The Magnitude of a Graph

GlaMS Example Showcase

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NOTIONS OF SIZE ACROSS MATHS

Structure	sets	f.d. vector spaces	topological spaces
Size function	cardinality	dimension	Euler characteristic

$$\chi(A \times B) = \chi(A) \cdot \chi(B)$$

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$$

$(A, B \subseteq X)$
+ assumptions

Magnitude

Magnitude is an invariant of enriched categories that generalises these notions of size.

In particular, we can talk about the magnitude of a metric space and, a fortiori, of a graph.

OUTLINE

1. Define the magnitude of a metric space.
2. Specialise to graphs.
3. See what magnitude can tell us.
4. See what magnitude can't tell us.

Magnitude of a Metric Space

- Let X be a finite metric space*.
- The similarity matrix of X is an X-by-X matrix with entries

$$S_X(a,b) = e^{-d(a,b)}$$

- If S_X is invertible, we say the magnitude of X is the sum of the entries of S_X^{-1}.

$$\#X = \sum_{a,b\in X} S_X^{-1}(a,b) \in \mathbb{R}$$

* we allow $d(a,b) = \infty$.

All Scales at Once

- Given $t > 0$, we can cosider the metric space tX.

$$d_{tX}(a,b) = t \cdot d_X(a,b)$$

- This gives a magnitude function of X.

$$\begin{aligned} (0,\infty) &\dashrightarrow \mathbb{R} \\ t &\longmapsto \#tX \end{aligned}$$

Graphs as Metric Spaces

A graph* G can be seen as a metric space with the shortest-path metric.

- $d(a,b) \in \mathbb{N} \cup \{\infty\}$; $\quad S_{tG}(a,b) = e^{-t\,d(a,b)}$.

$\Rightarrow$ entries of S_{tG}^{-1}, and hence $\#tG$, are rational functions of e^{-t} over $\mathbb{Q}$.

* finite, undirected, no loops, no parallel edges.

Graphs as Metric Spaces

For convenience, write $q := e^{-t}$. $(q^{\infty} = 0)$

Then $Z_G(a,b) = q^{d(a,b)}$, so $Z_G(q)$ a matrix over $\mathbb{Z}[q] \subset \mathbb{Q}(q)$.

If Z_G is invertible, we define the magnitude of G as

$$\# G(q) = \sum_{a,b \in G} \left(Z_G(q) \right)^{-1}(a,b) \in \mathbb{Q}(q).$$

Lemma. Z_G is always invertible over $\mathbb{Q}(q)$. Hence, $\#G$ is always defined and we can see it as a power series in $\mathbb{Z}[[q]]$.

Lemma. Let G and H be graphs, then

- $\#(G \sqcup H) = \#G + \#H$,
- $\#(G \,\square\, H) = \#G \cdot \#H$.

$\square$:= cartesian product of graphs $\neq$ categorical product

Computing Magnitude

Let $u = (1\ 1\ \ldots\ 1)^*$. Then

$$\#G(q) = \mathrm{sum}\left(Z_G(q)^{-1}\right) = u^* Z_G(q)^{-1} u$$

Definition. A weighting on G is a function (vector) $w_G : V(G) \to \mathbb{Q}(q)$, such that

$$\sum_{b \in V(G)} q^{d(a,b)} w_G(b) = 1 \quad \forall\, a \in V(G)$$

$$\iff \quad Z_G w_G = u.$$

Computing Magnitude

Lemma. If ω_G is a weighting on G, then

$$\# G = \sum_{a \in V(G)} \omega_G(a).$$

Basic Examples

Let G be a discrete graph (no edges).

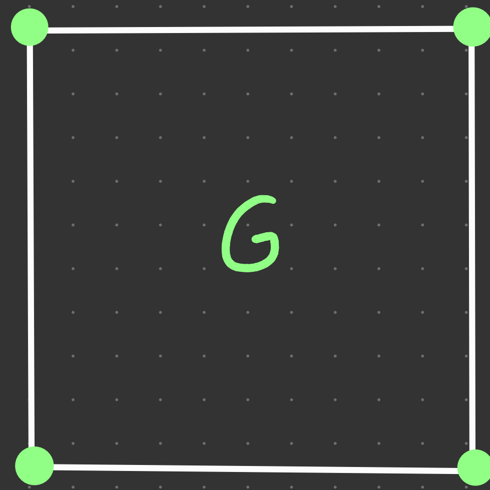
Then $d(a,b) = \infty \Rightarrow q^{d(a,b)} = 0, \quad a \neq b.$

$\therefore \quad Z_G(q) = \mathbb{I} \Rightarrow \#G(q) = |V(G)|.$

In fact, $Z_G(0) = \mathbb{I}$ for any G, so

$$\#G(0) = |V(G)|.$$

Basic Examples



BASIC EXAMPLES

Call a graph homogeneous if its group of automorphisms act transitively on the vertices.

Proposition. If G is homogeneous then

$$\# G(q) = \frac{V(G)}{\sum_{b \in V(G)} q^{d(a,b)}} \quad \text{for any } a \in V(G).$$

ALTERNATIVE FORMULA

Lemma. For any graph G,

$$\#G(q) = \sum_{k=0}^{\infty} (-1)^k \sum_{x_0 \neq x_1 \neq \cdots \neq x_k} q^{d(x_0,x_1) + \cdots + d(x_{k-1},x_k)}$$

where $x_0, \ldots, x_k \in V(G)$.

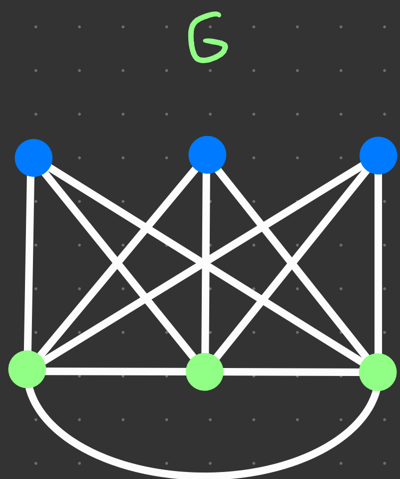
Corollary. The coefficient of q in $\#G(q) \in \mathbb{Z}[\![q]\!]$ is $-2|E(G)|$. Hence,

$$|V(G)| = \#G(0) \quad \text{and} \quad |E(G)| = -\frac{1}{2} \frac{d}{dq} \#G(q)\Big|_{q=0}.$$

Lemma. If $\omega_G(a)$ has no poles at 1 for all $a \in V(G)$, then

$$\#G(1) = \left| \left\{ \begin{array}{c} \text{connected components} \\ \text{of } G \end{array} \right\} \right| .$$

A Special Graph



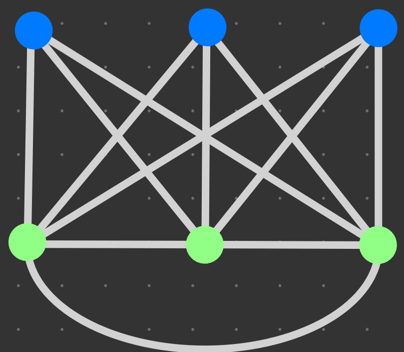
w = weight of blue vertices

v = weight of green vertices

$$\begin{cases} 1 = w + 3qv + 2q^2 w \\ 1 = v + q(2v + 3w) \end{cases}$$

$$\#G(q) = \frac{6}{1+4q}$$

A Special Graph



$$\#G(q) = \frac{6}{1+4q}$$

$$\#G(1) = \frac{6}{5} \neq \#\text{connected components}$$

But $\#K_5(q) = \dfrac{5}{1+4q}$, since K_5 is homogeneous.

Hence, $\#K_5^{\sqcup 6}(q) = \dfrac{30}{1+4q} = \#G^{\sqcup 5}(q)$.

$\Rightarrow$ magnitude cannot tell # connected components

THANK
YOU