Pushforward monads

Adrián Doña Mateo (www.maths.ed.ac.uk/~adona)

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University of Edinburgh

The main idea

Given a functor $G\colon \mathcal{C} \to \mathcal{D}$ and a monad T on $\mathcal{C}$, the **pushforward of T along** G is $G_\# T = \operatorname{Ran}_G GT$, which is canonically a monad on $\mathcal{D}$.

Today's main examples:

- The pushforward of $\mathcal{P}$ along **FinSet** $\hookrightarrow$ **Set** gives the theory of continuous lattices.
- The pushforward of 1 along **Field** $\hookrightarrow$ **Set** gives the theory of products of fields.

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Pushing a monad forward along a functor

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Well-known answer

If $F \dashv G$, then GTF is a monad on $\mathcal{D}$.

E.g. if $T = 1$, this is the monad induced by the adjunction $F \dashv G$.

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Little-known answer

When $\mathrm{Ran}_G\, GT$ exists, then it is canonically a monad on $\mathcal{D}$.

The pushforward of a monad

Definition

The **pushforward** of T along G is $G_{\#}T := \operatorname{Ran}_G GT$, when the latter exists.

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "G"] \arrow[d, "T"'] & \mathcal{D} \arrow[d, dashed, "\operatorname{Ran}_G GT"] \arrow[dl, phantom, "\overset{\kappa}{\Swarrow}"] \\
\mathcal{C} \arrow[r, "G"] & \mathcal{D}
\end{tikzcd}
$$

This comes with a monad structure, which I will now describe.

The monad structure of $G_{\#}T$

Let $G\colon \mathcal{C} \to \mathcal{D}$ and $T\colon \mathcal{C} \to \mathcal{C}$, and define a category $\mathcal{K}(G, T)$:

- The objects are pairs (S, σ) fitting into a diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "G"] \arrow[d, "T"'] & \mathcal{D} \arrow[d, "S"] \arrow[dl, phantom, "\overset{\sigma}{\swArrow}"] \\ \mathcal{C} \arrow[r, "G"'] & \mathcal{D} \end{tikzcd}$$

- A morphism $(S, \sigma) \to (S', \sigma')$ is a natural transformation $\alpha\colon S \Rightarrow S'$ such that $\sigma = \sigma' \circ \alpha G$.

The monad structure of $G_{\#}T$

If T is a monad, $\mathcal{K}(G, T)$ becomes strict monoidal.

The monoidal product of (S, σ) and (S', σ') and the monoidal unit are

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "G"] \arrow[d, "T"] \arrow[dd, bend right=60, "T"'] \arrow[dd, phantom, bend right=30, "\overset{\mu^T}{\Leftarrow}"] & \mathcal{D} \arrow[d, "S'"] \arrow[dl, phantom, "\overset{\sigma'}{\swArrow}"] \\
\mathcal{C} \arrow[r, "G"] \arrow[d, "T"] & \mathcal{D} \arrow[d, "S"] \arrow[dl, phantom, "\overset{\sigma}{\swArrow}"] \\
\mathcal{C} \arrow[r, "G"] & \mathcal{D}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{C} \arrow[r, "G"] \arrow[d, bend right, "T"'] \arrow[d, bend left, "1_{\mathcal{C}}"] \arrow[d, phantom, "\overset{\eta^T}{\Leftarrow}"] \arrow[dr, phantom, "/\!/"] & \mathcal{D} \arrow[d, "1_{\mathcal{D}}"] \\
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\mathcal{C} \arrow[r, "G"] \arrow[d, "T"] \arrow[dr, phantom, "\sigma \swArrow"] \arrow[l, phantom, "\overset{\mu^T}{\Leftarrow}"] & \mathcal{D} \arrow[d, "S"] \\
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\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{C} \arrow[r, "G"] \arrow[d, bend right=50, "T"'] \arrow[d, bend left=50, "1_{\mathcal{C}}"] \arrow[d, phantom, "\overset{\eta^T}{\Leftarrow}"] \arrow[dr, phantom, "="] & \mathcal{D} \arrow[d, "1_{\mathcal{D}}"] \\
\mathcal{C} \arrow[r, "G"] & \mathcal{D}
\end{tikzcd}
$$

$G_{\#}T = \mathrm{Ran}_G\, GT$ is the terminal object of $\mathcal{K}(G, T)$, so it has a unique monoid structure. This gives $G_{\#}T$ a canonical monad structure.

Pushforward along a right adjoint

Example

If $F \dashv G$, then $G_{\#}T = GTF$.

Pushforward along a right adjoint

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If $F \dashv G$, then $G_\# T = GTF$.

Proof. Recall that

right Kan extending along a right adjoint $=$ precomposing with the left adjoint,

so

$$G_\# T = \operatorname{Ran}_G GT = GTF.$$

Codensity monads

Definition

Let $G\colon \mathcal{C} \to \mathcal{D}$ be a functor. $G_{\#}1_{\mathcal{C}}$ is the **codensity monad** of G.

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Examples

- The codensity monad of $\mathbf{FinSet} \hookrightarrow \mathbf{Set}$ is the *ultrafilter monad*, whose algebras are compact Hausdorff spaces.
- The codensity monad of $\mathbf{Vect}_k^{\mathrm{fd}} \hookrightarrow \mathbf{Vect}_k$ is the *double dualisation monad*, whose algebras are linearly compact vector spaces.
- The codensity monad of $\mathbf{FinGrp} \hookrightarrow \mathbf{Grp}$ is the *profinite completion monad*, whose algebras are profinite groups.

A universal property of the pushforward

The right Kan extension comparison transformation κ gives a functor K:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "G"] \arrow[d, "T"'] \arrow[dr, phantom, "\overset{\kappa}{\swarrow}"] & \mathcal{D} \arrow[d, "G_{\#}T = \mathrm{Ran}_G\, GT"] \\ \mathcal{C} \arrow[r, "G"'] & \mathcal{D} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathcal{C}^T \arrow[r, "K"] \arrow[d, "U^T"] & \mathcal{D}^{G_{\#}T} \arrow[d, "U^{G_{\#}T}"] \\ \mathcal{C} \arrow[r, "G"] & \mathcal{D} \end{tikzcd}$$

Hence, K is an arrow in $\mathbf{CAT}/\mathcal{D}$.

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Let $\mathbf{Alg}\colon \mathbf{Mnd}(\mathcal{D})^{\mathrm{op}} \to \mathbf{CAT}/\mathcal{D}$ be the functor $S \mapsto \mathcal{D}^S$.

Theorem (Street)

K is a universal arrow from GU^T to $\mathbf{Alg}$.

A universal property of the pushforward

Theorem (Street) (continued)

More explicitly, we have an isomorphism, natural in S,

$$\mathbf{Mnd}(\mathcal{D})(S, G_{\#}T) \cong (\mathbf{CAT}/\mathcal{D})\left(\begin{array}{c} \mathcal{C}^T \\ \downarrow GU^T \\ \mathcal{D} \end{array}, \begin{array}{c} \mathcal{D}^S \\ \downarrow U^S \\ \mathcal{D} \end{array} \right).$$

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Motto

$U^{G_{\#}T}$ is the *universal monadic replacement* of GU^T.

A universal property of the pushforward

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Corollary

$G_\# T \cong (GU^T)_\# 1$, i.e. $G_\# T$ is the codensity monad of GU^T.

Pushing forward along $\mathbf{FinSet} \hookrightarrow \mathbf{Set}$

Some monads on Set and FinSet

Let $i\colon \mathbf{FinSet} \to \mathbf{Set}$ be the inclusion.

What happens when we push a monad forward along i?

Some monads on Set and FinSet

Let $i\colon \mathbf{FinSet} \to \mathbf{Set}$ be the inclusion.
What happens when we push a monad forward along i?

Any monad on **FinSet** is the restriction of a monad on **Set** (just take the pushforward), so we pick monads that restrict to **FinSet**.

Monad on **Set**	Algebras	On **FinSet**
$P_E := (-) + E$ for a finite set E	E-pointed sets	P^{f}_E
$A_M := M \times (-)$ for a finite monoid M	left M-sets	A^{f}_M
The powerset monad $\mathcal{P}$	suplattices	$\mathcal{P}^{\mathrm{f}}$

Pushing forward along FinSet $\hookrightarrow$ Set

Question

What is $i_\# T^{\mathrm{f}}$ for each of these monads?

Pushing forward along **FinSet** $\hookrightarrow$ **Set**

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What is $i_\# T^{\mathrm{f}}$ for each of these monads?

Two hints:

- The unit $1 \to T^{\mathrm{f}}$ gives a map of monads $i_\# 1 \to i_\# T^{\mathrm{f}}$.
  Recall that $U := i_\# 1$ is the **ultrafilter monad**.
- Because T^{f} is the restriction of T, we also get a map of monads $T \to i_\# T^{\mathrm{f}}$.

This gives forgetful functors $\mathbf{Set}^{i_\# T^{\mathrm{f}}} \to \mathbf{Set}^{U}$ and $\mathbf{Set}^{i_\# T^{\mathrm{f}}} \to \mathbf{Set}^{T}$.

Pushing forward along FinSet $\hookrightarrow$ Set

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Intuition

$i_\# T^{\mathrm{f}}$-algebras have an underlying T-algebra structure and compact Hausdorff topology, compatible in some way.

The case of P^{f}_E and A^{f}_M

Proposition (D)

There are distributive laws (which are isomorphisms)

$$UP_E \cong P_E U \quad \text{and} \quad UA_M \cong A_M U.$$

The case of P_E^{f} and A_M^{f}

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$$UP_E \cong P_E U \quad \text{and} \quad UA_M \cong A_M U.$$

This makes UP_E and UA_M monads on **Set**.

Theorem (D)

$$i_{\#}P_E^{\mathrm{f}} \cong UP_E \quad \text{and} \quad i_{\#}A_M^{\mathrm{f}} \cong UA_M.$$

The case of $\mathcal{P}^{\mathrm{f}}$

The filter monad F restricts to $\mathcal{P}^{\mathrm{f}}$ on **FinSet**. This gives a map $F \to i_{\#}\mathcal{P}^{\mathrm{f}}$.

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Proof sketch. Take

$$\varphi \in i_{\#}\mathcal{P}^{\mathrm{f}}(X) = \lim_{X \to n} \mathcal{P}n.$$

For $A \subseteq X$, let $\chi_A \colon X \to \{\bot, \top\}$ be the characteristic function.
Get a filter $\mathcal{F}$ on X by setting

$$A \in \mathcal{F} \iff \pi_{\chi_A}(\varphi) \subseteq \{\top\}.$$

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The F-algebras are **continuous lattices**, i.e. a certain kind of complete lattices with a compatible compact Hausdorff topology.

The definition of a continuous lattice is complicated.
We get it naturally from $\mathcal{P}$ and **FinSet** $\hookrightarrow$ **Set**.

The codensity monad of Field $\hookrightarrow$ Ring

The monad K

Let $i \colon \mathbf{Field} \to \mathbf{Ring}$ be the inclusion, and let $K := i_{\#}1$ be its codensity monad.
Recall that i is famously **not monadic** (**Field** doesn't have products).
U^K is its monadic replacement.

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For $R \in \textbf{Ring}$, we have

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A map from a ring to a field factors through the fraction field $\mathrm{Frac}(R/\mathfrak{p})$ for a unique prime ideal $\mathfrak{p}$. Hence,

$$KR = \prod_{\mathfrak{p} \in \operatorname{Spec} R} \mathrm{Frac}(R/\mathfrak{p}).$$

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To understand μ^K_R, we need to understand $\operatorname{Spec} KR$.

Proposition

The prime ideals of a product of fields are all maximal, and they correspond to ultrafilters on the indexing set.

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The prime ideals of a product of fields are all maximal, and they correspond to ultrafilters on the indexing set.

The multiplication μ^K_R only depends on those components indexed by primes corresponding to *principal ultrafilters*.

The category of K-algebras

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- Since **Ring** is complete and U^K creates limits, **Ring**^{K} is complete.

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Field has equalisers, but not products. It turns out that's all it's missing!

Theorem (D)

Ring$^K \cong$ **Prod**(**Field**) over **Ring**.

Proof sketch. K-algebra axioms $\implies$ every K-algebra is a product of fields, with a unique structure map (the projection onto the components that correspond to principal ultrafilters).

Pushing forward to Set

Let R denote the free ring monad on **Set**. What is $U^R_{\#}K$?

$$\begin{tikzcd} \mathbf{Prod}(\mathbf{Field}) \arrow[r, bend left, "U^K"] \arrow[r, phantom, "\top"] & \mathbf{Ring} \arrow[l, bend left, "F^K"] \arrow[r, bend left, "U^R"] \arrow[r, phantom, "\top"] & \mathbf{Set} \arrow[l, bend left, "F^R"] \end{tikzcd}$$

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U^R is a right adjoint $\implies U^R_\# K = U^R_\#(i_\# 1) = (U^R i)_\# 1$,
so $U^R_\# K$ is the codensity monad of the forgetful functor $\mathbf{Field} \to \mathbf{Set}$.

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Proposition (D)

$\mathbf{Prod}(\mathbf{Field})$ has and $U^R U^K$ preserves reflexive coequalisers.

The codensity monad of $\mathbf{Field} \to \mathbf{Set}$

Corollary (Kennison & Gildenhuys, Diers)

$\mathbf{Prod}(\mathbf{Field}) \to \mathbf{Set}$ is monadic and the corresponding monad is the codensity monad of $\mathbf{Field} \to \mathbf{Set}$.

The codensity monad of Field $\to$ Set

Corollary (Kennison & Gildenhuys, Diers)

Prod(**Field**) $\to$ **Set** is monadic and the corresponding monad is the codensity monad of **Field** $\to$ **Set**.

The theory of products of fields is the 'smallest' algebraic theory containing the theory of fields.

The codensity monad of Field $\to$ Set

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Prod(**Field**) $\to$ **Set** is monadic and the corresponding monad is the codensity monad of **Field** $\to$ **Set**.

The theory of products of fields is the 'smallest' algebraic theory containing the theory of fields.

This is a monad *without rank* with many interesting operations.
E.g. there are n-ary operations that vanish on all fields with fewer than n elements algebraically independent over the prime subfield.

Future work

- Pushforwards of lifted/restricted monads seem to give composite monads induced by different kinds of distributive laws.
  Can pushforwards be a generalisation of the latter?
- What happens if one pushes forward a monad along another?
  E.g. the category of algebras of the codensity monad of $(\cdot) + 1$ on $\mathbf{Set}$ is a 'modification' of the product completion of $\mathbf{Set}_*$.
- Is there a pushforward construction in the world of relative monads?

Thank you!

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Constants in Prod(Field)

- Constants: $\mathbb{Q} \times \mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{F}_5 \times \mathbb{F}_7 \times \cdots$

Given a field k, with $\operatorname{char} k = p$. The constant c in k is just c_p.

Operations in Prod(Field)

- n-ary operations: $\prod_{\mathfrak{p}\in \mathsf{Spec}\,\mathbb{Z}[t_1,\dots,t_n]} \mathsf{Frac}(\mathbb{Z}[t_1,\dots,t_n]/\mathfrak{p})$

Let k be a field, and θ an n-ary operation θ. A choice of n elements of k is equivalent to a ring homomorphism $h\colon \mathbb{Z}[t_1,\dots,t_n] \to k$. Then $\mathfrak{p} := \ker h$ is a prime ideal of $\mathbb{Z}[t_1,\dots,t_n]$, and applying θ to the elements $h(t_1),\dots,h(t_n)$ gives the image of $\theta_{\mathfrak{p}}$ under the rightmost morphism of

$$\begin{tikzcd}
\mathbb{Z}[t_1,\dots,t_n] \arrow[r, "q"] \arrow[d, "h"] & \mathbb{Z}[t_1,\dots,t_n]/\mathfrak{p} \arrow[r, "l"] \arrow[d] & \mathsf{Frac}(\mathbb{Z}[t_1,\dots,t_n]/\mathfrak{p}) \arrow[d] \\
k \arrow[r, equal] & k \arrow[r, equal] & k
\end{tikzcd}$$

Operations in Prod(Field)

Let $\tau \in \prod_{\mathfrak{p} \in \operatorname{Spec} \mathbb{Z}[t]} \operatorname{Frac}(\mathbb{Z}[t]/\mathfrak{p})$ be the unary operation with

- for each $p = 0$ or prime, set $\tau_{(p)} = 1$;
- $\tau_{\mathfrak{p}} = 0$ for every other $\mathfrak{p} \in \operatorname{Spec} \mathbb{Z}[t]$.

For k a field and $x \in k$, $\tau(x) = 1$ iff x is transcendental over the prime subfield of k.