Pushforward monads

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1 Oct 2024, CATNIP

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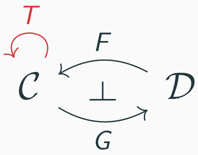
Pushing a monad forward along an adjunction

Let T be a monad on $\mathcal{C}$, and suppose we have an **adjunction**:



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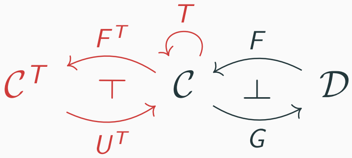
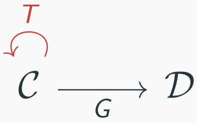


How can we get a monad on $\mathcal{D}$?

The composite adjunction $F^T F \dashv G U^T$ gives a monad on $\mathcal{D}$.

Pushing a monad forward along a functor

Let $\color{red}{T}$ be a monad on $\mathcal{C}$, and suppose we have a **functor**:



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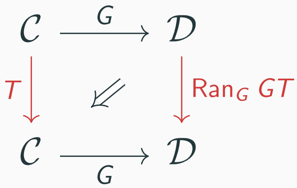

How can we get a monad on $\mathcal{D}$?

If $\mathrm{Ran}_G\, GT$ exists, then it has a canonical monad structure, and we call it the **pushforward of** T **along** G, denoted by $G_\# T$.

The universal property of the pushforward

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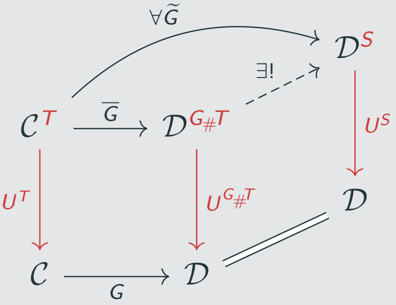
- An **object** of **Mnd** is a category equipped with a monad, e.g. $(\mathcal{C}, T)$.
- A **morphism** $(\mathcal{C}, T) \to (\mathcal{D}, S)$ is a *lax transformation of monads* or, equivalently, a pair of functors $(G, \overline{G})$ such that the following square commutes:

$$\begin{array}{ccc} \mathcal{C}^T & \xrightarrow{\overline{G}} & \mathcal{D}^S \\ {\scriptstyle U^T}\downarrow & & \downarrow{\scriptstyle U^S} \\ \mathcal{C} & \xrightarrow[G]{} & \mathcal{D} \end{array}$$

The universal property of the pushforward

Theorem (Street)

Let $G\colon \mathcal{C} \to \mathcal{D}$ be a functor, and T be a monad on $\mathcal{C}$. When $G_{\#}T$ exists, there is a canonical morphism $(G, \overline{G})\colon (\mathcal{C}, T) \to (\mathcal{D}, G_{\#}T)$ in **Mnd**, *and:*



The universal property of the pushforward

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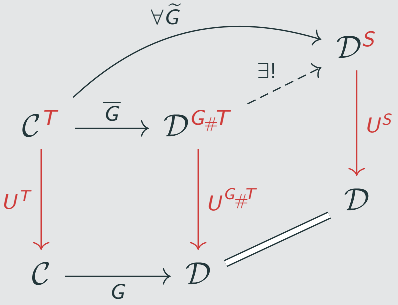

Thus, $\mathcal{D}^{G_{\#}T}$ is the **initial monadic-over-$\mathcal{D}$ replacement of** $\mathcal{C}^T$.

Examples

G	T	$\mathcal{C}^T$	$G_{\#}T$	$\mathcal{D}^{G_{\#}T}$

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fdVect$_k$ $\to$ **Vect**$_k$	id	**fdVect**$_k$	double dualisation	**lcVect**$_k$
Field $\to$ **Ring**	id	**Field**	product of residue fields	**Prod**(**Field**)

Thank you!

For more details and references:

Adrián Doña Mateo, *Pushforward monads*
https://arxiv.org/abs/2406.15256