

Pushforward monads

Adrián Doña Mateo (www.maths.ed.ac.uk/~adona)

University of Edinburgh

17 October 2025, CATNIP

Categorical universal algebra

There are a number of ways of studying universal algebra using category theory:

- Lawvere theories
- PROPs and operads
- Monads

Categorical universal algebra

There are a number of ways of studying universal algebra using category theory:

- Lawvere theories $\rightarrow$ algebras in **any** category with finite products
- PROPs and operads
- Monads

Categorical universal algebra

There are a number of ways of studying universal algebra using category theory:

- Lawvere theories $\rightarrow$ algebras in **any** category with finite products
- PROPs and operads $\rightarrow$ algebras in **any** symmetric monoidal category
- Monads

Categorical universal algebra

There are a number of ways of studying universal algebra using category theory:

- Lawvere theories $\rightarrow$ algebras in **any** category with finite products
- PROPs and operads $\rightarrow$ algebras in **any** symmetric monoidal category
- Monads $\rightarrow$ algebras in a **fixed** category

Changing the base category

Suppose we have two categories, and a monad on the first one:

$$T \hookrightarrow \mathcal{C} \qquad \mathcal{D}$$

How can we consider **T -algebras in $\mathcal{D}$** without leaving the world of monads?

Changing the base category

Suppose we have two categories, and a monad on the first one:

$$\mathcal{T} \hookrightarrow \mathcal{C} \xrightarrow{G} \mathcal{D}$$

How can we consider **T -algebras in $\mathcal{D}$** without leaving the world of monads?

- We'll need a way to relate $\mathcal{C}$ and $\mathcal{D}$, so we may ask for a functor G .

Changing the base category

Suppose we have two categories, and a monad on the first one:

$$\mathcal{T} \subset \mathcal{C} \xrightarrow{G} \mathcal{D}$$

How can we consider **T -algebras in $\mathcal{D}$** without leaving the world of monads?

- We'll need a way to relate $\mathcal{C}$ and $\mathcal{D}$, so we may ask for a functor G .

If G and $\mathcal{D}$ are *nice enough*, this is enough to construct a monad on $\mathcal{D}$ from T !

Pushforward monads

Let T be a monad and G be a functor.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{G} & \mathcal{D} \\ \tau \downarrow & & \\ \mathcal{C} & \xrightarrow{G} & \mathcal{D} \end{array}$$

Pushforward monads

Let T be a monad and G be a functor.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{G} & \mathcal{D} \\ T \downarrow & \epsilon \swarrow & \downarrow \text{Ran}_G GT \\ \mathcal{C} & \xrightarrow{G} & \mathcal{D} \end{array}$$

Definition (Street)

If $\text{Ran}_G GT$ exists, then it is the **pushforward of T along G** , denoted by $G_{\#}T$.

Pushforward monads

Let T be a monad and G be a functor.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{G} & \mathcal{D} \\ T \downarrow & \epsilon \swarrow & \downarrow \text{Ran}_G GT \\ \mathcal{C} & \xrightarrow{G} & \mathcal{D} \end{array}$$

Definition (Street)

If $\text{Ran}_G GT$ exists, then it is the **pushforward of T along G** , denoted by $G_{\#}T$.

There a unique way to make $G_{\#}T$ a monad on $\mathcal{D}$ so that ϵ becomes a *lax morphism of monads*.

Morphisms of monads

Let T be a monad on $\mathcal{C}$ and S be a monad on $\mathcal{D}$.

Definition

A **lax morphism** $(F, \varphi) : (\mathcal{C}, T) \rightarrow (\mathcal{D}, S)$ is a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ and a natural transformation $\varphi : SF \rightarrow FT$ which *intertwines* the units and multiplications.

Morphisms of monads

Let T be a monad on $\mathcal{C}$ and S be a monad on $\mathcal{D}$.

Definition

A **lax morphism** $(F, \varphi) : (\mathcal{C}, T) \rightarrow (\mathcal{D}, S)$ is a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ and a natural transformation $\varphi : SF \rightarrow FT$ which *intertwines* the units and multiplications.

Equivalently, a lax morphism (F, φ) amounts to a **lift of F** to the respective categories of algebras:

$$\begin{array}{ccc} \mathcal{C}^T & \xrightarrow{F\varphi} & \mathcal{D}^S \\ U^T \downarrow & & \downarrow U^S \\ \mathcal{C} & \xrightarrow{F} & \mathcal{D} \end{array}$$

T -algebras in $\mathcal{D}$

In what sense are the $G_{\#}T$ -algebras like T -algebras in $\mathcal{D}$?

T -algebras in $\mathcal{D}$

In what sense are the $G_{\#}T$ -algebras like T -algebras in $\mathcal{D}$?

Theorem (Street)

$(G, \epsilon) : (\mathcal{C}, T) \rightarrow (\mathcal{D}, G_{\#}T)$ is the *initial lax morphism* from $(\mathcal{C}, T)$ to a monad on $\mathcal{D}$ whose functor part is G .

$$\begin{array}{ccc} \mathcal{C}^T & \xrightarrow{G^{\epsilon}} & \mathcal{D}^{G_{\#}T} \\ U^T \downarrow & & \downarrow U^{G_{\#}T} \\ \mathcal{C} & \xrightarrow{G} & \mathcal{D} \end{array}$$

T -algebras in $\mathcal{D}$

In what sense are the $G_{\#}T$ -algebras like T -algebras in $\mathcal{D}$?

Theorem (Street)

$(G, \epsilon) : (\mathcal{C}, T) \rightarrow (\mathcal{D}, G_{\#}T)$ is the *initial lax morphism* from $(\mathcal{C}, T)$ to a monad on $\mathcal{D}$ whose functor part is G .

A commutative diagram illustrating the relationship between different monads and functors. The diagram consists of the following components:

- Top row: $\mathcal{C}^T$ on the left, $\mathcal{D}^{G_{\#}T}$ in the middle, and $\mathcal{D}^S$ on the right.
- Bottom row: $\mathcal{C}$ on the left, $\mathcal{D}$ in the middle, and $\mathcal{D}$ on the right. The two $\mathcal{D}$'s are connected by a double line.
- Vertical arrows: $U^T : \mathcal{C}^T \rightarrow \mathcal{C}$, $U^{G_{\#}T} : \mathcal{D}^{G_{\#}T} \rightarrow \mathcal{D}$, and $U^S : \mathcal{D}^S \rightarrow \mathcal{D}$.
- Horizontal arrows: $G^\epsilon : \mathcal{C}^T \rightarrow \mathcal{D}^{G_{\#}T}$ and $G : \mathcal{C} \rightarrow \mathcal{D}$.
- A curved arrow labeled $\forall \overline{G}$ connects $\mathcal{C}^T$ to $\mathcal{D}^S$.

T -algebras in $\mathcal{D}$

In what sense are the $G_{\#}T$ -algebras like T -algebras in $\mathcal{D}$?

Theorem (Street)

$(G, \epsilon) : (\mathcal{C}, T) \rightarrow (\mathcal{D}, G_{\#}T)$ is the *initial lax morphism* from $(\mathcal{C}, T)$ to a monad on $\mathcal{D}$ whose functor part is G .

A commutative diagram illustrating the initial lax morphism property. The diagram consists of two rows of objects and three columns. The top row objects are $\mathcal{C}^T$, $\mathcal{D}^{G_{\#}T}$, and $\mathcal{D}^S$. The bottom row objects are $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{D}$. Arrows are as follows: a curved teal arrow from $\mathcal{C}^T$ to $\mathcal{D}^S$ labeled $\forall \overline{G}$; a solid teal arrow from $\mathcal{C}^T$ to $\mathcal{D}^{G_{\#}T}$ labeled G^ϵ ; a dashed orange arrow from $\mathcal{D}^{G_{\#}T}$ to $\mathcal{D}^S$ labeled $\exists!$; a solid teal arrow from $\mathcal{C}$ to $\mathcal{D}$ labeled G ; a solid teal arrow from $\mathcal{D}^{G_{\#}T}$ to $\mathcal{D}$ labeled $U^{G_{\#}T}$; a solid teal arrow from $\mathcal{D}$ to $\mathcal{D}$ labeled with a double line; a solid teal arrow from $\mathcal{C}^T$ to $\mathcal{C}$ labeled U^T ; and a solid teal arrow from $\mathcal{D}^S$ to $\mathcal{D}$ labeled U^S .

First examples

If $G : \mathcal{C} \rightarrow \mathcal{D}$ has a left adjoint F , then computing pushforwards along G is easy:

$$G_{\#}T = \text{Ran}_G GT = GTF$$

First examples

If $G : \mathcal{C} \rightarrow \mathcal{D}$ has a left adjoint F , then computing pushforwards along G is easy:

$$G_{\#}T = \text{Ran}_G GT = GTF$$

Example

Let T be the monad on **Ab** whose algebras are rings, and $G : \mathbf{Ab} \rightarrow \mathbf{Set}$ be the forgetful functor. Then $G_{\#}T$ is the free ring monad on **Set**.

First examples

If $G : \mathcal{C} \rightarrow \mathcal{D}$ has a left adjoint F , then computing pushforwards along G is easy:

$$G_{\#}T = \text{Ran}_G GT = GTF$$

Example

Let T be the monad on **Ab** whose algebras are rings, and $G : \mathbf{Ab} \rightarrow \mathbf{Set}$ be the forgetful functor. Then $G_{\#}T$ is the free ring monad on **Set**.

Example

If $F \dashv G$, then $G_{\#}1$ is the monad induced by the adjunction.

Codensity monads

The pushforward of the identity monad has its own name:

Definition

The **codensity monad** of $G : \mathcal{C} \rightarrow \mathcal{D}$ is $G_{\#}1$.

Codensity monads

The pushforward of the identity monad has its own name:

Definition

The **codensity monad** of $G : \mathcal{C} \rightarrow \mathcal{D}$ is $G_{\#}1$.

The universal property of pushforwards says that $U^{G_{\#}1} : \mathcal{D}^{G_{\#}1} \rightarrow \mathcal{D}$ is the **monadic reflection** of G .

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{G^e} & \mathcal{D}^{G_{\#}1} \\ & \searrow G & \downarrow U^{G_{\#}1} \\ & & \mathcal{D} \end{array}$$

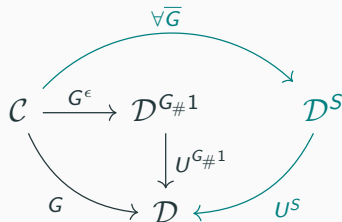
Codensity monads

The pushforward of the identity monad has its own name:

Definition

The **codensity monad** of $G : \mathcal{C} \rightarrow \mathcal{D}$ is $G_{\#}1$.

The universal property of pushforwards says that $U^{G_{\#}1} : \mathcal{D}^{G_{\#}1} \rightarrow \mathcal{D}$ is the **monadic reflection** of G .



Codensity monads

The pushforward of the identity monad has its own name:

Definition

The **codensity monad** of $G : \mathcal{C} \rightarrow \mathcal{D}$ is $G_{\#}1$.

The universal property of pushforwards says that $U^{G_{\#}1} : \mathcal{D}^{G_{\#}1} \rightarrow \mathcal{D}$ is the **monadic reflection** of G .

A commutative diagram illustrating the universal property of the codensity monad. The diagram consists of four nodes: $\mathcal{C}$ (top left), $\mathcal{D}^{G_{\#}1}$ (top middle), $\mathcal{D}^S$ (top right), and $\mathcal{D}$ (bottom middle). The arrows are as follows: a solid teal arrow from $\mathcal{C}$ to $\mathcal{D}^{G_{\#}1}$ labeled G^ϵ ; a dashed orange arrow from $\mathcal{D}^{G_{\#}1}$ to $\mathcal{D}^S$ labeled $\exists!$; a solid teal curved arrow from $\mathcal{C}$ to $\mathcal{D}$ labeled G ; a solid teal curved arrow from $\mathcal{D}^S$ to $\mathcal{D}$ labeled U^S ; a solid teal curved arrow from $\mathcal{C}$ to $\mathcal{D}^S$ labeled $\forall \overline{G}$; and a solid teal arrow from $\mathcal{D}^{G_{\#}1}$ to $\mathcal{D}$ labeled $U^{G_{\#}1}$.

Monadic reflection of FinSet

Example

What is the monadic reflection of the inclusion $i : \mathbf{FinSet} \hookrightarrow \mathbf{Set}$?

Monadic reflection of FinSet

Example

What is the monadic reflection of the inclusion $i : \mathbf{FinSet} \hookrightarrow \mathbf{Set}$?

$i_{\#}1$ is the **ultrafilter monad** β , and $\mathbf{Set}^{\beta} = \mathbf{CHaus}$

Monadic reflection of FinSet

Example

What is the monadic reflection of the inclusion $i : \mathbf{FinSet} \hookrightarrow \mathbf{Set}$?

$i_{\#}1$ is the **ultrafilter monad** β , and $\mathbf{Set}^{\beta} = \mathbf{CHaus}$

This is remarkable! Through this general categorical machinery, we obtain **CHaus** from the concept of finiteness of a set **alone**.

More examples of codensity monads

G	$G_{\#}1$	$\mathcal{D}^{G_{\#}1}$
$\mathbf{fdVect}_k \hookrightarrow \mathbf{Vect}_k$	double dualisation	linearly compact vector spaces
$\mathbf{FinGrp} \hookrightarrow \mathbf{Grp}$	profinite completion	profinite groups
$\mathbf{Field} \hookrightarrow \mathbf{Ring}$	product of residue fields	$\mathbf{Prod}(\mathbf{Field})$

Pushforwards along fully faithful functors

Suppose $G : \mathcal{C} \hookrightarrow \mathcal{D}$ is fully faithful.

Definition

A monad S on $\mathcal{D}$ **restricts along** G if, for all $c \in \mathcal{C}$, there is some $S'c \in \mathcal{C}$ such that $SGc \cong GS'c$.

Pushforwards along fully faithful functors

Suppose $G : \mathcal{C} \hookrightarrow \mathcal{D}$ is fully faithful.

Definition

A monad S on $\mathcal{D}$ **restricts along** G if, for all $c \in \mathcal{C}$, there is some $S'c \in \mathcal{C}$ such that $SGc \cong GS'c$. These choices assemble into a monad $G^\#S$ on $\mathcal{C}$ with $SG \cong G(G^\#S)$, called the **restriction of S along G** .

Pushforwards along fully faithful functors

Suppose $G : \mathcal{C} \hookrightarrow \mathcal{D}$ is fully faithful.

Definition

A monad S on $\mathcal{D}$ **restricts along** G if, for all $c \in \mathcal{C}$, there is some $S'c \in \mathcal{C}$ such that $SGc \cong GS'c$. These choices assemble into a monad $G^\#S$ on $\mathcal{C}$ with $SG \cong G(G^\#S)$, called the **restriction of S along G** .

Example

- For any G , we have $G^\#1 = 1$.

Pushforwards along fully faithful functors

Suppose $G : \mathcal{C} \hookrightarrow \mathcal{D}$ is fully faithful.

Definition

A monad S on $\mathcal{D}$ **restricts along** G if, for all $c \in \mathcal{C}$, there is some $S'c \in \mathcal{C}$ such that $SGc \cong GS'c$. These choices assemble into a monad $G^\#S$ on $\mathcal{C}$ with $SG \cong G(G^\#S)$, called the **restriction of S along G** .

Example

- For any G , we have $G^\#1 = 1$.
- The identity, powerset, ultrafilter, $E + -$, and $M \times -$ monads restrict along $\mathbf{FinSet} \hookrightarrow \mathbf{Set}$.

Pushforwards along fully faithful functors

Let $\mathbf{Mnd}(\mathcal{D})^{\text{res}G}$ denote the full subcategory of $\mathbf{Mnd}(\mathcal{D})$ on those monads which restrict along G .

Theorem (D)

If G is fully faithful and pushforwards along it exist, then the pushforward construction gives a *reflection*

$$\mathbf{Mnd}(\mathcal{C}) \begin{array}{c} \xleftarrow{G^\#} \\ \perp \\ \xrightarrow{G_\#} \end{array} \mathbf{Mnd}(\mathcal{D})^{\text{res}G} \hookrightarrow \mathbf{Mnd}(\mathcal{D}).$$

Examples in Set

The hypotheses of the theorem apply as soon as G is fully faithful and representably small, and $\mathcal{D}$ is complete.

Examples in Set

The hypotheses of the theorem apply as soon as G is fully faithful and representably small, and $\mathcal{D}$ is complete. This includes:

Example

- $\mathbf{FinSet} \hookrightarrow \mathbf{Set}$;

Examples in Set

The hypotheses of the theorem apply as soon as G is fully faithful and representably small, and $\mathcal{D}$ is complete. This includes:

Example

- **FinSet** $\hookrightarrow$ **Set**;
- **Set** _{$\geq \kappa$} $\hookrightarrow$ **Set**, where κ is any cardinal and **Set** _{$\geq \kappa$} is the category of sets with cardinality at least κ .

Monads on nonempty sets

Let $i : \mathbf{Set}_{\geq 1} \hookrightarrow \mathbf{Set}$. Since monads on $\mathbf{Set}$ preserve monos, **every monad restricts along i** .

Monads on nonempty sets

Let $i : \mathbf{Set}_{\geq 1} \hookrightarrow \mathbf{Set}$. Since monads on $\mathbf{Set}$ preserve monos, **every monad restricts along i** .

$$\begin{array}{ccc} & \xleftarrow{G^\#} & \\ \mathbf{Mnd}(\mathbf{Set}_{\geq 1}) & \perp & \mathbf{Mnd}(\mathbf{Set}) \\ & \xrightarrow{G_\#} & \end{array}$$

Monads on nonempty sets

Let $i : \mathbf{Set}_{\geq 1} \hookrightarrow \mathbf{Set}$. Since monads on $\mathbf{Set}$ preserve monos, **every monad restricts along i** .

$$\begin{array}{ccc} & \xleftarrow{G^\#} & \\ \mathbf{Mnd}(\mathbf{Set}_{\geq 1}) & \perp & \mathbf{Mnd}(\mathbf{Set}) \\ & \xrightarrow{G_\#} & \end{array}$$

This gives a monad on $\mathbf{Mnd}(\mathbf{Set})$. For $T \in \mathbf{Mnd}(\mathbf{Set})$, we have

- $G_\# G^\# T(X) = TX$ for all $X \in \mathbf{Set}_{\geq 1}$;
- $G_\# G^\# T(\emptyset)$ is the set of **pseudoconstants** of T .

Monads on nonempty sets

Let $i : \mathbf{Set}_{\geq 1} \hookrightarrow \mathbf{Set}$. Since monads on $\mathbf{Set}$ preserve monos, **every monad restricts along i** .

$$\begin{array}{ccc} & \xleftarrow{G^\#} & \\ \mathbf{Mnd}(\mathbf{Set}_{\geq 1}) & \perp & \mathbf{Mnd}(\mathbf{Set}) \\ & \xrightarrow{G_\#} & \end{array}$$

This gives a monad on $\mathbf{Mnd}(\mathbf{Set})$. For $T \in \mathbf{Mnd}(\mathbf{Set})$, we have

- $G_\# G^\# T(X) = TX$ for all $X \in \mathbf{Set}_{\geq 1}$;
- $G_\# G^\# T(\emptyset)$ is the set of **pseudoconstants** of T .

$G_\# G^\#$ -algebras are monads all of whose pseudoconstants are actual constants.

Monads on finite sets

Let $i : \mathbf{FinSet} \hookrightarrow \mathbf{Set}$.

- Every monad on **FinSet** is the restriction of some monad on **Set**.

Monads on finite sets

Let $i : \mathbf{FinSet} \hookrightarrow \mathbf{Set}$.

- Every monad on **FinSet** is the restriction of some monad on **Set**.
- Every monad T on **FinSet** has **at least two** extensions to a monad on **Set**.

Monads on finite sets

Let $i : \mathbf{FinSet} \hookrightarrow \mathbf{Set}$.

- Every monad on **FinSet** is the restriction of some monad on **Set**.
- Every monad T on **FinSet** has **at least two** extensions to a monad on **Set**.
 - $i_{\#}T = \text{Ran}_i iT$, the pushforward of T along i ;

Monads on finite sets

Let $i : \mathbf{FinSet} \hookrightarrow \mathbf{Set}$.

- Every monad on **FinSet** is the restriction of some monad on **Set**.
- Every monad T on **FinSet** has **at least two** extensions to a monad on **Set**.
 - $i_{\#}T = \text{Ran}_i iT$, the pushforward of T along i ;
 - $\text{Lan}_i iT$ the finitary monad determined by T .

Monads on finite sets

Let $i : \mathbf{FinSet} \hookrightarrow \mathbf{Set}$.

- Every monad on **FinSet** is the restriction of some monad on **Set**.
- Every monad T on **FinSet** has **at least two** extensions to a monad on **Set**.
 - $i_{\#}T = \text{Ran}_i iT$, the pushforward of T along i ;
 - $\text{Lan}_i iT$ the finitary monad determined by T .

Example

The covariant powerset monad on **FinSet** has at least three extensions: the powerset monad, the finite powerset monad (the finitary extension), and the filter monad (the pushforward).

Compact Hausdorff algebras

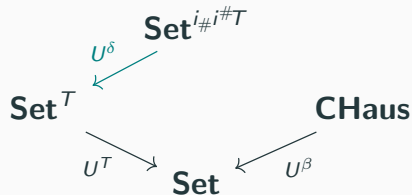
Let T be a monad on **Set** that restricts along $i : \mathbf{FinSet} \hookrightarrow \mathbf{Set}$. Using the reflection theorem, we get two monad maps:

$$\begin{array}{ccc} & \mathbf{Set}^{i_{\#}i^{\#}T} & \\ \mathbf{Set}^T & \searrow^{U^T} & \mathbf{CHaus} \\ & \mathbf{Set} & \swarrow_{U^{\beta}} \end{array}$$

Compact Hausdorff algebras

Let T be a monad on **Set** that restricts along $i : \mathbf{FinSet} \hookrightarrow \mathbf{Set}$. Using the reflection theorem, we get two monad maps:

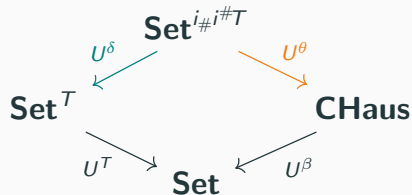
- the unit of the reflection gives
 $\delta : T \rightarrow i_{\#}i^{\#}T$;



Compact Hausdorff algebras

Let T be a monad on **Set** that restricts along $i : \mathbf{FinSet} \hookrightarrow \mathbf{Set}$. Using the reflection theorem, we get two monad maps:

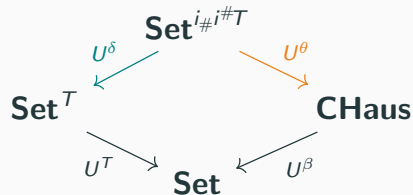
- the unit of the reflection gives $\delta : T \rightarrow i_{\#}i^{\#}T$;
- applying $i_{\#}$ to the unit of $i^{\#}T$ gives $\theta : i_{\#}1 \rightarrow i_{\#}i^{\#}T$.



Compact Hausdorff algebras

Let T be a monad on **Set** that restricts along $i : \mathbf{FinSet} \hookrightarrow \mathbf{Set}$. Using the reflection theorem, we get two monad maps:

- the unit of the reflection gives
 $\delta : T \rightarrow i_{\#}i^{\#}T$;
- applying $i_{\#}$ to the unit of $i^{\#}T$ gives
 $\theta : i_{\#}1 \rightarrow i_{\#}i^{\#}T$.

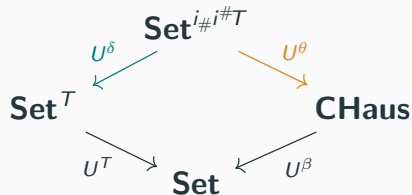


Every $i_{\#}i^{\#}T$ -algebra has a underlying T -algebra structure and an underlying compact Hausdorff topology.

Compact Hausdorff algebras

Let T be a monad on **Set** that restricts along $i : \mathbf{FinSet} \hookrightarrow \mathbf{Set}$. Using the reflection theorem, we get two monad maps:

- the unit of the reflection gives
 $\delta : T \rightarrow i_{\#}i^{\#}T$;
- applying $i_{\#}$ to the unit of $i^{\#}T$ gives
 $\theta : i_{\#}1 \rightarrow i_{\#}i^{\#}T$.



Every $i_{\#}i^{\#}T$ -algebra has a underlying T -algebra structure and an underlying compact Hausdorff topology. This situation is similar to having a **distributive law** between T and β .

Specific examples

T	$i_{\#}T$	$i_{\#}T$ -algebras
1	β	compact Hausdorff spaces
const. at 1	const. at 1	1-element sets
$E + -$	$(E + -)\beta$	E -pointed compact Hausdorff spaces
$M \times -$	$(M \times -)\beta$	compact Hausdorff spaces with a discrete M action
2^{2^-}	2^{2^-}	complete atomic Boolean algebras
$\mathcal{P}$	filter monad	continuous lattices

The rank of pushforwards

Every monad T on **FinSet** has a **unique** finitary extension to a monad on **Set**.
It follows that $i_{\#}T$ is finitary iff it **agrees** with this extension.

The rank of pushforwards

Every monad T on **FinSet** has a **unique** finitary extension to a monad on **Set**. It follows that $i_{\#}T$ is finitary iff it **agrees** with this extension.

In fact, the situation is pretty dire:

Theorem (D)

*If T is a consistent monad on **FinSet**, then $i_{\#}T$ has **no rank**.*

- A monad has rank iff it is λ -ary for some regular cardinal λ .
- The only two inconsistent monads are the one that is constant at 1, and its unique submonad which sends $\emptyset$ to $\emptyset$.

Summary

Given a monad T on $\mathcal{C}$ and a *nice* functor $G : \mathcal{C} \rightarrow \mathcal{D}$,
one gets a **pushforward monad** $G_{\#}T$ on $\mathcal{D}$ which is suitably universal.

Summary

Given a monad T on $\mathcal{C}$ and a *nice* functor $G : \mathcal{C} \rightarrow \mathcal{D}$,
one gets a **pushforward monad** $G_{\#}T$ on $\mathcal{D}$ which is suitably universal.

If G is fully faithful, there is an **reflection** $\mathbf{Mnd}(\mathcal{C}) \begin{array}{c} \xleftarrow{G_{\#}} \\ \perp \\ \xrightarrow{G_{\#}} \end{array} \mathbf{Mnd}(\mathcal{D})^{\text{res } G}$.

Summary

Given a monad T on $\mathcal{C}$ and a *nice* functor $G : \mathcal{C} \rightarrow \mathcal{D}$,
one gets a **pushforward monad** $G_{\#}T$ on $\mathcal{D}$ which is suitably universal.

If G is fully faithful, there is an **reflection** $\mathbf{Mnd}(\mathcal{C}) \begin{matrix} \xleftarrow{G^{\#}} \\ \perp \\ \xrightarrow{G_{\#}} \end{matrix} \mathbf{Mnd}(\mathcal{D})^{\text{res } G}$.

For $i : \mathbf{FinSet} \rightarrow \mathbf{Set}$, the monad $i_{\#}T$ **almost never has rank**.

Future directions

- The pushforward construction makes sense in any bicategory:

Future directions

- The pushforward construction makes sense in any bicategory:
 - $\mathcal{V}\text{-Mat}$, $\mathcal{V}\text{-Prof}$ and $\mathbf{Span}(\mathcal{C})$ often have all extensions.

Future directions

- The pushforward construction makes sense in any bicategory:
 - $\mathcal{V}\text{-Mat}$, $\mathcal{V}\text{-Prof}$ and $\mathbf{Span}(\mathcal{C})$ often have all extensions.
 - Monads in $\mathbf{MND}(\mathbf{Cat})$ are **distributive laws**, so one should be able to push them forward. *Problem: how to compute right extensions in $\mathbf{MND}(\mathbf{Cat})$?*

Future directions

- The pushforward construction makes sense in any bicategory:
 - $\mathcal{V}\text{-Mat}$, $\mathcal{V}\text{-Prof}$ and $\mathbf{Span}(\mathcal{C})$ often have all extensions.
 - Monads in $\mathbf{MND}(\mathbf{Cat})$ are **distributive laws**, so one should be able to push them forward. *Problem: how to compute right extensions in $\mathbf{MND}(\mathbf{Cat})$?*
- What properties of monads are preserved by the pushforward process?
E.g. idempotence, strength, commutativity, etc.

Thank you!

References

- A. Doña Mateo. Pushforward monads. *Theory and Applications of Categories*, 44(30):927–963, 2025.
- R. Garner. The Vietoris Monad and Weak Distributive Laws. *Applied Categorical Structures*, 28(2):339–354, 2020.
- J. F. Kennison and D. Gildenhuys. Equational completion, model induced triples and pro-objects. *Journal of Pure and Applied Algebra*, 1(4):317–346, 1971.
- T. Leinster. Codensity and the ultrafilter monad. *Theory and Applications of Categories*, 28(13):332–370, July 2013.
- R. Street. The formal theory of monads. *Journal of Pure and Applied Algebra*, 2(2):149–168, 1972.