Pushforward monads

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Categorical universal algebra

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- Lawvere theories
- PROPs and operads
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- Monads $\rightarrow$ algebras in a **fixed** category

Changing the base category

Suppose we have two categories, and a monad on the first one:

$$T \circlearrowright \mathcal{C} \qquad\qquad \mathcal{D}$$

How can we consider **T-algebras in $\mathcal{D}$** without leaving the world of monads?

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- We'll need a way to relate $\mathcal{C}$ and $\mathcal{D}$, so we may ask for a functor G.

If G and $\mathcal{D}$ are *nice enough*, this is enough to construct a monad on $\mathcal{D}$ from T!

Pushforward monads

Let T be a monad and G be a functor.

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Definition (Street)

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There a unique way to make $G_\# T$ a monad on $\mathcal{D}$ so that ϵ becomes a *lax morphism of monads*.

Morphisms of monads

Let T be a monad on $\mathcal{C}$ and S be a monad on D.

Definition

A **lax morphism** $(F, \varphi) : (\mathcal{C}, T) \to (\mathcal{D}, S)$ is a functor $F : \mathcal{C} \to \mathcal{D}$ and a natural transformation $\varphi : SF \to FT$ which *intertwines* the units and multiplications.

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Equivalently, a lax morphism (F, φ) amounts to a **lift of** F to the respective categories of algebras:

$$\begin{tikzcd} \mathcal{C}^T \arrow[r, "F^\varphi"] \arrow[d, "U^T"'] & \mathcal{D}^S \arrow[d, "U^S"] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

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Let T be the monad on **Ab** whose algebras are rings, and $G : \mathbf{Ab} \to \mathbf{Set}$ be the forgetful functor. Then $G_{\#}T$ is the free ring monad on **Set**.

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Example

If $F \dashv G$, then $G_\# 1$ is the monad induced by the adjunction.

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This is remarkable! Through this general categorical machinery, we obtain **CHaus** from the concept of finiteness of a set **alone**.

More examples of codensity monads

G	$G_{\#}1$	$\mathcal{D}^{G_{\#}1}$
$\mathbf{fdVect}_k \hookrightarrow \mathbf{Vect}_k$	double dualisation	linearly compact vector spaces
$\mathbf{FinGrp} \hookrightarrow \mathbf{Grp}$	profinite completion	profinite groups
$\mathbf{Field} \hookrightarrow \mathbf{Ring}$	product of residue fields	$\mathbf{Prod}(\mathbf{Field})$

Pushforwards along fully faithful functors

Suppose $G : \mathcal{C} \hookrightarrow \mathcal{D}$ is fully faithful.

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Example

- For any G, we have $G^{\#}1 = 1$.
- The identity, powerset, ultrafilter, $E + -$, and $M \times -$ monads restrict along $\mathbf{FinSet} \hookrightarrow \mathbf{Set}$.

Pushforwards along fully faithful functors

Let $\mathbf{Mnd}(\mathcal{D})^{\mathrm{res}G}$ denote the full subcategory of $\mathbf{Mnd}(\mathcal{D})$ on those monads which restrict along G.

Theorem (D)

If G is fully faithful and pushforwards along it exist, then the pushforward construction gives a reflection

$$\begin{tikzcd} \mathbf{Mnd}(\mathcal{C}) \arrow[r, hook, bend right, "G_{\#}"'] \arrow[r, phantom, "\perp"] & \mathbf{Mnd}(\mathcal{D})^{resG} \arrow[l, bend right, "G^{\#}"'] \arrow[r, hook] & \mathbf{Mnd}(\mathcal{D}). \end{tikzcd}$$

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- $\mathbf{FinSet} \hookrightarrow \mathbf{Set}$;
- $\mathbf{Set}_{\geq\kappa} \hookrightarrow \mathbf{Set}$, where κ is any cardinal and $\mathbf{Set}_{\geq\kappa}$ is the category of sets with cardinality at least κ.

Monads on nonempty sets

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This gives a monad on $\mathbf{Mnd}(\mathbf{Set})$. For $T \in \mathbf{Mnd}(\mathbf{Set})$, we have

- $G_\# G^\# T(X) = TX$ for all $X \in \mathbf{Set}_{\geq 1}$;
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$G_{\#}G^{\#}$-algebras are monads all of whose pseudoconstants are actual constants.

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Example

The covariant powerset monad on **FinSet** has at least three extensions: the powerset monad, the finite powerset monad (the finitary extension), and the filter monad (the pushforward).

Compact Hausdorff algebras

Let T be a monad on **Set** that restricts along $i : \mathbf{FinSet} \hookrightarrow \mathbf{Set}$. Using the reflection theorem, we get two monad maps:

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 & \mathbf{Set}^{i_\# i^\# T} & \\
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Every $i_\# i^\# T$-algebra has a underlying T-algebra structure and an underlying compact Hausdorff topology. This situation is similar to having a **distributive law** between T and β.

Specific examples

T	$i_\# T$	$i_\# T$-algebras
1	β	compact Hausdorff spaces
const. at 1	const. at 1	1-element sets
$E + -$	$(E + -)\beta$	E-pointed compact Hausdorff spaces
$M \times -$	$(M \times -)\beta$	compact Hausdorff spaces with a discrete M action
2^{2^-}	2^{2^-}	complete atomic Boolean algebras
$\mathcal{P}$	filter monad	continuous lattices

The rank of pushforwards

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In fact, the situation is pretty dire:

Theorem (D)

If T is a consistent monad on **FinSet**, *then $i_\# T$ has no rank.*

- A monad has rank iff it is λ-ary for some regular cardinal λ.
- The only two inconsistent monads are the one that is constant at 1, and its unique submonad which sends $\varnothing$ to $\varnothing$.

Summary

Given a monad T on $\mathcal{C}$ and a *nice* functor $G : \mathcal{C} \to \mathcal{D}$,
one gets a **pushforward monad** $G_{\#}T$ on $\mathcal{D}$ which is suitably universal.

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If G is fully faithful, there is an **reflection** $\mathbf{Mnd}(\mathcal{C}) \underset{G_{\#}}{\overset{G^{\#}}{\underset{\hookrightarrow}{\overset{\leftarrow}{\perp}}}} \mathbf{Mnd}(\mathcal{D})^{\mathrm{res}G}$.

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If G is fully faithful, there is an **reflection** $\mathbf{Mnd}(\mathcal{C}) \underset{G_{\#}}{\overset{G^{\#}}{\underset{\perp}{\rightleftarrows}}} \mathbf{Mnd}(\mathcal{D})^{\mathrm{res}G}$.

For $i : \mathbf{FinSet} \to \mathbf{Set}$, the monad $i_{\#}T$ **almost never has rank**.

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  - Monads in **MND**(**Cat**) are **distributive laws**, so one should be able to push them forward. *Problem: how to compute right extensions in* **MND**(**Cat**)*?*
- What properties of monads are preserved by the pushforward process? E.g. idempotence, strength, commutativity, etc.

Thank you!

References

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