MAGNITUDE HOMOLOGY IN EUCLIDEAN SPACE

STRUCTURE & SYMMETRY DAY

28 FEB 2024

ADRIÁN DOÑA MATEO

Magnitude homology is a homology theory of (certain) enriched categories and, in particular, of metric spaces.

> "SINGULAR HOMOLOGY DETECTS THE EXISTENCE OF HOLES. MAGNITUDE HOMOLOGY MEASURES THEIR SIZE."

OUTLINE

1. WHAT IS MAGNITUDE HOMOLOGY?
2. STRAIGHT METRIC SPACES & THEIR MAG. HOM.
3. MAG. HOM. EQUIVALENCES IN EUCLIDEAN SPACE

WHAT IS MAGNITUDE HOMOLOGY?

Let (X,d) be a metric space.

The **length** of $(x_0, \dots, x_n) \in X^{n+1}$ is
$$\ell(x_0, \dots, x_n) = \sum_{i=1}^{n} d(x_{i-1}, x_i)$$

The set of **proper chains** of X of degree n and length ℓ is
$$P_n^\ell(X) = \{ (x_0, \dots, x_n) \in X^{n+1} \mid x_i \neq x_{i+1} \text{ for } 0 \leq i < n \ \& \ \ell(x_0, \dots, x_n) = \ell \}$$

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The magnitude chain complex of X is

$$MC_n(X)=\bigoplus_{\ell\geq 0} MC_n^\ell(X),$$

where $MC_n^\ell(X)=\mathbb{Z}P_n^\ell(X)$.

With boundary map $\partial_n: MC_n^\ell(X)\longrightarrow MC_{n-1}^\ell(X)$ the alternating sum $\sum_{i=0}^{n}(-1)^i\partial_n^i$, where

$$\partial_n^i(x_0,\dots,x_n)=\begin{cases}(x_0,\dots,\widehat{x_i},\dots,x_n) & \text{if length unchanged,}\\ 0 & \text{otherwise.}\end{cases}$$

WHAT IS MAGNITUDE HOMOLOGY?

The magnitude homology of X, $MH_n^\ell(X)$ is the homology of the $MC_*^\ell(X)$ chain complex.

Let $\underline{\mathrm{Met}}_1$ denote the category of metric spaces and distance-decreasing maps.

MC_n^ℓ and MH_n^ℓ are functors

$$\underline{\mathrm{Met}}_1 \longrightarrow \underline{\mathrm{Ab}}$$

$f: X \to Y$ gives

$$f_*(x_0, \dots, x_n) = \begin{cases} (f(x_0), \dots, f(x_n)) & \text{if length unchanged,} \\ 0 & \text{o/w} \end{cases}$$

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The boundary map

$$\partial_1 : MC_1^\ell(X) \longrightarrow MC_0^\ell(X)$$

is 0, so

$$\begin{aligned} MH_0^\ell(X) &= MC_0^\ell(X) \\ &= \mathbb{Z}P_0^\ell(X) \\ &= \begin{cases} \mathbb{Z}X & \text{if } \ell = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Also, the 1-cycles are all of $MC_1^\ell(X)$.

Ex: DEGREES 0 AND 1

The boundary map

$$\partial_2 : MC_2^\ell(X) \longrightarrow MC_1^\ell(X)$$

sends (x_0, x_1, x_2) to $-(x_0, x_2)$ if

$$d(x_0, x_2) = d(x_0, x_1) + d(x_1, x_2),$$

and to 0 otherwise.

Say x_1 is **between** x_0 and x_2 when the triangle inequality is an equality.

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Say x_1 is **between** x_0 and x_2 when the triangle inequality is an equality.

Say x_0 and x_2 are **adjacent** if there is no x_1 between them.

Then
$$MH_1^\ell(X) =$$
$$\mathbb{Z}\{(x_0, x_1) \mid d(x_0, x_1) = \ell \ \&\ x_0 \text{ and } x_1 \text{ are adjacent}\}.$$

Theorem (Leinster & Shulman [1])

$MH_1(X) = 0$ iff X is Menger convex.

WHY MAGNITUDE HOMOLOGY?

One can define the magnitude homology of an enriched category.

- For categories enriched over finite sets, MH is the simplicial homology of the nerve.

- We view metric spaces as a particular kind of $\mathbb{R}^+$-enriched category.

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- For categories enriched over finite sets, MH is the simplicial homology of the nerve.

- We view metric spaces as a particular kind of $\mathbb{R}^+$-enriched category.

Under favourable conditions, magnitude homology categorifies magnitude.

Theorem (Leinster & Shulman [1])

If X is a finite metric space,

$$\mathrm{Mag}_{\mathbb{Q}((q^{\mathbb{R}}))}(X) = \sum_{n=0}^{\infty} (-1)^n \left(\sum_{\ell \geq 0} \mathrm{rk}\, MH_n^{\ell}(X)\, q^{\ell} \right)$$

BUT what about infinite metric spaces?

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STRAIGHT METRIC SPACES

The closed interval from x to z in X is the set

$$[x,z] = \{ y \in X \mid y \text{ is between } x \text{ and } z \}.$$

The open interval is

$$(x,z) = [x,z] \setminus \{x,z\}.$$

These intervals are partially ordered by setting

$$y \leq y' \Leftrightarrow y \in [x,y'] \Leftrightarrow y' \in [y,z].$$

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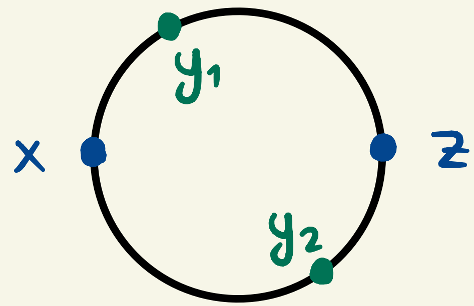
Say X is geodetic if $[x,z]$ is totally ordered for all $x,z \in X$.

Ex: any subspace of $\mathbb{R}^d$ is geodetic.

Non-ex: the circle, with the geodesic metric is not geodetic.



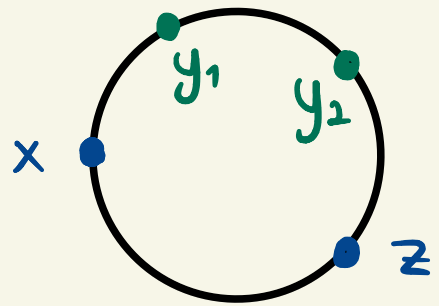
STRAIGHT METRIC SPACES

A **4-cut** in X is a tuple $(x, y_1, y_2, z) \in X^4$ such that

$$y_1 \in (x, y_2) \ \& \ y_2 \in (y_1, z)$$

but $y_1, y_2 \notin (x, z)$.

Ex:



Non-ex: any subspace of $\mathbb{R}^d$ has no 4-cuts.

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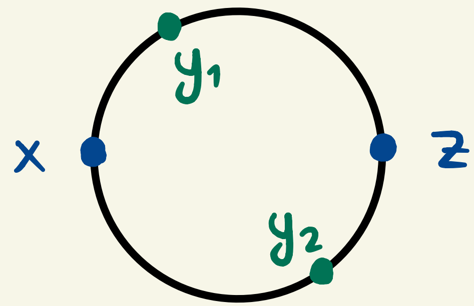

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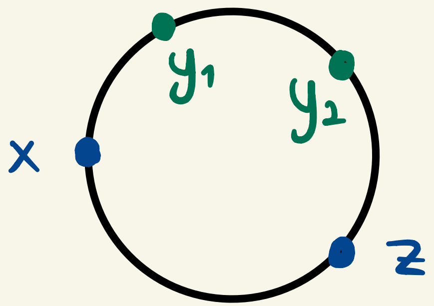

Non-ex: any subspace of $\mathbb{R}^d$ has no 4-cuts.

Say X is **straight** if it is geodetic and has no 4-cuts.

Ex: any subspace of $\mathbb{R}^d$ is straight.

Ex: any complete graph or tree is straight.

STRAIGHT MAGNITUDE HOMOLOGY

Kaneta & Yoshinaga [2] gave an explicit description of the magnitude homology of straight metric spaces.

Say $(x_0, \ldots, x_n) \in P_n^\ell(X)$ is a thin frame if

- $x_i \notin (x_{i-1}, x_{i+1})$ for $0 < i < n$,
- $(x_i, x_{i+1}) = \emptyset$ for $0 \leq i < n$.

Let $T_n^\ell(X) \subseteq P_n^\ell(X)$ be the set of thin frames.

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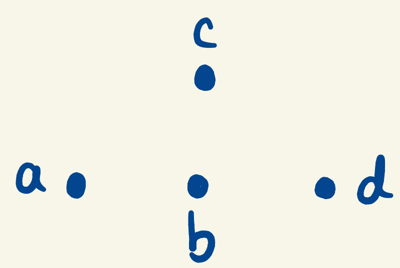
Ex: Let X consist of 4 points in $\mathbb{R}^d$



- (a,c,d) is a thin frame,
- (a,b,d) is NOT,
- (a,d,c) is NOT.

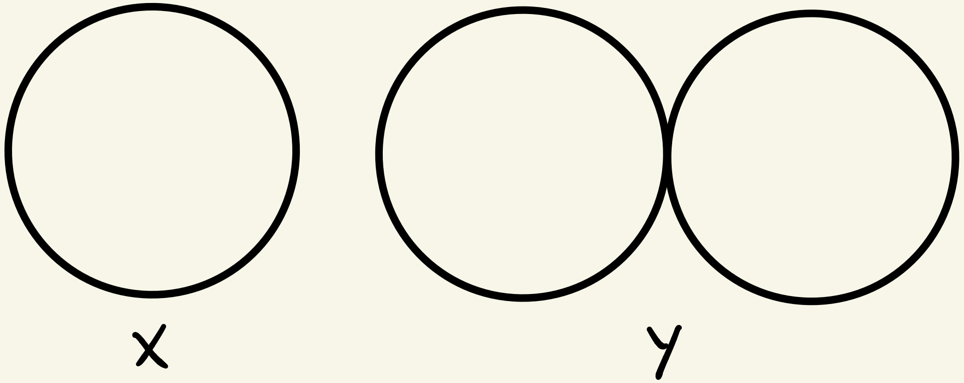
Theorem ([2])
If X is straight, then

$$MH_n^\ell(X) \cong \mathbb{Z}T_n^\ell(X).$$

Moreover, natural in X.

STRAIGHT MAGNITUDE HOMOLOGY

This has some striking consequences. Ex:



X and Y have isomorphic magnitude homology in all degrees and gradings.

BUT there is no map $f: X \to Y$ inducing this isomorphism.

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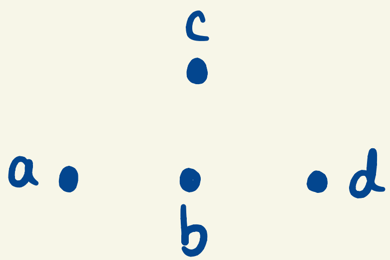

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INNER BOUNDARIES

For straight metric spaces, magnitude homology only cares about the points showing up in thin frames. These are all adjacent to something.

We focus on the set of such points.

The **inner boundary** of X is

$$\rho X = \{ x \in X \mid x \text{ is adjacent to some } y \in X \}$$

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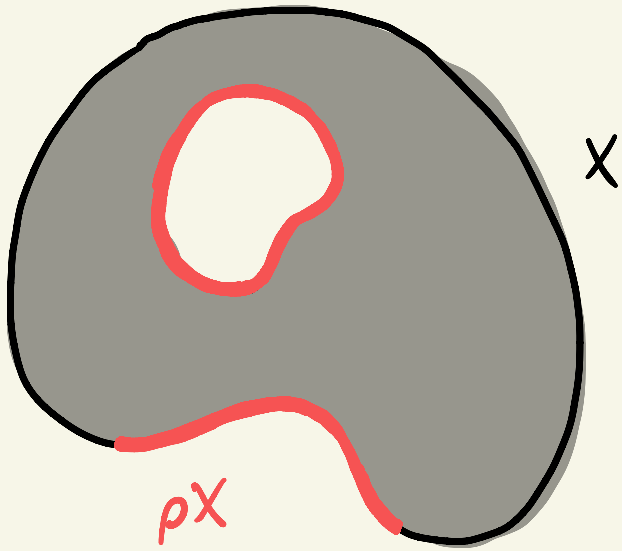
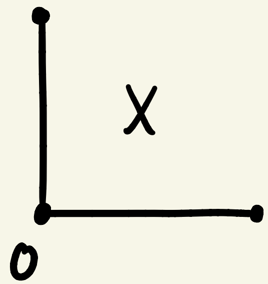


- If $X \subseteq \mathbb{R}^d$, then ρX is contained in the topological boundary of X.
- ρX need not be closed in X:



$\Rightarrow \rho X = X \setminus \{0\}$.

INNER BOUNDARIES

Theorem

Let X straight and $f,g: X \to Y$.

$$f(x) = g(x) \ \forall x \in \rho X \quad \Rightarrow$$

$$f_*, g_* : MC^*_{*>0}(X) \to MC^*_{*>0}(Y)$$

are chain homotopic.

Corollary

Let X straight, $\rho X \subseteq A \subseteq X$ and $r: X \to A$ a distance-decreasing retraction. Then

$$MH^*_{*>0}(X) \cong MH^*_{*>0}(A).$$

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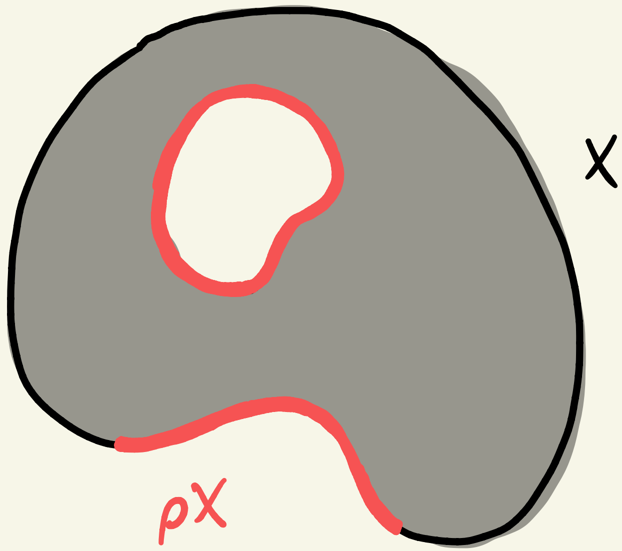

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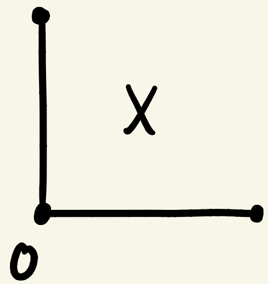

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PROOF: We have

$$\rho X \subseteq A \underset{r}{\overset{\iota}{\rightleftarrows}} X$$

with $r\iota = 1_A$. Since $\rho X \subseteq A$, ιr fixes ρX, so the theorem implies $r_* \iota_*$ is the identity. $\square$

Ex: Let $X \subseteq \mathbb{R}^d$ closed and convex. Sending $y \in \mathbb{R}^d$ to the unique closest point in X gives a retraction

$$\mathbb{R}^d \setminus \operatorname{int} X \to \operatorname{bd} X.$$

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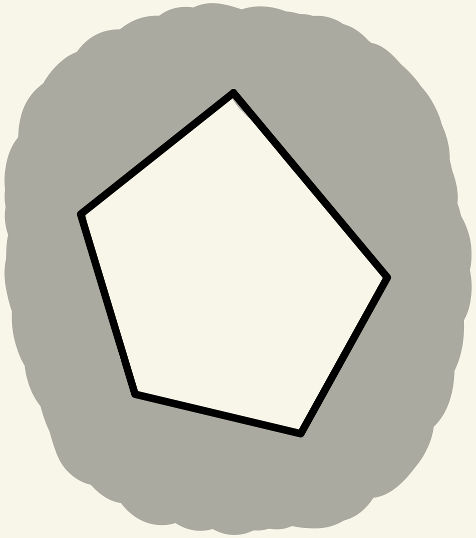
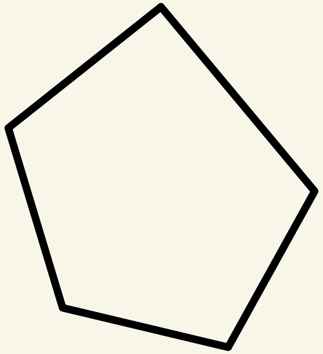


$\mathbb{R}^d \setminus \operatorname{int} X$



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Since the inner boundary of $\mathbb{R}^d \setminus \operatorname{int} X$ is precisely $\operatorname{bd} X$. We have that these two spaces are 'magnitude homology equivalent'!

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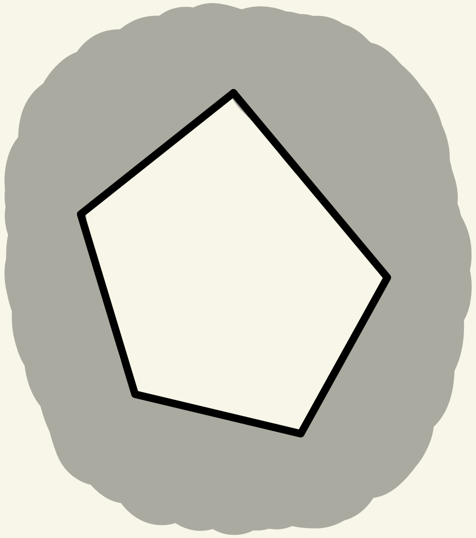

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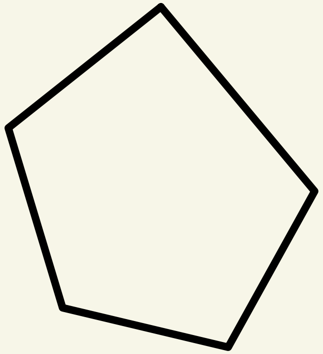

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Theorem

Let X straight, $f : X \longrightarrow X$.

TFAE:

(i) f_* is the identity on $MH^*_{*>0}(X)$,

(ii) f_* is the identity on $MH^*_n(X)$ for some $n > 0$,

(iii) f restricts to the identity on ρX.

In particular, if Y is also straight, then $f : X \rightleftarrows Y : g$ are inverse mag. hom. equiv. iff $gf|_{\rho X} = 1_{\rho X}$ and $fg|_{\rho Y} = 1_{\rho Y}$.

MAGNITUDE HOMOLOGY EQUIVALENCE

In fact, we can fully characterise when two closed subsets of $\mathbb{R}^d$ are mag. hom. equiv. using a purely geometric condition.

The **core** of $X \subseteq \mathbb{R}^d$ is

$$\mathrm{core}(X) = \overline{\mathrm{conv}}(\rho X) \cap X.$$

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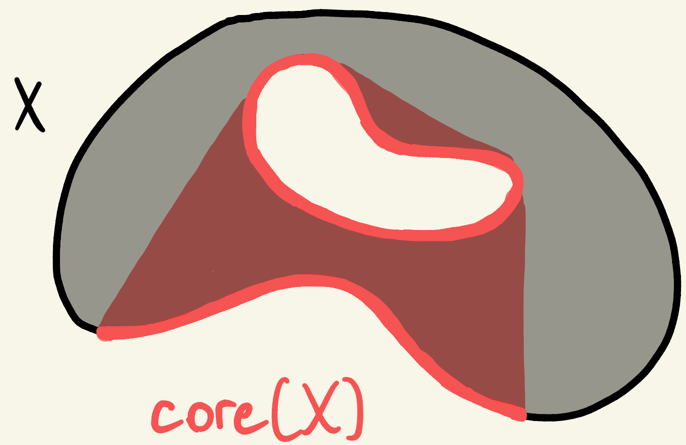

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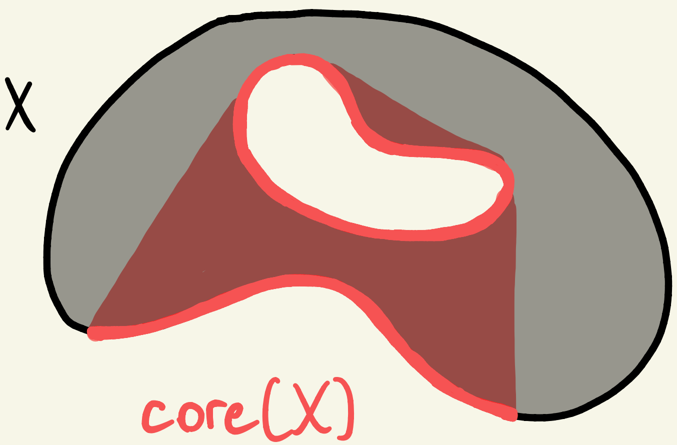
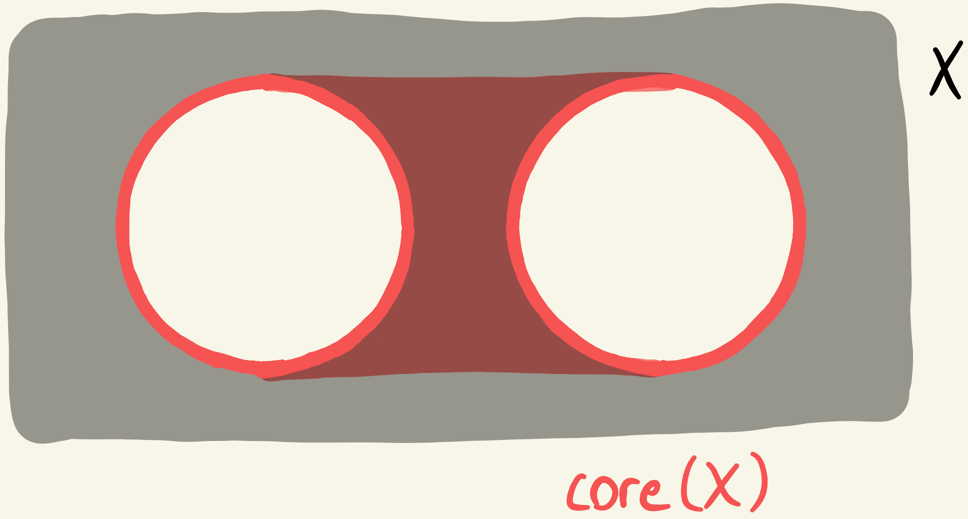




Theorem
Let X and $Y \subseteq \mathbb{R}^d$ closed and nonempty. Then X and Y are magnitude homology equivalent iff

$$\mathrm{core}(X) \cong \mathrm{core}(Y).$$

MAGNITUDE HOMOLOGY EQUIVALENCE

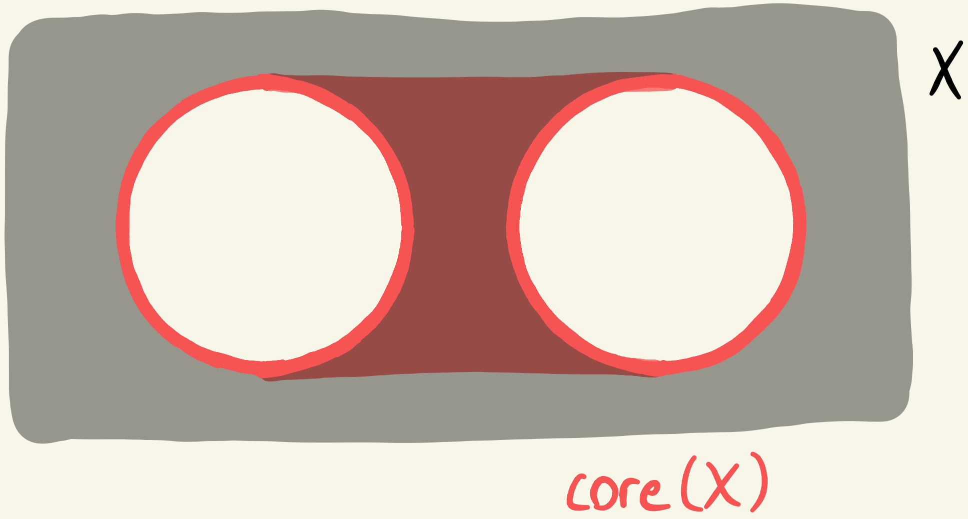
PROOF SKETCH: $(\Leftarrow)$

Using the metric projection, we get a distance-decreasing retraction $X \longrightarrow \mathrm{core}(X)$. So we get composite equivalences

$$X \simeq \mathrm{core}(X) \cong \mathrm{core}(Y) \simeq Y.$$

$(\Rightarrow)$ The maps in an equivalence restrict to isometries $\rho X \cong \rho Y$. The structure of $\mathbb{R}^d$ then gives $\overline{\mathrm{conv}}(\rho X) \cong \overline{\mathrm{conv}}(\rho Y)$ and, with some work, $\mathrm{core}(X) \cong \mathrm{core}(Y)$.

$\square$



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REFERENCES

[1] Leinster & Shulman. "Magnitude homology of enriched categories and metric spaces". 2021.

[2] Kaneta & Yoshinaga. "Magnitude homology of metric spaces and order complexes". 2021.

[3] Leinster. "Codensity and the ultrafilter monad." 2013.