U}$. By Theorem 5.1, there exist two stability conditions $\sigma_1 \in \Theta_{\mathfrak{E}} \cap \mathcal{W}_{\mathcal{E}_2}^+$ and $\sigma_2 \in \mathcal{W}_{\mathcal{E}_0}^-$. If ϕ_k denotes the phase function in σ_k ($k = 1, 2$), we must have

$$\phi_1(i_*\mathcal{E}_0[2]) > \phi_1(i_*\mathcal{E}_2) > \phi_1(i_*\mathcal{E}_1[1])$$

and

$$\phi_2(i_*\mathcal{E}_1[1]) > \phi_2(i_*\mathcal{E}_0[2]) > \phi_2(i_*\mathcal{E}_2).$$

But then $\sigma_1 \in \Theta_{\mathfrak{E}} \cap \Theta_{\tau_2^{\pm 1}\mathfrak{E}} \cap \bar{U}$ and $\sigma_2 \in \Theta_{\mathfrak{E}} \cap \Theta_{\tau_1^{\pm 1}\mathfrak{E}} \cap \bar{U}$. Since the subsets Θ are open, this is enough to conclude that $\Theta_{\mathfrak{E}} \cap \Theta_{\tau_j^{\pm 1}\mathfrak{E}} \cap U \neq \emptyset$, for $j = 1, 2$. This shows the claim.

In general, let $\sigma \in \Theta_{\mathfrak{S}} \cap U$, for an ordered quivery collection $\mathfrak{S} = \{S_0, S_1, S_2\}$. Then, proceeding as above, $\Theta_{\mathfrak{S}} \cap U$ is connected and we can find a stability condition $\bar{\sigma}$ in the closure of the $\widetilde{\mathrm{GL}}_2(\mathbb{R})$ -orbit of σ such that S_0, S_1 , and S_2 are $\bar{\sigma}$ -stable of the same phase, that is $\bar{\sigma} \in \Theta_{\mathfrak{S}} \cap \bar{U}$. But then, by (15), $\bar{\sigma} \in \bar{\Theta}_{\mathfrak{E}}$, for some exceptional collection \mathfrak{E} of vector bundles on \mathbb{P}^2 . This gives $\Theta_{\mathfrak{E}} \cap \Theta_{\mathfrak{S}} \cap U \neq \emptyset$ and this intersection is connected, which completes the proof. \square

We conclude the section by making a comparison between Stab_a and $\mathrm{Stab}^\dagger(\mathcal{D}_0)$. To this end, we define Stab_g as the set of geometric stability conditions which, up to the action of \mathbb{C} , are of the form $\sigma_{a,b}$ with

$$\begin{aligned} \Im(a) &> 0 \\ \Re(b) &> -B\Re(a) + \frac{1}{2}B^2, \end{aligned}$$

where as in Definition 2.4, $B := -\frac{\Im(b)}{\Im(a)}$. By Theorem 2.5, all pairs $(a, b) \in \mathbb{C}^2$ satisfying the above inequalities are actual stability conditions. This implies that Stab_g is an open, connected, and simply-connected subset of $\mathrm{Stab}(\mathcal{D}_0)$. Moreover, up to the action of $\widetilde{\mathrm{GL}}_2(\mathbb{R})$, we can assume the central charge of a stability condition in Stab_g to take the form (see [ABL07])

$$Z^{t,m}(-) = - \int_{\mathbb{P}^2} e^{-(t+im)h} \mathrm{ch}(-),$$

for $t, m \in \mathbb{R}$, $m > 0$, and h the class of a line in \mathbb{P}^2 . In such a case, we denote the corresponding stability condition by $\sigma^{t,m}$.

Let $\mathfrak{E} = \{\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2\}$ be an exceptional collection of vector bundles on \mathbb{P}^2 . Set $\mathrm{ch}(\mathcal{E}_j) = (r_j, d_j, c_j)$, $\mu_j := d_j/r_j$, and $\Delta_j := \frac{1}{2} \left(1 - \frac{1}{r_j^2}\right)$ ($j = 1, 2, 3$). Note that $\mu_0 < \mu_1 < \mu_2 < \mu_0 + 3$.

Proposition 6.11. *We have $\sigma = \sigma^{t,m} \in \Theta_{\mathfrak{E}} \cap \text{Stab}_g$ only if (t, m) is contained in the open semicircle with center $(C, 0)$, where*

$$(16) \quad C := \frac{1}{2}(\mu_0 + \mu_2) + \frac{\Delta_0 - \Delta_2}{\mu_2 - \mu_0}$$

and radius $R := \sqrt{\rho}$, where

$$(17) \quad \rho := \left(\frac{\Delta_0 - \Delta_2}{\mu_2 - \mu_0} \right)^2 + \frac{1}{4}(\mu_2 - \mu_0)^2 - (\Delta_0 + \Delta_2) > 0.$$

Proof. First of all, let $\sigma = \sigma^{t,m} \in \Theta_{\mathfrak{E}} \cap \text{Stab}_g$. Then, by Lemma 6.5,

- (a) $\phi(i_*\mathcal{E}_0) < \phi(i_*\mathcal{E}_1) < \phi(i_*\mathcal{E}_2)$ and
- (b) $\phi(i_*\mathcal{E}_0) + 1 = \phi(i_*\mathcal{E}_0[1]) < \phi(i_*\mathcal{E}_2)$.

As a consequence of (b), (t, m) lies in the region bounded by

$$\frac{\Im Z^{t,m}(i_*\mathcal{E}_0)}{\Re Z^{t,m}(i_*\mathcal{E}_0)} = \frac{\Im Z^{t,m}(i_*\mathcal{E}_2)}{\Re Z^{t,m}(i_*\mathcal{E}_2)},$$

Making it explicit, we have

$$m^2 + \left(t - \frac{r_0c_2 - r_2c_0}{d_2r_0 - d_0r_2} \right)^2 = -2 \frac{d_0c_2 - d_2c_0}{d_2r_0 - d_0r_2} + \left(\frac{r_0c_2 - r_2c_0}{d_2r_0 - d_0r_2} \right)^2.$$

As observed in Appendix A,

$$\frac{c_j}{r_j} = \frac{1}{2r^2} - \frac{1}{2} + \frac{\mu_j^2}{2} = -\Delta_j + \frac{\mu_j^2}{2},$$

for $j = 0, 1, 2$. Substituting we immediately deduce (16) and (17). The fact that $\rho > 0$ is again a straightforward computation, using

$$0 = \chi(\mathcal{E}_2, \mathcal{E}_0) = r_0r_2 \left(1 - \frac{3}{2}(\mu_2 - \mu_0) + \frac{1}{2}(\mu_2 - \mu_0)^2 - (\Delta_0 + \Delta_2) \right).$$

□

Using Lemma 6.7 and a deformation argument it can be proved that the statement of the previous proposition is actually an *if and only if*.

Remark 6.12. *By [GR87, Prop. 5.1], we have*

$$\frac{\Delta_0 - \Delta_2}{\mu_2 - \mu_0} = \frac{3}{2} \cdot \frac{r_0^2 - r_2^2}{r_0^2 + r_2^2 + (c_2r_0 - c_0r_2)^2} \in \left[-\frac{3}{2}, \frac{3}{2} \right].$$

Hence, if $m > 3/\sqrt{2}$, then $\sigma^{t,m} \notin \Theta_{\mathfrak{E}}$.

7. SIMPLY-CONNECTEDNESS

We can now prove the simply-connectedness of $\text{Stab}^\dagger(\mathcal{D}_0)$:

Theorem 7.1. *The connected component $\text{Stab}^\dagger(\mathcal{D}_0)$ is simply-connected.*

The idea of the proof is very simple: by an elementary topological argument, using what we proved in the previous section, we first reduce Theorem 7.1 to proving that Stab_a is simply-connected. To show this last assertion, we associate to every loop in Stab_a a word in the generators of the affine braid group B_3 . Then the simply-connectedness of Stab_a will be equivalent to the fact that B_3 acts freely on the set of ordered quivery subcategories, and that, for every relation in B_3 , we can find a corresponding loop that is contractible.

The main reason we involve Bridgeland’s description of the set Stab_a is the following: The loci of degenerate stability conditions appearing in Remark 6.9 are rather implicit in our description of $\text{Stab}^\dagger(\mathcal{D}_0)$; however, they are essential for the simply-connectedness of the space.

Remark 7.2. *Following [Bri06], let $\text{Stab}_n \subset \text{Stab}^\dagger(\mathcal{D}_0)$ be the subset of normalized stability conditions with $Z(k(x)) = -1$. Denote by $\overline{\text{Stab}}$ the quotient $\text{Stab}^\dagger(\mathcal{D}_0)/\mathbb{C}$ by the \mathbb{C} -action, which must also be simply-connected. By the results of the previous section, there always exist semi-stable objects of class $[k(x)]$ in $\text{Stab}^\dagger(\mathcal{D}_0)$; hence $Z(k(x))$ is never zero. It follows that the subset Stab_n already surjects onto $\overline{\text{Stab}}$. This surjection is a Galois covering $\overline{\text{Stab}} \cong \text{Stab}_n/\mathbb{Z}$, where the action by $n \in \mathbb{Z}$ is given as the shift $[2n]$; by the simply-connectedness of $\overline{\text{Stab}}$, it follows that $\text{Stab}_n \cong \overline{\text{Stab}} \times \mathbb{Z}$.*

In particular, there is a connected component of “very normalized” stability conditions $\text{Stab}_{vn} \subset \text{Stab}_n$ containing the geometric stability conditions where the skyscraper sheaves are semistable of phase 1. It is a global slicing with respect to the \mathbb{C} -action, and simply-connected. It is invariant under spherical twists and tensoring with line bundles (i.e., invariant under the subgroup $\Gamma_1(3) \subset \text{Aut } \mathcal{D}_0$ of Theorem 8.1). Presumably, Bridgeland’s Conjecture 1.2 in [Bri06] could be modified to use this connected component Stab_{vn} rather than its open subset $\text{Stab}_n^0(X)$ in the notation of [Bri06].

Lemma 7.3. *Let X be a topological space such that*

$$X = A \cup \bigcup_{n \in \mathbb{N}} B_n$$

where

- A and all B_n are open, connected, and simply-connected;
- $A \cap B_n$ is non-empty and connected, for all $n \in \mathbb{N}$;

- $B_n \cap B_m = \emptyset$, for $n \neq m$.

Then X is simply connected.

Proof. This is an easy application of the classical Seifert–Van Kampen Theorem. Indeed, consider a continuous loop. By compactness, there exists an integer N such that the loop is fully contained in

$$X_N := A \cup \bigcup_{n \leq N} B_n.$$

But an easy inductive argument shows that X_N is connected and simply-connected. Hence the loop can be contracted in X_N and so in X . \square

To prove Theorem 7.1, we use the previous lemma with $A := \text{Stab}_a$, and the countable family B_n as $\Phi(U)$, for Φ an autoequivalence of \mathcal{D}_0 which belongs to the subgroup generated by spherical twists $\text{ST}_{\mathcal{F}}$ (\mathcal{F} an exceptional vector bundle on \mathbb{P}^2). Clearly U and so all $\Phi(U)$ are open, connected, and simply-connected and $U \cap \Phi(U) = \emptyset$ unless $U = \Phi(U)$.

By Corollary 6.8 and Corollary 6.10, Theorem 7.1 follows then from the next proposition.

Proposition 7.4. *Stab_a is simply-connected.*

Before proving Proposition 7.4, we need a few Lemmata.

Lemma 7.5. *Let $\mathfrak{S} = \{S_0, S_1, S_2\}$ and $\mathfrak{R} = \{R_0, R_1, R_2\}$ be two ordered quivery collections. Assume that $\Theta_{\mathfrak{S}} \cap \Theta_{\mathfrak{R}} \neq \emptyset$. Then, either $\Theta_{\mathfrak{S}} = \Theta_{\mathfrak{R}}$, or there exists a stability condition $\bar{\sigma} = (\bar{Z}, \bar{\mathcal{P}}) \in \partial\Theta_{\mathfrak{S}} \cap \Theta_{\mathfrak{R}}$ such that the image of \bar{Z} is contained in a line.*

Proof. By hypothesis, either $\Theta_{\mathfrak{S}} = \Theta_{\mathfrak{R}}$, or there exists a stability condition $\sigma \in \partial\Theta_{\mathfrak{S}} \cap \Theta_{\mathfrak{R}}$. Now, we proceed as in the proof of Corollary 6.10: for every stability condition in $\Theta_{\mathfrak{R}}$ there exists a sequence $G_k \in \widetilde{\text{GL}}_2(\mathbb{R})$ ($k \in \mathbb{N}$) such that $\sigma \cdot G_k \rightarrow \bar{\sigma}$, where $\bar{\sigma} = (\bar{Z}, \bar{\mathcal{P}})$ is a stability condition in $\Theta_{\mathfrak{R}}$ such that the image of \bar{Z} is contained in a line. But then $\bar{\sigma} \in \partial\Theta_{\mathfrak{S}} \cap \Theta_{\mathfrak{R}}$, as wanted. \square

Lemma 7.6. *Let $\mathfrak{S} = \{S_0, S_1, S_2\}$ and $\mathfrak{R} = \{R_0, R_1, R_2\}$ be two ordered quivery collections. If $\bar{\sigma} = (\bar{Z}, \bar{\mathcal{P}}) \in \partial\Theta_{\mathfrak{S}} \cap \Theta_{\mathfrak{R}}$ is such that the image of \bar{Z} is contained in a line, then there exists $\gamma = \gamma_s \cdot \dots \cdot \gamma_1 \in B_3$, $\gamma_k \in \{\tau_0^{\pm 1}, \tau_1^{\pm 1}, \tau_2^{\pm 1}\}$ for all $k \in \{1, \dots, s\}$, such that $\mathfrak{R} = \gamma\mathfrak{S}$ and there exist real numbers $0 = a_0 < a_1 < \dots < a_s < a_{s+1} = 1$ and a continuous path $\alpha: [0, 1] \rightarrow \text{Stab}_a$ such that $\alpha([a_k, a_{k+1})) \subseteq \Theta_{\gamma_k \dots \gamma_1 \mathfrak{S}} \cap \Theta_{\mathfrak{S}}$ and $\alpha(1) = \bar{\sigma}$.*

Proof. It follows immediately from the proof of Proposition 6.6. \square

Lemma 7.7. *Let \mathfrak{S} be an ordered quivery collection. Then, for all $\gamma \in \{\tau_0^{\pm 1}, \tau_1^{\pm 1}, \tau_2^{\pm 1}\}$, $\Theta_{\mathfrak{S}} \cup \Theta_{\gamma\mathfrak{S}}$ is simply-connected.*

Proof. For simplicity, we assume $\gamma = \tau_1$. By Lemma 6.5 and the Seifert-Van Kampen Theorem, we only need to show that $\Theta_{\mathfrak{E}} \cap \Theta_{\tau_1 \mathfrak{E}}$ is connected. But, using Lemma 6.5 again, as well as the remark following it, we have

$$\Theta_{\mathfrak{E}} \cap \Theta_{\tau_1 \mathfrak{E}} = \mathcal{C}_{\mathfrak{E}} \cap \{(m_0, m_1, m_2, \phi_0, \phi_1, \phi_2) \in \mathbb{R}^6 : \phi_1 > \phi_0, \phi_1 > \phi_2\},$$

which is clearly connected. \square

Proof. (Proposition 7.4) Take a continuous loop $\alpha: [0, 1] \rightarrow \text{Stab}_a$. Then, by Lemma 7.6, there exist real numbers $0 = a_0 < a_1 < \dots < a_m = 1$, $m \in \mathbb{N}$, and ordered spherical collections

$$\{\mathfrak{M}_k = \{M_0^k, M_1^k, M_2^k\}\}_{k \in \{1, \dots, m\}}$$

with \mathfrak{M}_{k+1} obtained from \mathfrak{M}_k by an element $\gamma_{k+1} \in \{\tau_0^{\pm 1}, \tau_1^{\pm 1}, \tau_2^{\pm 1}\}$, such that $\alpha([a_{k-1}, a_k]) \subseteq \Theta_k := \Theta_{\mathfrak{M}_k}$ for $k \in \{1, \dots, m\}$ and $\alpha(0) \in \Theta_m \cap \Theta_1$. Thus we can assign a word $W(\alpha) = \gamma_m \dots \gamma_1$ in the generators of B_3 to every loop α . Using Lemma 7.7 we deduce that the homotopy class $[\alpha]$ of α is determined by $W(\alpha)$, and that $[\alpha]$ is in fact determined by element in the free group L generated by $r, \tau_0, \tau_1, \tau_2$ associated to $W(\alpha)$.

Now assume more specifically that the stability conditions $(Z, \mathcal{P}) = \alpha(0) = \alpha(1)$ is given by $\mathcal{P}((0, 1] = \mathcal{A}_{\mathfrak{E}_1}$, $Z(i_* \mathcal{O}_{\mathbb{P}^2}) = Z(i_* \Omega_{\mathbb{P}^2}(1)[1]) = Z(i_* \mathcal{O}_{\mathbb{P}^2}(-1)[2]) = -\frac{1}{3}$. Then any heart $\mathcal{P}((\phi, \phi + 1])$ is a shift of $\mathcal{A}_{\mathfrak{E}_1}$. As an ordered quivery collection is determined, up to reordering, by its heart, we have

$$r^j W(\alpha)(\mathfrak{M}_1) = \mathfrak{M}_1,$$

for some j . Since the braid group B_3 acts freely on the set of ordered quivery subcategories (by [Bri05b, Thm. 5.6]), we have $r^j W(\alpha) = \text{id}_{B_3}$ in B_3 . Due to the description of B_3 in terms of generators and relations, it follows that we have an identity in L of the form

$$W(\alpha) = r^{-j} (h_1 R_1^{\pm 1} h_1^{-1}) \dots (h_s R_s^{\pm 1} h_s^{-1}),$$

with $R_1, \dots, R_s \in \{r \tau_i r^{-1} \tau_{i+1}^{-1}, \tau_i \tau_{i+1} \tau_i^{-1} \tau_{i+1}^{-1} \tau_i^{-1} \tau_{i+1}^{-1}, r^3\}$ and $h_1, \dots, h_s \in L$ arbitrary elements.

By Lemma 7.8, loops with associated words $\tau_i \tau_{i+1} \tau_i^{-1} \tau_{i+1}^{-1} \tau_i^{-1}$ (or its inverse) can be contracted in Stab_a . This implies that α can be contracted in general, and so Stab_a is simply-connected. \square

Lemma 7.8. *Let α be a loop with word $W(\alpha) = \tau_i \tau_{i+1} \tau_i^{-1} \tau_{i+1}^{-1} \tau_i^{-1}$. Then α is contractible.*

Proof. We may assume $i = 1$. We will say that a loop α “runs through the regions U_1, \dots, U_m ” for open subsets $U_i \subset \text{Stab}_a$ if there are $0 = a_0 < a_1 < \dots < a_m < 1$ with $\alpha([a_{k-1}, a_k]) \subset U_k$ and $\alpha([a_m, 1]) \subset U_1$.

By assumption, the loop α runs through the regions $\Theta_1, \dots, \Theta_6$ given by $\Theta_k = \Theta_{\mathfrak{M}_k}$ and

$$\begin{aligned} \mathfrak{M}_1 &= \{S_0, S_1, S_2\} & \mathfrak{M}_4 &= \{\Gamma, \text{ST}_{S_0}^{-1}S_1[1], S_0[2]\} \\ \mathfrak{M}_2 &= \{S_0, \text{ST}_{S_1}^{-1}S_2, S_1[1]\} & \mathfrak{M}_5 &= \{\text{ST}_{S_0}^{-1}S_1, \text{ST}_{S_0}^{-1}S_2, S_0[2]\} \\ \mathfrak{M}_3 &= \{\Gamma, S_0[1], S_1[1]\} & \mathfrak{M}_6 &= \{\text{ST}_{S_0}^{-1}S_1, S_0[1], S_2\} \end{aligned}$$

for some ordered quivery collection S_0, S_1, S_2 , where $\Gamma = \text{ST}_{S_0}^{-1}\text{ST}_{S_1}^{-1}S_2$.

First of all observe that, by Lemma 6.5,

$$\Theta_1 \cap \Theta_2 \cap \Theta_3 \neq \emptyset$$

and it is homeomorphic to the locus in Θ_1 given by those stability conditions having phases such that $\phi(S_1) < \phi(S_0) < \phi(S_2)$ and $\phi(S_0) < \phi(\text{ST}_{S_1}^{-1}S_2)$. This implies that we can replace α by a loop, which we will denote again α , such that α runs through the regions $\Theta_1, \Theta_3, \Theta_4, \Theta_5, \Theta_6$. Repeating the same argument on $\Theta_4 \cap \Theta_5 \cap \Theta_6$, we can replace it by a loop that runs through $\Theta_1, \Theta_3, \Theta_4, \Theta_6$.

Let $t_1 \in (0, 1)$ be such that $\alpha([0, t_1]) \subset \Theta_1$ and $\alpha(t_1) = (Z_1, \mathcal{P}_1) \in \Theta_3 \cap \partial\Theta_1$. By Lemma 7.5, we can assume the image of Z_1 to be contained in the real line. In such a case, we have $\phi_1(S_0) = \phi_1(S_1) = 0$ and $\phi_1(S_2) = 1$. At the same time, $\text{ST}_{S_0}^{-1}S_1$ is semistable as well of phase 0. Hence, $\alpha(t_1) \in \partial\Theta_6$ and, in particular, $\Theta_3 \cap \Theta_6 \neq \emptyset$.

Let $t_3 \in (t_1, 1)$ be such that $\alpha(t_3) \in \Theta_1 \cap \partial\Theta_6$ and $\alpha((t_3, 1]) \subset \Theta_1$. The intersection $\Theta_1 \cap \partial\Theta_6$ can be easily described using Lemma 7.7: in particular, it is connected and simply-connected, being homeomorphic to the region

$$\mathcal{C}_{\mathfrak{M}_1} \cap \{(m_0, m_1, m_2, \phi_0, \phi_1, \phi_2) \in \mathbb{R}^6 : \phi_1 = \phi_0, \phi_0 \leq \phi_2 \leq \phi_0 + 1\}.$$

Hence, we can replace α by a homotopic path for which $\alpha(t_1) = \alpha(t_3)$.

What we proved so far is that our original loop α is homotopic to a loop which can be decomposed as a loop α' contained in $\Theta_1 \cup \Theta_3$ (which is nothing but $\alpha([0, t_1])$ composed with $\alpha([t_3, 1])$, since $\alpha(t_1) = \alpha(t_3)$) and another loop α'' which runs through $\Theta_3, \Theta_4, \Theta_6$ (which is nothing but $\alpha([t_1, t_3])$).

Now, let $t_2 \in (t_1, t_3)$ be such that $\alpha(t_2) \in \Theta_4 \cap \partial\Theta_3$. Consider the loop β which is the same as the loop α'' but with base point $\alpha(t_2)$. Arguing as before, we can assume that β can be decomposed as a loop α''' contained in $\Theta_4 \cup \Theta_6$ and another loop α'''' contained in $\Theta_3 \cup \Theta_6$.

Summing up, to prove that α is contractible, we only need to prove that all regions $\Theta_1 \cup \Theta_3, \Theta_4 \cup \Theta_6$, and $\Theta_3 \cup \Theta_6$ are simply-connected. Again, by Lemma 6.5 and the Seifert–Van Kampen Theorem, it is sufficient to show that the intersections $\Theta_1 \cap \Theta_3, \Theta_4 \cap \Theta_6$, and $\Theta_3 \cap \Theta_6$ are connected. For $\Theta_1 \cap \Theta_3$, observe that it corresponds to the locus in $\mathcal{C}_{\mathfrak{M}_1}$ in which Γ is stable

and $\phi(S_0), \phi(S_1) < \phi(\Gamma)$. This can be easily proved to be connected by proceeding in a similar way as in the last part of the proof of Lemma 6.7: in this situation we use the $\widetilde{\mathrm{GL}}_2(\mathbb{R})$ to fix the values of $Z(\Gamma)$ and $Z(S_1)$. The region $\Theta_1 \cap \Theta_3$ is then, up to the action of $\widetilde{\mathrm{GL}}_2(\mathbb{R})$, cut out by half-planes, and so it is connected. The intersection $\Theta_4 \cap \Theta_6$ is analogous. Finally, $\Theta_3 \cap \Theta_6$ corresponds to the locus in $\mathcal{C}_{\mathfrak{m}_3}$ in which S_2 is stable, $\phi(S_0[1]) < \phi(S_1[1])$, and $\phi(S_2) < \phi(\mathrm{ST}_{S_0}^{-1}S_1)$, which is again connected by a similar argument. This completes the proof of the lemma. \square

8. GROUP OF AUTOEQUIVALENCES

Let $\mathrm{Aut} \mathcal{D}_0$ be the group of autoequivalences of \mathcal{D}_0 up to isomorphism of functors, and let $\mathrm{Aut}^\dagger \mathcal{D}_0$ be the subgroup of $\mathrm{Aut} \mathcal{D}_0$ preserving the connected component $\mathrm{Stab}^\dagger(\mathcal{D}_0)$.

The numerical K -group of \mathcal{D}_0 is $K(\mathcal{D}_0)/K^\perp = K(\mathcal{D}_0)/\mathbb{Z} \cdot [k(x)] \cong \mathbb{Z}^{\oplus 2}$. Since the Euler form is skew-symmetric, there is a natural map

$$(18) \quad \mathrm{Aut} \mathcal{D}_0 \rightarrow \mathrm{SL}(2, \mathbb{Z})$$

given by sending an autoequivalence to its induced action on the numerical K -group. Crucial for us will be the congruence subgroup $\Gamma_1(3) \subset \mathrm{SL}(2, \mathbb{Z})$ of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \pmod{3}$$

It has generators $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$ with a single relation given by $(ST)^3 = 1$.

Theorem 8.1.

$$\mathrm{Aut}^\dagger \mathcal{D}_0 \cong \mathbb{Z} \times \Gamma_1(3) \times \mathrm{Aut}(X).$$

We start by identifying the subgroup $\Gamma_1(3) \subset \mathrm{Aut}^\dagger(\mathcal{D}_0)$: As observed in [Asp05, Section 7.3.6], there is a relation

$$(19) \quad (\mathrm{ST}_{\mathcal{O}_{\mathbb{P}^2}} \circ (_ \otimes \pi^* \mathcal{O}(1)))^3 \cong \mathrm{Id}.$$

Due to the description of $\Gamma_1(3)$ by generators and relations, this induces a map $\Gamma_1(3) \rightarrow \mathrm{Aut}^\dagger \mathcal{D}_0$; as the induced action on K/K^\perp is faithful, the map is injective, and we obtain:

Proposition 8.2. *The subgroup of $\mathrm{Aut}^\dagger \mathcal{D}_0$ generated by $\mathrm{ST}_{\mathcal{O}_{\mathbb{P}^2}}$ and $_ \otimes \pi^* \mathcal{O}(1)$ is isomorphic to $\Gamma_1(3)$.*

Alternatively, one can prove that the above composition is isomorphic to the generator of the natural \mathbb{Z}_3 -action on $\mathbb{D}_0^b([\mathbb{C}^3/\mathbb{Z}_3])$ via the derived McKay correspondence of [BKR01].

Lemma 8.3. *For any exceptional vector bundle \mathcal{E} on \mathbb{P}^2 , its associated spherical twist $\mathrm{ST}_{\mathcal{E}}$ is contained in the subgroup $\Gamma_1(3)$ generated by $\mathrm{ST}_{\mathcal{O}_{\mathbb{P}^2}}$ and $_ \otimes \pi^* \mathcal{O}(1)$.*

Proof. It is sufficient to show that there exists $g \in \Gamma_1(3)$ with $\mathcal{E} \cong g(\mathcal{O}_{\mathbb{P}^2})$. By [GR87], \mathcal{E} is contained in a mutation of the exceptional collection $\{i_* \mathcal{O}_{\mathbb{P}^2}(-1), i_* \Omega_{\mathbb{P}^2}(1), i_* \mathcal{O}_{\mathbb{P}^2}\}$ on \mathbb{P}^2 . Remark 6.2 completes the proof. \square

Restricting to the subgroup $\mathrm{Aut}^\dagger \mathcal{D}_0$ makes it possible to control autoequivalences via the following proposition:

Proposition 8.4. *Let $\Phi \in \mathrm{Aut} \mathcal{D}_0$ be an autoequivalence such that there exist two geometric stability conditions σ, σ' with $\Phi(\sigma) = \sigma'$. Then Φ is isomorphic to the composition of an automorphism of X with $_ \otimes \mathcal{O}(n)[k]$.*

Proof. From the description of geometric stability conditions it follows that Φ sends skyscraper sheaves $k(x)$, $x \in \mathbb{P}^2$, to shifts of skyscraper sheaves. More precisely, after replacing Φ by $\Phi \circ _ [k]$ for some $k \in \mathbb{Z}$, we may assume that for every $x \in \mathbb{P}^2$ there is $x' \in \mathbb{P}^2$ with $\Phi(k(x)) \cong k(x')$.

Now, the push-forwards $\mathcal{O}_{\mathbb{P}^2}(n)$ of line bundles on \mathbb{P}^2 are characterized as the only simple objects satisfying $\mathbf{R}\mathrm{Hom}(E, k(x)) \cong \mathbb{C} \oplus \mathbb{C}[-1]$ for all $x \in \mathbb{P}^2$. They are naturally ordered by the existence of morphisms. Hence, after composing Φ with the tensorization by a line bundle, we may assume $\Phi(\mathcal{O}_{\mathbb{P}^2}(n)) \cong \mathcal{O}_{\mathbb{P}^2}(n)$ for all $n \in \mathbb{Z}$.

Then Φ induces an automorphism of $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1)) \cong \mathbb{C}^3$, which is given by an element of $\mathrm{GL}(3)$. After composing Φ with an element of $\mathrm{Aut} \mathbb{P}^2 \cong \mathrm{PGL}(3)$, we may assume that the induced automorphism on $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1))$ is given by a scalar multiplication. Due to the compatibility of Φ with the composition

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1)) \otimes \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}(1), k(x)) \rightarrow \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}, k(x))$$

it follows that $\Phi(k(x)) \cong k(x)$ for all $x \in \mathbb{P}^2$.

The claim now follows from Proposition C.3. \square

Now Theorem 8.1 follows easily: Given $\Phi \in \mathrm{Aut}^\dagger \mathcal{D}_0$, pick an arbitrary geometric stability condition $\sigma \in U$. By Corollary 5.2, there exists a stability condition $\sigma' \in \bar{U}$ and a composition Ψ of spherical twists associated to exceptional vector bundles with $\Psi \circ \Phi(\sigma) = \sigma'$. The stability condition $\Psi \circ \Phi(\sigma)$ has no semistable objects of class $[k(x)]$; thus actually $\sigma' \in U$. By Proposition 8.4 and Lemma 8.3, Φ is contained in the group generated by $\Gamma_1(3)$, shifts, and $\mathrm{Aut}(X)$.

As the actions by \mathbb{Z} , by $\Gamma_1(3)$, and by $\mathrm{Aut}(X)$ commute, we get a surjective map

$$\mathbb{Z} \times \Gamma_1(3) \times \mathrm{Aut}(X) \rightarrow \mathrm{Aut}^\dagger \mathcal{D}_0.$$

Since $\mathbb{Z} \times \text{Aut}(X)$ acts by ± 1 on K/K^\perp , its intersection with $\Gamma_1(3)$ is trivial, and the above map is an isomorphism.

9. Π -STABILITY AND GLOBAL MIRROR SYMMETRY

In this section, we outline how our results fit into expectations coming from mirror symmetry for the local \mathbb{P}^2 . Mirror symmetry for the local \mathbb{P}^2 has been discussed in many places of the mathematical physics literature, see e.g. [AGM94, DG00]; our presentation follows [ABK08] and [Asp05] most closely.

9.1. Monodromy and autoequivalences. The family of mirror partners to the local \mathbb{P}^2 can be constructed explicitly from the following family of genus one curves: The equation

$$X_0^3 + X_1^3 + X_2^3 - 3\psi X_0 X_1 X_2 = 0$$

cuts out a surface $S \subset \mathbb{P}^2 \times \mathbb{C}$. At $\psi^3 = 1$ and $\psi = \infty$, the fibers are singular; all other fibers of S over \mathbb{C} are smooth genus one curves. There is a μ_3 -action on S given by $X_0 \mapsto \omega^{-1} X_0$ and $\psi \mapsto \omega \psi$, and leaving the other variables invariant, where $\omega = \exp(2\pi i/3)$. Let \mathcal{Y} be the quotient

$$(S \setminus \{\psi^3 = 1\})/\mu_3$$

of the union of the smooth fibers by the group action; then \mathcal{Y} is a family of smooth elliptic curves over $(\mathbb{C} - \mu_3)/\mu_3$.

In fact, the base is the moduli space $\mathcal{M}_{\Gamma_1(3)} \cong (\mathbb{C} - \mu_3)/\mu_3$ of elliptic curves with $\Gamma_1(3)$ -level structure. We can also think of $\mathcal{M}_{\Gamma_1(3)}$ as \mathbb{P}^1 with the points $z = -\frac{1}{27}$ and $z = 0$ removed, and a stacky \mathbb{Z}_3 point at $z = \infty$ (where we set $z = -\frac{1}{(3\psi)^3}$). The fundamental group of $\mathcal{M}_{\Gamma_1(3)}$ is $\Gamma_1(3)$. It is generated by the loops $\gamma_{-\frac{1}{27}}, \gamma_0$ around $-\frac{1}{27}$ and 0; as their composition is a loop around $z = \infty$, they satisfy $(\gamma_{-\frac{1}{27}} \gamma_0)^3 = 1$.

Given any $z \in \mathcal{M}_{\Gamma_1(3)}$, one can determine a basis of first homology $H_1(\mathcal{Y}_z)$ of the fibers by choosing a path from z to $-\frac{1}{27}$ and from z to ∞ ; the basis is then given by the two corresponding vanishing cycles \bar{A}_z and \bar{B}_z . This basis yields an identification of $\pi_1(\mathcal{M}_{\Gamma_1(3)})$ as a subgroup of $SL_2(\mathbb{Z})$ by its monodromy action on the first homology H_1 of the fibers of \mathcal{Y} . Explicitly, we get

$$\gamma_{-\frac{1}{27}} \equiv \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}, \quad \gamma_0 \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

A well-known principle of mirror symmetry states that monodromies in the mirror family \mathcal{Y} lift to autoequivalences in the derived category \mathcal{D}_0 : it is implied by homological mirror symmetry and has been applied and verified e.g. in [ST01, Hor05]. Theorem 8.1 gives another incarnation of

this principle, as the action of $\Gamma_1(3) \subset \text{Aut } \mathcal{D}_0$ on $K(\mathcal{D}_0)/K^\perp \cong \mathbb{Z}^{\oplus 2}$ matches the action of $\Gamma_1(3) \cong \pi_1(\mathcal{M}_{\Gamma_1(3)})$ if we identify $ST_{\mathcal{O}}$ with $\gamma_{-\frac{1}{27}}$ and $_ \otimes \mathcal{O}(1)$ with γ_0 .

9.2. Period integrals and $\text{Stab}^\dagger(\mathcal{D}_0)$. However, in the spirit of [Bri09], there is also more geometric connection between the mirror moduli space $\mathcal{M}_{\Gamma_1(3)}$ and the space of stability conditions. The periods in this mirror construction are given by integrals over the meromorphic differential form $\lambda = \ln \frac{X_2}{X_3} \frac{dX_1}{X_1}$. More precisely, let $\mathcal{Y}^0 \subset \mathcal{Y}$ be the complement of the set of poles of λ , and $\tilde{\mathcal{Y}}^0$ be the cover on which $\ln \frac{X_2}{X_3}$ is well-defined. Following [ABK08], one can choose a family of cycles $A_z, B_z \in H_1(\tilde{\mathcal{Y}}_z^0)$ that project to $\bar{A}_z, \bar{B}_z \in H_1(\mathcal{Y}_z)$, and a third family of cycles $C_z \in H_1(\mathcal{Y}_z^0)$ and define the period integrals as:

$$\Pi(z) = \begin{pmatrix} \int_B \lambda \\ \int_A \lambda \\ \int_C \lambda \end{pmatrix}$$

The authors show that if A, B, C are chosen appropriately, then the action of $\pi_1(\mathcal{M}_{\Gamma_1(3)}) \cong \Gamma_1(3)$ on these 3 periods matches the action of $\Gamma_1(3) \subset \text{Aut } \mathcal{D}_0$ on $K(\mathcal{D}_0) \cong \mathbb{Z}^{\oplus 3}$.

We will now ignore the construction of period integrals and instead just consider their Picard-Fuchs equation; with $\theta_z := z \frac{d}{dz}$ it is given by

$$(20) \quad (\theta_z^3 + 3z\theta_z(3\theta_z + 1)(3\theta_z + 2))\Pi = 0,$$

and has singularities at $z = 0$, $z = -\frac{1}{27}$ and $z = \infty$.

Using an *Ansatz* and solving for the coefficients of the power series, one can find expansions of three linearly independent solutions around $z = 0$ and $\psi = 0$, respectively (see also [ABK08, Section 6] and [Asp05, Section 7.3]).

Around $z = 0$, we make the standard branch choice of $\ln z$ for $z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, and get as expansions (compare with [ABK08, p. 37])

$$\begin{aligned} \omega_0(z) &= 1 \\ \omega_1(z) &= \frac{1}{2\pi i} \left(\ln z + 3 \sum_{n=1}^{\infty} \frac{(3n-1)!}{n!^3} (-z)^n \right) \\ \omega_2(z) &= \frac{1}{(2\pi i)^2} \left((\ln z)^2 + 6 \ln z \cdot \sum_{n=1}^{\infty} \frac{(3n-1)!}{n!^3} (-z)^n + \sum_{n=1}^{\infty} l_n z^n \right) \end{aligned}$$

where the differential equation defines the l_n recursively:

$$l_n = -\frac{1}{n^3} \left((3n-1)(3n-2)(3n-3)l_{n-1} + 18 \cdot (-1)^n \frac{(3n-1)!}{n!^3} \cdot n^2 - 18 \cdot (-1)^n \frac{(3n-4)!}{(n-1)!^3} (27n^2 - 36n + 11) \right)$$

Similarly, the power series expansion of a basis of solutions nearby $z = \infty$ are given by $\varpi_0(\psi) = 1$ and:

$$\begin{aligned} \varpi_1(\psi) &= \frac{1}{2\pi i} \sum_{\substack{n=1 \\ 3 \nmid n}}^{\infty} \frac{\Gamma\left(\frac{n}{3}\right)}{\Gamma(n+1)\Gamma\left(1-\frac{n}{3}\right)^2} (3\psi)^n \\ \varpi_2(\psi) &= \frac{1}{2\pi i} \sum_{\substack{n=1 \\ 3 \nmid n}}^{\infty} \frac{\Gamma\left(\frac{n}{3}\right)}{\Gamma(n+1)\Gamma\left(1-\frac{n}{3}\right)^2} (3e^{\frac{2\pi i}{3}}\psi)^n \end{aligned}$$

Here we use $\psi = -\frac{1}{3\sqrt[3]{z}}$ with the branch choice $\frac{2\pi}{3} < \arg \psi < \frac{4\pi}{3}$ for $|\arg z| < \pi$.

Following Aspinwall, we define the solutions $a(z), b(z)$ of (20) for $z \in \mathbb{C}^* \setminus \mathbb{R}_{<0}$ by setting

$$(21) \quad a(z) = \omega_1(z) - \frac{1}{2}, \quad b(z) = -\frac{1}{2}\omega_2(z) + \frac{1}{2}\omega_1(z) - \frac{1}{4}$$

for $|z| < \frac{1}{27}$ and analytic continuation. This analytic continuation is computed explicitly in [Asp05, Eqn. (286)] and [ABK08, Eqn. (6.22)], and gives the following expansion of $a(z)$ and $b(z)$ around $z = \infty$:

$$(22) \quad a(z) = \varpi_1(z) - \frac{1}{2}, \quad b(z) = \frac{1}{3}(\varpi_1(z) - \varpi_2(z) - 1)$$

Theorem 9.3. *Fix a universal cover $\widetilde{\mathcal{M}}_{\Gamma_1(3)} \rightarrow \mathcal{M}_{\Gamma_1(3)}$ together with its $\Gamma_1(3)$ -action of deck transformations and choose a fundamental domain $D \subset \widetilde{\mathcal{M}}_{\Gamma_1(3)}$ that projects isomorphically onto $\mathbb{C}^* \setminus \mathbb{R}_{<0}$. Then there is an embedding $I: \widetilde{\mathcal{M}}_{\Gamma_1(3)} \rightarrow \text{Stab}^\dagger(\mathcal{D}_0)$ defined by the following properties:*

(a) *For $(Z(z), \mathcal{P}(z)) = I(z)$, the central charge is given by*

$$Z(E)(z) = -c(E) + a(z) \cdot d(E) + b(z) \cdot r(E),$$

for all $E \in K(\mathcal{D}_0)$ (where we identify $a(z), b(z)$ with their analytic continuations from D to $\widetilde{\mathcal{M}}_{\Gamma_1(3)}$).

(b) *For $z \in D$, the stability condition $I(z)$ is geometric with $k(x)$ having phase 1.*

On the boundary of $I(D)$ we have several interesting special points: The point $z = 0$ corresponds to the *large volume limit point*, where $\mathfrak{S}(a) \rightarrow +\infty$ and the central charge is approximately given by $Z(E) = -\int_{\mathbb{P}^2} \text{ch}(E)e^{-ah}$, where h is the class of a line in \mathbb{P}^2 ; the limit stability condition as $z \rightarrow 0$ can be described as a polynomial stability condition of [Bay09]. The point $z = \infty$ is the *orbifold point*: the heart of the bounded t -structure is $\mathcal{A}_1 \cong \text{Coh}_0[\mathbb{C}^3/\mathbb{Z}_3]$ and the three simple objects of \mathcal{A}_1 have the same central charge equal to $-1/3$; this point is fixed under the \mathbb{Z}_3 -action on \mathcal{D}_0 given by relation (19) (i.e., by tensor product in $\text{Coh}_0[\mathbb{C}^3/\mathbb{Z}_3]$ with a non-trivial one-dimensional \mathbb{Z}_3 -representation). Finally, when $\psi = \omega$ (resp. $\psi = \omega^2$) and so $z = -\frac{1}{27}$ (these are called *conifold points*), we have a singularity: indeed, $Z(\mathcal{O}_{\mathbb{P}^2}) = 0$, resp. $Z(\mathcal{O}_{\mathbb{P}^2}(-1)) = 0$, depending on whether we approach $-\frac{1}{27}$ from above or below.

The proof of the theorem is based on the following two observations:

- (a) For all $z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, the complex numbers $a(z), b(z)$ satisfy the inequalities of Definition 2.4.
- (b) The monodromy action of $\Gamma_1(3)$ on the solutions $a(z), b(z)$ (computed, for example, in [Asp05]) is compatible with the $\Gamma_1(3)$ -action on Stab^\dagger .

We first verified Observation (a) by explicit computations using the computer algebra package SAGE [S⁺09].¹ A complete argument is sketched in Appendix D. To prove Theorem 9.3 we only need to show Observation (b):

Proof. (Theorem 9.3) By Observation (a) and Theorem 2.5, we obtain an embedding $I: D \hookrightarrow U \subset \text{Stab}^\dagger(\mathcal{D}_0)$. By Bridgeland's deformation result, the extension of I to $\widetilde{\mathcal{M}}_{\Gamma_1(3)}$ is unique, if it exists.

Now, we can extend I to the $\Gamma_1(3)$ -translates of D uniquely by requiring it to be $\Gamma_1(3)$ -equivariant. Hence, it remains to check that this extension of I glues along the translates of $\partial D \subset \widetilde{\mathcal{M}}_{\Gamma_1(3)}$, and is compatible with the requirement that $a(z), b(z)$ are solutions to the Picard-Fuchs equation.

Let $\gamma_0 \in \Gamma_1(3)$ be the loop going in positive direction around the origin $z = 0$, and γ_∞ the loop around $z = \infty$ acting on ψ by $\psi \mapsto e^{\frac{2\pi i}{3}}\psi$. Then, by the $\Gamma_1(3)$ -equivariance, it is in fact enough to check the glueing along $\overline{D} \cap \gamma_0(\overline{D})$ lying above $(-\frac{1}{27}, 0) \subset \mathbb{C}$ in the z -plane, and along $\overline{D} \cap \gamma_\infty(\overline{D})$, lying above the line segments $(0, 1) \cdot e^{\frac{2}{3}\pi i}$ and $(0, 1) \cdot e^{\frac{4}{3}\pi i}$ in the ψ -plane.

The action of γ_0 on the solutions is given by

$$\omega_1(z) \mapsto \omega_1(z) + 1, \quad \omega_2(z) \mapsto \omega_2(z) + 2\omega_1(z) + 1.$$

¹The program used to test the inequalities is available for download from the authors' homepages. It implements the power series expansion around $z = 0$ and $\psi = 0$ and tests the inequalities for random complex numbers in their respective convergence domains.

The action of $_ \otimes \mathcal{O}(1)$ on the set of geometric stability conditions $\sigma_{a,b}$ of Theorem 2.5 is given by $a \mapsto a+1$ and $b \mapsto b-a-\frac{1}{2}$. Using the expansions in equation (21), we see that the induced action of γ_0 on $a(z)$ and $b(z)$ matches exactly; hence the definition of $a(z), b(z)$ on $\gamma_0(D)$ by analytic continuation agrees with the implicit definition given by the requirement that I is γ_0 -equivariant; on the other hand, when $\arg(z) = \pi$ we have $B = 0$ and $b(-\frac{1}{27}) = 0$ (see app. D), and it follows that $a(z), b(z)$ still satisfy the inequalities of Definition 2.4 for $z \in \overline{D} \cap \gamma_0(\overline{D})$, i.e. for $z = (-\frac{1}{27}, 0)$ with $\arg(z) = \pi$. Then Theorem 2.5 implies that I glues along this boundary component of D within the geometric chamber.

Similarly, the action of γ_∞ on the space of solutions is computed in terms of the expansions around $z = \infty$ as

$$\varpi_1(z) \mapsto \varpi_2(z), \quad \varpi_2(z) \mapsto -\varpi_1(z) - \varpi_2(z).$$

The central charges of the three simple objects in the quiver category \mathcal{A}_1 are given by

$$\begin{aligned} Z(\mathcal{O}_{\mathbb{P}^2})(z) &= b(z) = \frac{1}{3}\varpi_1(z) - \frac{1}{3}\varpi_2(z) - \frac{1}{3} \\ Z(\Omega_{\mathbb{P}^2}(1)[1])(z) &= -2b(z) + a(z) - \frac{1}{2} = \frac{1}{3}\varpi_1(z) + \frac{2}{3}\varpi_2(z) - \frac{1}{3} \\ Z(\mathcal{O}_{\mathbb{P}^2}(-1)[2])(z) &= b(z) - a(z) - \frac{1}{2} = -\frac{2}{3}\varpi_1(z) - \frac{1}{3}\varpi_2(z) - \frac{1}{3} \end{aligned}$$

The autoequivalence $(\text{ST}_{\mathcal{O}_{\mathbb{P}^2}} \circ (_ \otimes \pi^* \mathcal{O}(1)))^{-1}$ permutes these 3 objects and preserves the heart of the t-structure \mathcal{A}_1 ; hence it is easy to see that its action on the central charge matches the monodromy γ_∞ . \square

APPENDIX A. BOUNDS ON STABLE CHERN CLASSES AFTER DREZET-LE POTIER

We give a brief review and a reformulation of the main result of [DLP85]. We recall that for a torsion-free sheaf \mathcal{F} on \mathbb{P}^2 , its slope is defined by $\mu(\mathcal{F}) = \frac{d(\mathcal{F})}{r(\mathcal{F})}$, giving the following notion of stability:

Definition A.1. *A torsion-free sheaf \mathcal{F} on \mathbb{P}^2 is called slope-stable if the inequality $\mu(\mathcal{F}') < \mu(\mathcal{F})$ holds for all saturated subsheaves $\mathcal{F}' \subset \mathcal{F}$.*

A vector bundle E on \mathbb{P}^2 is exceptional if it is simple and rigid. Let \mathfrak{A} be the set of all $\alpha \in \mathbb{Q}$ such that there exists an exceptional vector bundle on \mathbb{P}^2 with slope α . For any $\alpha \in \mathfrak{A}$, we call its rank r_α be the smallest integer $r > 0$ such that $r\alpha \in \mathbb{Z}$. We call $\Delta_\alpha := \frac{1}{2} \left(1 - \frac{1}{r_\alpha^2}\right)$ its discriminant. It follows from Riemann-Roch that the rank and discriminant of an exceptional vector bundle with slope α (if it exists) are given by these formulas; similarly, any non-exceptional vector bundle satisfies $\Delta \geq \frac{1}{2}$.

For two rational numbers with $3 + \alpha + \beta \neq 0$, Drezet and Le Potier define the operation

$$\alpha.\beta := \frac{\alpha + \beta}{2} + \frac{\Delta_\beta - \Delta_\alpha}{3 + \alpha - \beta}$$

Let \mathfrak{D} be the set of rational numbers of the form $\frac{p}{2^q}$ for $p \in \mathbb{Z}, q \in \mathbb{Z}_{>0}$. One defines a function $\epsilon: \mathfrak{D} \rightarrow \mathbb{Q}$ inductively by $\epsilon(n) := n$ for $n \in \mathbb{Z}$ and

$$\epsilon\left(\frac{2p+1}{2^{q+1}}\right) := \epsilon\left(\frac{p}{2^q}\right).\epsilon\left(\frac{p+1}{2^q}\right)$$

Theorem A.2. [DLP85, Théorème A and chapitre 5] *The set \mathfrak{A} of exceptional slopes is equal to the image $\epsilon(\mathfrak{D})$.*

Now define

$$P(X) := 1 + \frac{3}{2}X + \frac{1}{2}X^2$$

$$p(x) := \begin{cases} P(-|x|) & |x| < 3 \\ 0 & \text{otherwise} \end{cases}$$

and, for any $\alpha \in \mathfrak{A}$,

$$p_\alpha(x) := p(x - \alpha) - \Delta_\alpha$$

If α, β are of the form given in Theorem A.2, then p_α and p_β are monotone decreasing and increasing, respectively; they intersect in the point $(\alpha.\beta, \Delta_{\alpha.\beta})$.

Theorem A.3. [DLP85] *Given an integer $r > 0$ and rationals $\mu, \Delta \in \mathbb{Q}$, there exists a stable vector bundle \mathcal{E} on \mathbb{P}^2 with rank r , slope μ , and discriminant Δ if and only if*

- (a) $r\mu \in \mathbb{Z}$ and $r(P(\mu) - \Delta) \in \mathbb{Z}$, and
- (b) For every $\alpha \in \mathfrak{A}$ with $r_\alpha < r$ and $|\alpha - \mu| < 3$, we have $\Delta \geq p_\alpha(\mu)$.

A proof of the previous theorem can be found in [LP97, Thm. 16.2.1]. This leads us to define (cf. [LP97, Sect. 16.4]) $\delta_\infty^{DP}: \mathbb{R} \rightarrow [1/2, 1]$ as

$$\delta_\infty^{DP} := \sup \{p_\alpha : \alpha \in \mathfrak{A}\}.$$

The necessary and sufficient condition for the existence of non-exceptional stable sheaves can then be written as

$$\Delta \geq \delta_\infty^{DP}(\mu).$$

The first part of Theorem 2.3 now follows immediately. For the last assertion, let (μ_n, Δ_n) be a sequence of distinct points in S_E that converges in \mathbb{R}^2 to (μ, Δ) . For every $\alpha \in \mathfrak{A}$, we have $\Delta_n \geq p_\alpha(\mu_n)$ for all $n \gg 0$ (in fact, this holds whenever $\alpha \neq \mu_n$). By continuity, $\Delta \geq p_\alpha(\mu)$, and thus $\Delta \geq \delta_\infty^{DP}(\mu)$, i.e. the accumulation point (μ, Δ) is contained in S_∞ .

APPENDIX B. BRIDGELAND'S STABILITY CONDITIONS

In this section we give a brief review of stability conditions on derived categories, following [Bri07].

Let \mathcal{T} be a triangulated category with good properties, e.g. the bounded derived category of coherent sheaves on a smooth and projective variety or \mathcal{D}_0 . A *stability condition* σ on \mathcal{T} consists in a pair (Z, \mathcal{P}) , where $Z : K(\mathcal{T}) \rightarrow \mathbb{C}$ (*central charge*) is an additive map and $\mathcal{P}(\phi) \subseteq \mathcal{T}$ are full, additive subcategories ($\phi \in \mathbb{R}$) satisfying:

- (a) for any $0 \neq E \in \mathcal{P}(\phi)$ we have $Z(E) \neq 0$ and $Z(E)/|Z(E)| = \exp(i\pi\phi)$;
- (b) $\forall \phi \in \mathbb{R}, \mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$;
- (c) if $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j), j = 1, 2$, then $\text{Hom}(A_1, A_2) = 0$;
- (d) for any $E \in \mathcal{T}$ there is a sequence of real numbers $\phi_1 > \dots > \phi_n$ and a collection of triangles $E_{j-1} \rightarrow E_j \rightarrow A_j$ with $E_0 = 0, E_n = E$ and $A_j \in \mathcal{P}(\phi_j)$ for all j .

The collection of exact triangles in (d) is called the *Harder-Narasimhan filtration* of E . Each subcategory $\mathcal{P}(\phi)$ is extension-closed and abelian. Its nonzero objects are said to be *semistable* of phase ϕ in σ , and the simple objects (i.e., objects without proper subobjects or quotients) are said to be *stable*.

For any interval $I \subseteq \mathbb{R}$, $\mathcal{P}(I)$ is defined to be the extension-closed subcategory of \mathcal{T} generated by the subcategories $\mathcal{P}(\phi)$, for $\phi \in I$. Bridgeland proved that, for all $\phi \in \mathbb{R}$, $\mathcal{P}((\phi, \phi + 1])$ is the heart of a bounded t -structure on \mathcal{T} . The category $\mathcal{P}((0, 1])$ is called the *heart* of σ .

Remark B.1. Let $\mathbb{H} := \{z \in \mathbb{C} : z = |z| \exp(i\pi\phi), 0 < \phi \leq 1\}$. If $\mathcal{A} \subseteq \mathcal{T}$ is the heart of a bounded t -structure, then a group homomorphism $Z : K(\mathcal{T}) \rightarrow \mathbb{C}$ gives rise to a unique stability condition when the following two conditions are satisfied ([Bri07, Prop. 5.3]): (i) $Z(\mathcal{A} \setminus 0) \subseteq \mathbb{H}$ (Z is a stability function on \mathcal{A}); (ii) Harder-Narasimhan filtrations exist for objects in \mathcal{A} with respect to Z .

Condition (i) means that, for all $0 \neq A \in \mathcal{A}$, the requirement $Z(A) \in \mathbb{H}$ gives a well-defined phase $\phi(A) := (1/\pi) \arg(Z(A)) \in (0, 1]$. This defines a notion of phase-stability for objects in \mathcal{A} , and so of (semi)stable objects of \mathcal{A} . Then condition (ii) asks for the existence of finite filtrations for every object in \mathcal{A} in semistable ones with decreasing phases.

In particular, if \mathcal{A} is an abelian category of finite length (i.e., Artinian and Noetherian) with a finite number of simple objects $\{S_0, \dots, S_m\}$, then any group homomorphism $Z : K(\mathcal{T}) \rightarrow \mathbb{C}$ with $Z(S_i) \in \mathbb{H}$ for all i extends to a unique stability condition on \mathcal{T} .

We give an improved criterion for the existence of Harder-Narasimhan filtrations:

Proposition B.2. *Let $\mathcal{A} \subseteq \mathcal{T}$ be the heart of a bounded t -structure on \mathcal{T} and let $Z: K(\mathcal{T}) \rightarrow \mathbb{C}$ be a stability function on \mathcal{A} . Write $\mathcal{P}'(1) \subset \mathcal{A}$ for the full subcategory of objects with phase 1 with respect to Z , and assume that:*

- *The image of $\mathfrak{S}(Z)$ is a discrete subgroup of \mathbb{R} .*
- *For all $E \in \mathcal{A}$, any sequence of subobjects*

$$0 = A_0 \subseteq A_1 \subseteq \dots \subseteq A_j \subseteq A_{j+1} \subseteq \dots \subseteq E,$$

with $A_j \in \mathcal{P}'(1)$, stabilizes.

Then Harder-Narasimhan filtrations exist for objects in \mathcal{A} with respect to Z .

Proof. We use the same ideas as in [Bri08, Prop. 7.1], and want to apply [Bri07, Prop. 2.4].

First of all notice that, if

$$0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0$$

is an exact sequence in \mathcal{A} , then

$$0 \leq \mathfrak{S}Z(A) \leq \mathfrak{S}Z(E) \quad \text{and} \quad 0 \leq \mathfrak{S}Z(B) \leq \mathfrak{S}Z(E).$$

Let

$$\dots \subseteq E_{j+1} \subseteq E_j \subseteq \dots \subseteq E_1 \subseteq E_0 = E$$

be an infinite sequence of subobjects of an object E in \mathcal{A} with $\phi(E_{j+1}) > \phi(E_j)$, for all j . Since $\mathfrak{S}Z$ is discrete, there exists $N \in \mathcal{N}$ such that

$$0 \leq \mathfrak{S}Z(E_n) = \mathfrak{S}Z(E_{n+1}),$$

for all $n \geq N$. Consider the exact sequence in \mathcal{A}

$$0 \rightarrow E_{n+1} \rightarrow E_n \rightarrow F_{n+1} \rightarrow 0.$$

Then, by additivity of $\mathfrak{S}Z$, we have $\mathfrak{S}Z(F_{n+1}) = 0$, for all $n \geq N$. But this yields $\phi(F_{n+1}) = 1$, for all $n \geq N$ and so $\phi(E_{n+1}) \leq \phi(E_n)$, a contradiction. In this way, property (a) of [Bri07, Prop. 2.4] is satisfied.

Let

$$E = E_0 \twoheadrightarrow E_1 \twoheadrightarrow \dots \twoheadrightarrow E_j \twoheadrightarrow E_{j+1} \twoheadrightarrow \dots$$

be an infinite sequence of quotients of E in \mathcal{A} with $\phi(E_j) > \phi(E_{j+1})$, for all j . As before, $\mathfrak{S}Z(E_n) = \mathfrak{S}Z(E_{n+1})$, for all $n \geq N$. Consider the exact sequence in \mathcal{A}

$$0 \rightarrow F_n \rightarrow E_N \rightarrow E_n \rightarrow 0,$$

for $n \geq N$. Then $\Im Z(F_n) = 0$, i.e. $F_n \in \mathcal{P}'(1)$. Hence we have an infinite sequence of subobjects of E_N belonging to $\mathcal{P}'(1)$, a contradiction. Property (b) of [Bri07, Prop. 2.4] is then verified and the proposition is proved. \square

A stability condition is called *locally-finite* (see [Bri07, Sect. 5]) if there exists some $\epsilon > 0$ such that, for all $\phi \in \mathbb{R}$, each quasi-abelian subcategory $\mathcal{P}((\phi - \epsilon, \phi + \epsilon))$ is of finite length. In this way $\mathcal{P}(\phi)$ has finite length so that every object in $\mathcal{P}(\phi)$ has a finite Jordan–Hölder filtration into stable factors of the same phase. The set of stability conditions which are locally-finite will be denoted by $\text{Stab}(\mathcal{T})$. The stability conditions we consider also satisfy the additional conditions in the definition given in [KS08, Sect. 3.4] (in particular the *support property*). The local-finiteness condition will then be automatic.

The main result in [Bri07] endows $\text{Stab}(\mathcal{T})$ with a topology, induced by a metric $d(-, -)$ (see [Bri07, Prop. 8.1] for the explicit form of d), in such a way it becomes a complex manifold whose connected components are locally modeled on linear subspaces of $\text{Hom}(K(\mathcal{T}), \mathbb{C})$ via the map \mathcal{Z} sending a stability condition (Z, \mathcal{P}) to its central charge Z .

We recall the statement of Bridgeland’s deformation result in the situation where $\sigma = (\mathcal{P}, Z)$ is contained in a connected component $\text{Stab}^*(\mathcal{T})$ of maximal dimension, i.e. modeled on the whole $\text{Hom}(K(\mathcal{T}), \mathbb{C})$. In such a case, the metric

$$(23) \quad \|W\|_\sigma := \sup \left\{ \frac{|W(E)|}{|Z(E)|} : E \text{ is } \sigma\text{-stable} \right\}$$

is finite, and thus defines a topology on $\text{Hom}(K(\mathcal{T}), \mathbb{C})$.

Theorem B.3. [Bri07, Theorem 7.1][Bri08, Lemma 4.5] *In the situation of the previous paragraph, let $0 < \epsilon < 1/8$. Then, for any group homomorphism $W : K(\mathcal{T}) \rightarrow \mathbb{C}$ with*

$$\|W - Z\|_\sigma < \sin(\pi\epsilon),$$

there exists a unique (locally-finite) stability condition $\tau = (W, \mathcal{Q}) \in \text{Stab}^(\mathcal{T})$ with $d(\sigma, \tau) < \epsilon$.*

In particular this shows that the map $\mathcal{Z} : \text{Stab}^*(\mathcal{T}) \rightarrow \text{Hom}(K(\mathcal{T}), \mathbb{C})$ is a local homeomorphism.

Remark B.4. By [Bri07, Lemma 8.2], we have a left action on $\text{Stab}(\mathcal{T})$ by the autoequivalence group $\text{Aut}(\mathcal{T})$, and a right action by $\widetilde{\text{GL}}_2(\mathbb{R})$, the universal cover of the matrices in $\text{GL}_2(\mathbb{R})$ with positive determinant. The first action is defined, for $\Phi \in \text{Aut}(\mathcal{T})$, by $\Phi(Z, \mathcal{P}) = (Z \circ \phi_*^{-1}, \Phi(\mathcal{P}))$, where ϕ_* is the automorphism induced by Φ at the level of Grothendieck groups. The second one is the lift of the action of $\text{GL}_2(\mathbb{R})$ on $\text{Hom}(K(\mathcal{T}), \mathbb{C})$ (by

identifying $\mathbb{C} \cong \mathbb{R}^2$). Notice, in particular, that the additive group \mathbb{C} acts on $\text{Stab}(\mathcal{T})$, via the embedding $\mathbb{C} \hookrightarrow \widetilde{\text{GL}}_2(\mathbb{R})$.

APPENDIX C. WEAK AMPLE SEQUENCES AND TRIVIAL EQUIVALENCES

The notion of ample sequences in an abelian category \mathcal{A} was introduced by Bondal and Orlov in [BO01], and generalized to the notion of weak ample sequences in [IUU06, Appendix A]. We repeat their definition here:

Definition C.1. *A collection of objects $P_i \in \mathcal{A}, i \in \mathbb{Z}$ is called a weak ample sequence if for any pair of objects $F, G \in \mathcal{A}$, the following three conditions are satisfied for $i \ll 0$:*

- (a) *The evaluation morphism $\text{Hom}_{\mathcal{A}}(P_i, F) \otimes P_i \rightarrow F$ is surjective.*
- (b) *There exist l and a surjective morphism $\phi: P_i^{\oplus l} \rightarrow G$ such that the induced map $\text{Ext}_{\mathcal{A}}^j(G, F) \rightarrow \text{Ext}_{\mathcal{A}}^j(P_i^{\oplus l}, F)$ is the zero-map for all $j \neq 0$.*
- (c) $\text{Hom}_{\mathcal{A}}(F, P_i) = 0$.

Lemma C.2. *The collection $P_i := \mathcal{O}_{|i|\mathbb{P}^2}(4i)$ is a weak ample sequence in Coh_0 .*

Proof. The properties (a) and (c) are proven exactly as in [IUU06, Lemma 45].

To prove property (b), first note that there is an exact sequence

$$0 \rightarrow \pi^* \mathcal{O}(3i) \rightarrow \pi^* \mathcal{O}(4i) \rightarrow P_i \rightarrow 0.$$

If we choose $N \ll 0$ such that $H^k(X, F(j)) = 0$ for $j < 3N$ and $k \neq 0$, it follows by the long exact Ext-sequence that $\text{Ext}^k(P_i, F) = 0$ for $i < N$ and $k > 1$.

The remaining case, $k = 1$, follows as in [IUU06, Lemma 45]. \square

(Weak) ample sequences are a very useful tool for extending isomorphisms of functors (see [Orl97]): indeed, as observed in [IUU06, Proposition 44], an isomorphism between two functors restricted to a weak ample sequence in an abelian category \mathcal{A} extends to an isomorphism to the whole derived category $\text{D}^b(\mathcal{A})$. Here we will use them to prove the following result:

Proposition C.3. *Assume that $\Phi: \mathcal{D}_0 \rightarrow \mathcal{D}_0$ is an autoequivalence with $\Phi(\mathcal{O}_{\mathbb{P}^2}) \cong \mathcal{O}_{\mathbb{P}^2}$ and $\Phi(k(x)) \cong k(x)$ for all $x \in \mathbb{P}^2$. Then Φ is isomorphic to the pull-back by a scalar $u \in \mathbb{C}^* \subset \text{Aut } X$.*

The strategy of the proof is quite standard (see [Orl97, Kaw04, CS07]): given such an autoequivalence Φ , we will produce a sheaf $\mathcal{E} \in \text{Coh}(X \times X)$ and an isomorphism of functors on a weak ample sequence between Φ and

the Fourier–Mukai functor $\Phi_{\mathcal{E}}(_) := (p_1)_*(\mathcal{E} \otimes p_2^*(_))$ (here and in the sequel all functors are supposed to be derived and p_1, p_2 will always denote the two projections). As observed above, this isomorphism will extend to an isomorphism on the whole \mathcal{D}_0 . But then, the assumptions in Proposition C.3 will guarantee that \mathcal{E} is isomorphic, up to an automorphism of X , to the structure sheaf of the diagonal \mathcal{O}_{Δ_X} , and this will conclude the proof. Unfortunately, to work with the whole X may be slightly inconvenient: hence we will pass to the formal neighborhood of \mathbb{P}^2 inside X and work therein.

Notice that a proof could probably also be given in a more direct way, along the lines of [IUU06, Appendix A] and [IU05, Claim 3.8].

Let \widehat{X} be the formal completion of X along \mathbb{P}^2 , and denote again by $\pi: \widehat{X} \rightarrow \mathbb{P}^2$ the projection. For $n \in \mathbb{N}$, denote by X_n the n -th infinitesimal neighborhood of \mathbb{P}^2 inside X and $j_n: X_n \rightarrow \widehat{X}$, $j_n^m: X_n \rightarrow X_m$ ($n < m$) the natural inclusions ($X_0 = \mathbb{P}^2$). Then \mathcal{D}_0 can be embedded as full triangulated subcategory of $D^b(\widehat{X})$, the bounded derived category of coherent sheaves on \widehat{X} .

Consider the diagonal embedding $\Delta: \widehat{X} \rightarrow \widehat{X} \times \widehat{X}$ and set $\mathcal{O}_{\Delta_{\widehat{X}}} := \Delta_* \mathcal{O}_{\widehat{X}}$.

Step 1. By a local computation, we have

$$\begin{aligned} (j_n \times \text{id}_{\widehat{X}})^* \mathcal{O}_{\Delta_{\widehat{X}}} &\cong (\text{id}_{X_n} \times j_n)_* \mathcal{O}_{\Delta_{X_n}} \in \text{Coh}(X_n \times \widehat{X}), \\ (\text{id}_{\widehat{X}} \times j_n)^* \mathcal{O}_{\Delta_{\widehat{X}}} &\cong (j_n \times \text{id}_{X_n})_* \mathcal{O}_{\Delta_{X_n}} \in \text{Coh}(\widehat{X} \times X_n). \end{aligned}$$

Since \widehat{X} is quasi-projective, we can find a (possibly infinite) resolution of $\mathcal{O}_{\Delta_{\widehat{X}}}$ by a complex

$$C := \{C^{-l}\}_{l \in \mathbb{N}} := \left\{ L_l^{\oplus a_l} \boxtimes M_l^{\oplus b_l} \right\}_{l \in \mathbb{N}}$$

with L_l, M_l isomorphic to line bundles $\mathcal{O}_{\widehat{X}}(k_l) = \pi^*(\mathcal{O}_{\mathbb{P}^2})(k_l)$. By applying $(j_n \times \text{id})^*$ to C we have a new complex

$$C_n := \left\{ j_n^*(L_l)^{\oplus a_l} \boxtimes M_l^{\oplus b_l} \right\}_{l \in \mathbb{N}} \cong (j_n \times \text{id})^* \mathcal{O}_{\Delta_{\widehat{X}}}.$$

Similarly, we can find a resolution of $\mathcal{O}_{\Delta_{X_n}}$ by a complex

$$D_n := \{D_n^{-l}\}_{l \in \mathbb{N}} := \left\{ L_{n,l}^{\oplus a_{n,l}} \boxtimes M_{n,l}^{\oplus b_{n,l}} \right\}_{l \in \mathbb{N}}$$

with $L_{n,l}, M_{n,l}$ line bundles on X_n isomorphic to $\mathcal{O}_{X_n}(k_{n,l})$, for some $k_{n,l} \in \mathbb{Z}$. We have $C_n \cong (\text{id}_{X_n} \times j_n)_*(D_n)$.

Step 2. Let $\Phi: \mathcal{D}_0 \rightarrow \mathcal{D}_0$ be an autoequivalence as in Proposition C.3. Then a standard computation shows that Φ maps Coh_0 to itself. Moreover, it maps $\text{Coh}(X_n)$ into itself, i.e. there exist induced equivalences $\tilde{\Phi}_n: \text{Coh}(X_n) \rightarrow \text{Coh}(X_n)$ such that $\Phi \circ j_n \cong j_n \circ \tilde{\Phi}_n$.

For all $n \in \mathbb{N}$, apply $\tilde{\Phi}_n \boxtimes \text{id}$ to D_n to get a complex

$$\tilde{\mathcal{E}}_n := \left\{ \tilde{\Phi}_n(L_{n,l})^{\oplus a_{n,l}} \boxtimes M_{n,l}^{\oplus b_{n,l}} \right\}_{l \in \mathbb{N}} \in \text{D}^-(X_n \times X_n).$$

Consider the functor $\Phi_{\tilde{\mathcal{E}}_n}: \text{Coh}(X_n) \rightarrow \text{D}^-(X_n)$,

$$\Phi_{\tilde{\mathcal{E}}_n}(_) := (p_1)_* \left(\tilde{\mathcal{E}}_n \otimes p_2^*(_) \right).$$

Lemma C.4. *We have $(j_n)_* \Phi_{\tilde{\mathcal{E}}_n}(k(x)) \cong \Phi(k(x))$, for all $x \in \mathbb{P}^2$.*

Proof. First of all, notice that $H^p(X_n, M_{n,l} \otimes k(x)) = 0$, for all $p \neq 0$. Hence, we have

$$(24) \quad \begin{aligned} \Phi_{\tilde{\mathcal{E}}_n}(k(x)) &\cong \left\{ (p_1)_* \left(\tilde{\Phi}_n(L_{n,l})^{\oplus a_{n,l}} \boxtimes (M_{n,l}^{\oplus b_{n,l}} \otimes k(x)) \right) \right\} \\ &\cong \left\{ \tilde{\Phi}_n(L_{n,l})^{\oplus a_{n,l}} \otimes H^0(X_n, M_{n,l} \otimes k(x))^{\oplus b_{n,l}} \right\}. \end{aligned}$$

At the same time,

$$k(x) \cong \left\{ L_{n,l}^{\oplus a_{n,l}} \boxtimes H^0(X_n, M_{n,l} \otimes k(x))^{\oplus b_{n,l}} \right\}$$

and so, by applying $\tilde{\Phi}_n$,

$$(25) \quad \tilde{\Phi}_n(k(x)) \cong \left\{ \tilde{\Phi}_n(L_{n,l})^{\oplus a_{n,l}} \otimes H^0(X_n, M_{n,l} \otimes k(x))^{\oplus b_{n,l}} \right\}.$$

An easy check shows that the differentials of the complexes (24) and (25) are compatible. Hence we have the isomorphism $\Phi_{\tilde{\mathcal{E}}_n}(k(x)) \cong \tilde{\Phi}_n(k(x))$, and so $(j_n)_* \Phi_{\tilde{\mathcal{E}}_n}(k(x)) \cong \Phi(k(x))$, as wanted. \square

By Lemma C.4, we deduce immediately (using [Bri99, Lemma 4.3]) that $\tilde{\mathcal{E}}_n$ is a sheaf on $X_n \times X_n$, flat over X_n (with respect to p_2). Moreover, $\Phi_{\tilde{\mathcal{E}}_n}$ maps $\text{Coh}(X_n)$ into $\text{Coh}(X_n)$.

Step 3. Set $\mathcal{E}_n := (j_n \times \text{id}_{X_n})_* \tilde{\mathcal{E}}_n \in \text{Coh}(\hat{X} \times X_n)$. Then $\Phi_{\mathcal{E}_n} \cong (j_n)_* \Phi_{\tilde{\mathcal{E}}_n}$.

Lemma C.5. *There exist natural maps $\eta_n^m: \mathcal{E}_m \rightarrow \mathcal{E}_n$ in $\text{Coh}(\hat{X} \times X_n)$, for $n < m$.*

Proof. Consider the sheaf $\mathcal{F}_k := (\text{id}_{X_n} \times j_n)_* \tilde{\mathcal{E}}_n \in \text{Coh}(X_n \times \hat{X})$. Then $\mathcal{F}_k \cong (\tilde{\Phi} \boxtimes \text{id}_{\hat{X}})(C_k)$.

For $n < m$, the morphisms $\tilde{\Phi}(j_m^*(L_l)) \rightarrow \tilde{\Phi}(j_n^*(L_l))$ (induced by projections $j_m^*(L_l) \rightarrow j_n^*(L_l)$) induce morphisms $\mathcal{F}_m \rightarrow \mathcal{F}_n$. These last ones, induce morphisms $\tilde{\mathcal{E}}_m \rightarrow \tilde{\mathcal{E}}_n$ and so $\eta_n^m: \mathcal{E}_m \rightarrow \mathcal{E}_n$. \square

Set $\mathcal{E} := \lim \mathcal{E}_n \in \text{Coh}(\widehat{X} \times \widehat{X})$.

Lemma C.6. *There exists an isomorphism of functors*

$$\phi: \Phi_{\mathcal{E}}|_W \rightarrow \Phi|_W.$$

Proof. Define a subcategory W_m of $\text{Coh}(X_m)$ consisting of all sheaves $\mathcal{O}_{X_n}(h) := (j_n^m)_* j_n^*(\mathcal{O}_{\widehat{X}})(h)$ for $n < m$. We have to show that, for all $k \in \mathbb{N}$, there exist isomorphisms of functors

$$\phi_k: \Phi_{\mathcal{E}_k}|_{W_k} \rightarrow (\Phi \circ (j_n)_*)|_{W_n}$$

such that, for $n < m$, we have

$$\phi_m(\mathcal{O}_{X_n}(h)) = \phi_n(\mathcal{O}_{X_n}(h)) \circ \eta_n^m(\mathcal{O}_{X_n}(h)),$$

for all $h \in \mathbb{Z}$, where we denoted, by abuse of notation, η_n^m the natural transformation

$$\Phi_{\mathcal{E}_m}|_{W_n} \rightarrow \Phi_{\mathcal{E}_n}|_{W_n}$$

induced by Lemma C.5.

This follows directly from, e.g., [CS07, Sections 4.2,4.5]. Indeed, we can truncate the complex D_k to have $(j_k \times \text{id}_{X_k})_* \mathcal{O}_{\Delta_{X_k}}$ as a direct summand of a certain truncation of $(j_k \times \text{id}_{X_k})_* D_k$. Hence, even if we have an infinite resolution, we can limit ourself to consider only a finite number of objects $L_{k,l}, M_{k,l}$, for $0 \leq l \leq n_0, n_0 \gg 0$ independent of k .

Then, we define ϕ_k on the objects $P \in W_k$ such that $H^p(X_k, M_{k,i} \otimes P) = 0$, for all $p \neq 0$ and all $0 \leq l \leq n_0$. To extend to all W_k we use Beilinson's resolutions in \mathbb{P}^2 (pulled-back to X_k) and [Kaw04, Lemma 6.4].

Finally, the compatibilities follow easily from the definition of \mathcal{E}_k . \square

Step 4. Consider the functor $\Phi_{\mathcal{E}}$. A standard computation (see [Huy06, Corollary 5.23]) shows that, since $\Phi_{\mathcal{E}}(k(x)) \cong k(x)$ and $\Phi_{\mathcal{E}}(\mathcal{O}_{\mathbb{P}^2}) \cong \mathcal{O}_{\mathbb{P}^2}$, there exists a family of automorphisms $u_k: X_k \rightarrow X_k$ such that $u_0 = \text{id}$. It is not difficult to check that these are all compatible and so they induce an automorphism $u: \widehat{X} \rightarrow \widehat{X}$ (which corresponds to the multiplication by an element of \mathbb{C}^* on the fibers of π). By composing $\Phi_{\mathcal{E}}$ with u , we can assume $\mathcal{E} \cong \mathcal{O}_{\Delta_{\widehat{X}}}$. Hence we have an isomorphism of functors $\text{id}|_W \rightarrow \Phi|_W$. By [IUU06, Proposition 44], this induces an isomorphism of functors $\text{id} \rightarrow \Phi$, and Proposition C.3 is proved.

APPENDIX D. PROOF OF THE INEQUALITY FOR CENTRAL CHARGES

This appendix is a brief sketch of a complete proof of Observation (a) on page 42: On the fundamental domain $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, the functions $a(z), b(z)$ defined by equations (21) and (22) satisfy the inequalities of Definition 2.4. The general idea is to deduce the inequalities from inequalities for real or imaginary parts of holomorphic functions, which only need to be tested on

the boundary of the fundamental domain. Note that the boundary, expressed in z and ψ according to the convergence domains $|z| \leq \frac{1}{27}$ and $|\psi| < 1$ of our power series expansions, consists of two copies of $z \in [-1/27, 0]$, with the two natural branch choices of $\ln z$, and of the two ray segments $\psi \in [0, 1] \cdot e^{2\pi i/3}$ and $\psi \in [0, 1] \cdot e^{4\pi i/3}$. (We will refer to the boundary segments by $\arg(z) = \pm\pi$ etc.)

Step 1. We first show that $\Im(a) > 0$. For example by using the integral criteria, it can be shown easily that the series $\sum_{n=1}^{\infty} \frac{(3n-1)!}{n!^3} \frac{1}{27^n}$ converges to a real number less than $\frac{1}{\pi\sqrt{3}}$ (the exact value is 0.129...). Thus, for $|z| < 1/27$, $z \neq 0$, $|\arg(z)| < \pi$, we have

$$\Im(a(z)) \geq -\frac{1}{2\pi} \left(\ln \frac{1}{27} + 3 \sum_{n=1}^{\infty} \frac{(3n-1)!}{n!^3} \frac{1}{27^n} \right) > 0.$$

Passing to the ψ -coordinate, the inequality follows trivially on the two boundary rays from (22) and the definition of $\varpi_1(\psi)$.

Step 2. For $B = -\Im(b(z))/\Im(a(z))$ we have $-1 < B < 0$. More precisely, we will use $-1/2 \leq B < 0$ for $\Im(z) \geq 0$ (i.e., for $2\pi/3 < \arg(\psi) \leq \pi$ or $0 \leq \arg(z) < \pi$), and $-1 < B \leq -1/2$ for $\Im(z) \leq 0$ (i.e., for $\pi \leq \arg(\psi) < 4\pi/3$ or $-\pi < \arg(z) \leq 0$).

To show this, first notice that $\Im(Z(\mathcal{O}_{\mathbb{P}^2})(z)) = \Im(b(z)) > 0$: indeed, for $\arg(z) = \pi$ or $\arg(\psi) = \frac{2\pi}{3}$, it is trivial to check that $\Im(b(z)) = 0$. Similarly, the inequality holds strictly for $\arg(z) = -\pi$ or $\arg(\psi) = \frac{4\pi}{3}$, and it also holds around $z = 0$. Thus the strict inequality holds on the interior of the fundamental domain, and thus $B < 0$.

Similarly we can show $\Im(Z(\mathcal{O}_{\mathbb{P}^2}(-1)[2])(z)) = \Im(b(z) - a(z)) < 0$ and thus $-1 < B$. For the more precise statement, it is sufficient to look at the sign of $\Im(Z(\Omega_{\mathbb{P}^2}(1)[1])(z)) = \Im(b(z) - \frac{1}{2}a(z))$: once again the maximum principle shows that $\Im(Z(\Omega_{\mathbb{P}^2}(1)[1])(z)) \leq 0$, for $2\pi/3 < \arg(\psi) \leq \pi$ and $0 \leq \arg(z) < \pi$. This implies to $-\frac{1}{2} \leq B$. The case $\Im(z) \leq 0$ is analogous.

Step 3. Finally, to check that the other two inequalities of Definition 2.4 are satisfied, we show the following stronger statement:

$$(26) \quad -\Re(b(z)) - B\Re(a(z)) + \frac{B^2}{2} < \frac{3}{8} = \Delta_{-1/2},$$

for all $-1/2 \leq B \leq 0$, in the region $\Im(z) \geq 0$ (and an analogous statement, which we will skip, for $\Im(z) \leq 0$). By the claims of the previous step, this will imply Observation (a), as $\delta_{\infty}^{DP}(\mu) \geq \frac{1}{2}$, and as $\Delta_B \geq \frac{3}{8}$ for $B \notin \mathbb{Z}$. Also note that we only have to prove the inequality above for $B = 0$, and for $B = -\frac{1}{2}$.

First we note that $b(-1/27) = 0$ (with the choice of $\arg(-1/27) = \pi$); this can be deduced from the monodromy. From this, it follows that the series $\sum_{n=1}^{\infty} |l_n| \frac{1}{27^n}$ (from the definition, it is easy to see that $l_n = (-1)^n |l_n|$) converges to a real number less than 3 (the exact value is 1.558...). From this we can deduce (26) for the cases $\arg(z) = 0$ and $\arg(z) = \pi$: Setting $B = -\frac{1}{2}$ and $B = 0$ yields

$$\frac{1}{2}\Re(\omega_2(z)) + \frac{1}{8} < \frac{3}{8}, \quad \text{and} \quad \frac{1}{2}\Re(\omega_2(z)) - \frac{1}{2}\Re(\omega_1(z)) + \frac{1}{4} < \frac{3}{8}.$$

As $\Re(\omega_1(z)) = \frac{1}{2\pi}\Im(\ln(z))$, this would follow from

$$\frac{1}{2}\Re(\omega_2(z)) + \frac{1}{8} < \frac{3}{8} \quad \text{if } z < 0, \quad \text{and} \quad \frac{1}{2}\Re(\omega_2(z)) + \frac{1}{4} < \frac{3}{8} \quad \text{if } z > 0.$$

Finally using the definition of $\omega_2(z)$, both inequalities become

$$-\frac{1}{8\pi^2} \left((\ln|z|)^2 + 6 \ln|z| \cdot \sum_{n=1}^{\infty} \frac{(3n-1)!}{n!^3} (-z)^n + \sum_{n=1}^{\infty} |l_n| (-z)^n \right) < \frac{1}{8}.$$

But the quantity on the left is at most

$$-\frac{1}{8\pi^2} \left(\left(\ln \frac{1}{27} \right)^2 + 6 \ln \frac{1}{27} \cdot \sum_{n=1}^{\infty} \frac{(3n-1)!}{n!^3} \frac{1}{27^n} - \sum_{n=1}^{\infty} |l_n| \frac{1}{27^n} \right),$$

which is smaller than $\frac{1}{8}$ by the estimate of the last term mentioned earlier.

For the cases $\arg(\psi) = \pi$ and $\arg(\psi) = \frac{2\pi}{3}$, we first observe that, for $0 < \rho \leq 1$ and for any function $u: \mathbb{Z}_{>0} \rightarrow \{0, 1\}$ with $u(1) = 1$, we have

$$(27) \quad \begin{aligned} & \frac{1}{2\pi} \sum_{\substack{n=1 \\ 3 \nmid n}}^{\infty} \frac{\Gamma\left(\frac{n}{3}\right)}{\Gamma(n+1)\Gamma\left(1-\frac{n}{3}\right)^2} (3\rho)^n (-1)^{u(n)} \\ & \leq \rho \left(-\frac{3}{2\pi} \frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3}\right)^2} + \frac{1}{2\pi} \sum_{\substack{n=2 \\ 3 \nmid n}}^{\infty} \frac{\Gamma\left(\frac{n}{3}\right)}{\Gamma(n+1)\Gamma\left(1-\frac{n}{3}\right)^2} 3^n \right) < 0. \end{aligned}$$

Setting $B = 0$ and $B = -\frac{1}{2}$, and using the definition of $\varpi_1(\psi)$, the needed inequalities are

$$\frac{3}{8} > \frac{1}{3}\Re(\varpi_2(-\rho)) + \frac{1}{3},$$

for $\psi = -\rho$, and

$$\frac{3}{8} > \begin{cases} \frac{1}{6}\Re(\varpi_2(\rho e^{2\pi i/3})) + \frac{5}{24} \\ \frac{2}{3}\Re(\varpi_2(\rho e^{2\pi i/3})) + \frac{1}{3} \end{cases},$$

for $\psi = \rho e^{2\pi i/3}$, $0 < \rho \leq 1$. But, by (27),

$$\Re(\varpi_2(\rho e^{2\pi i/3})), \Re(\varpi_2(-\rho)) < 0.$$

Hence, (26) holds also for $\arg(\psi) = \pi$ and $\arg(\psi) = \frac{2\pi}{3}$, and the proof of Observation (a) is complete.

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