Soergel bimodules and HOMFLY-PT homology

Part III Essay

Adrián Doña Mateo

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1 Introduction

Mathematicians use different algebraic invariants to tell topological spaces apart. For example, every topological space has an associated sequence of abelian groups: its singular homology. In the study of knots up to isotopy, many of these invariants come in the form of polynomials. The Alexander polynomial, the Jones polynomial and the HOMFLY-PT polynomial, in order of appearance, are examples of these. These two kinds of invariants are in essence very different: the first kind is a sequence of algebraic structures, while the second kind is just a polynomial. In a highly influential paper [8], Khovanov constructed a series of cohomology modules for each oriented link diagram, and showed that this construction is invariant under link isotopy. Moreover, he showed that taking the graded Euler characteristic of this series gives the Jones polynomial. This was the first time that a link invariant of the second kind was shown to originate from a invariant of the first kind, a process that is known as *categorification.* This new invariant, which we now know as Khovanov homology, has sparked great interest in the past two decades. For instance, it was shown that it detects the unknot [13], a question that is still unanswered for the Jones polynomial.

Shortly after the discovery of Khovanov homology, other knot polynomials, including the Alexander polynomial, were found to be the Euler characteristic of certain knot homology theories. In [12], Khovanov and Rozansky showed that the same is true for the HOMFLY-PT polynomial using complexes of matrix factorisations. This construction was later recast into the language of Hochschild homology of Soergel bimodules in [11]. This essay develops this homology theory, introducing all the necessary background.

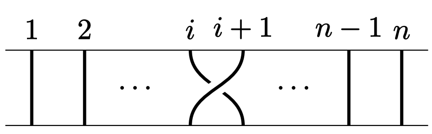
Section 2 gives an brief overview of braid groups and Hecke algebras, with a short introduction to the Kazhdan–Lusztig basis. Section 3 constructs the HOMFLY-PT polynomial, following [4], using the Ocneanu trace on the Hecke algebra. Section 4 introduces Soergel bimodules, as certain bimodules over a polynomial algebra. Discussion of the link homology of [11] properly begins in Section 5. Unlike in the original paper, which proves invariance by relating it to the homology defined in [12] using matrix factorisations, we will prove the result using Soergel bimodules exclusively. First, Rouquier complexes are introduced as a categorification of the standard basis of the Hecke algebra. Second, Hochschild homology is defined for algebras over a field, and Koszul complexes are used to compute the Hochschild homology some simple Soergel bimodules. Next, Khovanov's triply graded homology of links is constructed and a proof of its invariance is sketched. The essay finishes with a careful computation of the homology of the $(2,n)$-torus link. Appendix A provides a quick overview of graded rings and modules, mostly to establish the conventions used. Complexes and the homotopy category, which are ubiquitous in Section 5, are discussed in Appendix B. Examples are provided throughout and effort is put in making the details of calculations explicit.

2 Braid groups and Hecke algebras

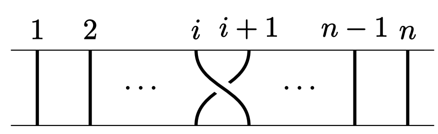
A braid consists of n points on a horizontal plane in Euclidean space connected by n strands to n points in another horizontal plane directly below it. The strands are only allowed to move downwards, i.e. if $\gamma : [0,1] \to \mathbf{R}^3$ is a normalized arc-length parameterisation of a strand then its z-coordinate is strictly monotone. Two braids can be concatenated by placing one below the other so that the endpoints coincide. This operation descends to isotopy classes of braids, which then form the *braid group* on n strands B_n. For an alternative definition of the braid group as a fundamental group see [1, §1.1]. We may take representatives of isotopy classes of braids where the n endpoints are equally spaced on a line. This allows us to embed $B_n \hookrightarrow B_{n+1}$ canonically by adding an endpoint after the previous n and a completely vertical strand connected to it. Artin proved that B_n is given by the presentation with generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$\begin{aligned} \sigma_i\sigma_j &= \sigma_j\sigma_i && \text{for } |i-j| \geq 2, \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} && \text{for } i \in \{1, \ldots, n-2\}. \end{aligned}$$

We may visualise these generators as follows.



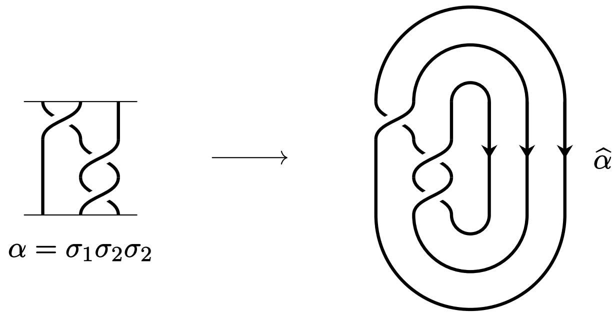
σ_i



σ_i^{-1}

If we add the relation $\sigma_i^2 = e$ for all i, we get a presentation of the symmetric group S_n. This gives a surjective homomorphism $B_n \to S_n$.

Given a braid $\alpha \in B_n$, its closure $\widehat{\alpha}$ is the oriented link given by attaching n more strands connecting each endpoint to the one below it as shown here:



Alexander proved that every tame link is isotopic to the closure of a braid. However, two different braids (even with different number of strands) may have isotopic closures. Markov addressed this problem by giving an algebraic criterion to detect when this happens.

Theorem 2.1. *Two braids have isotopic closures if and only if they are related by a sequence of the following moves:*

- *(M1)* *Change $\alpha \in B_n$ to $\beta\alpha\beta^{-1}$, for any $\beta \in B_n$.*
- *(M2)* *Change $\alpha \in B_n$ to $\alpha\sigma_n^{\pm 1} \in B_{n+1}$, or the inverse of this operation.*

Proofs of the results mentioned so far can be found in the first two chapters of [1]. By Theorem 2.1, if our goal is to find an invariant of links, it now suffices to find a function of $\bigcup_{n\geq 1} B_n$ that is invariant under (M1) and (M2). In order to define such a function, we first introduce an intermediate algebraic structure.

Definition 2.2. The *Hecke algebra* H_n is the unital associative algebra over $\mathbf{Z}[v, v^{-1}]$ generated by $\delta_1, \dots, \delta_{i-1}$ subject to the relations

$$\begin{aligned} \delta_i^2 &= (v^{-1} - v)\delta_i + 1 && \text{for } i \in \{1, \dots, n-1\}, \\ \delta_i\delta_j &= \delta_j\delta_i && \text{for } |i-j| \geq 2, \\ \delta_i\delta_{i+1}\delta_i &= \delta_{i+1}\delta_i\delta_{i+1} && \text{for } i \in \{1, \dots, n-2\}. \end{aligned} \tag{2.1}$$

Equivalently, H_n is the quotient of the group algebra of B_n over $\mathbf{Z}[v, v^{-1}]$ by the relations (2.1). As such, H_n is a representation of B_n where we send σ_i to δ_i. Given a commutative ring A and an invertible element $x \in A$, there is a unique ring homomorphism $\varphi : \mathbf{Z}[v, v^{-1}] \to A$ sending v to x. This gives A the structure of a $\mathbf{Z}[v, v^{-1}]$ algebra, and allows us to form the *specialisation* $(H_n)_\varphi := A \otimes_{\mathbf{Z}[v,v^{-1}]} H_n$ of H_n. For instance, taking $x = 1 \in \mathbf{Z}$, $(H_n)_\varphi$ becomes the group algebra $\mathbf{Z}[S_n]$. In this sense, H_n is seen as a deformation of the latter algebra.

Remark 2.3. Slightly different definitions of the Hecke algebra are in use throughout the literature. Essentially, they are all reparametrisations of our definitions. The explicit relation is given here for the benefit of the reader. Consider the homomorphism $\varphi : \mathbf{Z}[v, v^{-1}] \to \mathbf{Z}[q^{1/2}, q^{-1/2}]$ that sends v to $q^{-1/2}$. Then $(H_n)_\varphi$ is the Hecke algebra defined in [2], except that their generators are scaled by the invertible element $q^{1/2}$ instead, i.e. $T_i = q^{1/2}\delta_i$. In [4], this is further specialised by letting q be a nonzero complex number. Our choice of presentation is the one used in [17], which makes the statement of Theorem 2.4 below significantly simpler.

If we wish to rewrite a product of generators, relations (2.1) are as good as $\delta_i^2 = 1$, except that they create additional linear terms. Formally [17, Theorem 3.5], if W is a set of minimal-length words in $s_1, \ldots, s_{n-1}$ (the generators of S_n) representing all elements of S_n, then $\{\delta_w \mid w \in W\}$ is a $\mathbf{Z}[v, v^{-1}]$-linear basis for H_n, where $\delta_w = \delta_{i_1}\delta_{i_2}\ldots\delta_{i_m}$ if $w = s_{i_1}s_{i_2}\ldots s_{i_m} \in W$. In particular, this gives an obvious isomorphism of $\mathbf{Z}[v, v^{-1}]$-modules between H_n and $(\mathbf{Z}[v, v^{-1}])[S_n]$. A useful choice for W [4] is given by the words

$$\beta_1\beta_2\ldots\beta_n, \quad \text{where, for each } i,\ \beta_i = s_i s_{i-1}\ldots s_{i-k_i} \text{ for some } 0 \le k_i < i. \tag{2.2}$$

We call the corresponding basis of H_n the *standard basis*. It follows from (2.1) that each of the generators is invertible, with $\delta_i^{-1} = \delta_i + (v - v^{-1})$, and then so are all of the standard basis elements.

The Hecke algebra admits another interesting basis, introduced in [2]. First we define an $\mathbf{Z}$-linear ring automorphism $H_n \to H_n$, denoted $h \mapsto \overline{h}$, uniquely determined by setting $\overline{v} = v^{-1}$ and $\overline{\delta_i} = \delta_i^{-1}$ for each i. This self-inverse homomorphism is called the *Kazhdan–Lusztig involution.* They proved:

Theorem 2.4. *There is a unique set $\{b_w \mid w \in W\} \subseteq H_n$ with the property that, for any $w \in W$,*

(a) $\overline{b_w} = b_w$, and

(b) $b_w = \delta_w + \sum_{x<w} h_{x,w}\delta_x$ for some $h_{x,w} \in v\mathbf{Z}[v]$, where $<$ denotes the Bruhat order[1].

Such a set is a $\mathbf{Z}[v, v^{-1}]$-linear basis for H_n, called the Kazhdan–Lusztig basis*. The change of basis coefficients $h_{x,w}$ are called the* Kazhdan–Lusztig polynomials*.*

Example 2.5. The only $x \in W$ with $x < s_i$ is the identity, so there is some $f(v) \in v\mathbf{Z}[v]$ such that $b_{s_i} = \delta_i + f(v)$. Condition (a) forces

$$\delta_i + f(v) = \delta_i^{-1} + f(v^{-1}) = \delta_i + (v - v^{-1}) + f(v^{-1}),$$

which implies $f(v) = v$ and so $b_{s_i} = \delta_i + v$. This then gives, for instance,

$$b_{s_i}^2 = \delta_i^2 + 2v\delta_i + 1 = (v^{-1} - v)\delta_i + 2v\delta_i + 1 = vb_{s_i} + v^{-1}b_{s_i}. \tag{2.3}$$

The proof of Theorem 2.4 can be found in [17, §3], where the Kazhdan–Lusztig basis is constructed by induction on the Bruhat order. Since the δ_i generate H_n as an algebra, so do the $b_{s_i} = \delta_i + v$.

There is another important involution of H_n, which we call the *Kazhdan–Lusztig anti-involution.* This is the $\mathbf{Z}$-linear automorphism ω given by setting $\omega(v) = v^{-1}$ and $\omega(\delta_i) = \delta_i^{-1}$, as for the Kazhdan–Lusztig involution, but instead of ω being a ring automorphism we set $\omega(ab) = \omega(b)\omega(a)$ for any $a, b \in H_n$. Taking a reduced word for $w \in S_n$, it is easy to see that $\omega(\delta_w) = \delta_w^{-1}$. We also have

$$\omega(b_{s_i}) = \omega(\delta_i + v) = \delta_i^{-1} + v^{-1} = (\delta_i + v - v^{-1}) + v^{-1} = b_{s_i},$$

so that Kazhdan–Lusztig basis elements of the form b_{s_i} are also self-dual under ω. Using this anti-involution we can define two important functionals on the Hecke algebra: the standard trace and the standard form.

[1]The Bruhat order is defined for any Coxeter system in [17, §1.2.6]

Definition 2.6. The *standard trace* is the $\mathbf{Z}[v, v^{-1}]$-linear functional $\epsilon : H_n \to \mathbf{Z}[v, v^{-1}]$ that extracts the coefficient of δ_{id}, i.e. $\epsilon(1) = 1$ and $\epsilon(\delta_w) = 0$ for all $w \neq \mathrm{id}$.

Definition 2.7. The *standard form* is the $\mathbf{Z}$-bilinear form $(-,-) : H_n \times H_n \to \mathbf{Z}[v, v^{-1}]$ given by $(x, y) = \epsilon(\omega(x)y)$ for $x, y \in H_n$. This is $\mathbf{Z}[v, v^{-1}]$-linear in the second argument and satisfies $(vx, y) = v^{-1}(x, y)$. We call such a form *sesquilinear.*

It follows from the definition that if $\omega(z) = z$, then $(xz, y) = (x, zy)$, so in particular $(xb_{s_i}, y) = (x, b_{s_i}y)$. The standard form will play a crucial role in the theory of Soergel bimodules.

3 The Ocneanu trace and the HOMFLY-PT polynomial

We now define a function of $\bigcup_{n\geq 1} H_n$ that is invariant under the Markov move (M1) and *almost* invariant under (M2). This union should be seen as the direct limit of the sequence of embeddings $H_n \hookrightarrow H_{n+1}$ given by sending δ_i to δ_i. A function f that is invariant under (M1) is called a *trace*, since this condition is equivalent to requiring that $f(\alpha\beta) = f(\beta\alpha)$. The following trace was introduced by Ocneanu in [3], although the proof here is adapted from [4] using Remark 2.3.

Theorem 3.1. *There is unique $\mathbf{Z}[v, v^{-1}]$-linear map*

$$\mathrm{Tr} : \bigcup_{n\geq 1} H_n \to \mathbf{Z}[v, v^{-1}, z]$$

such that

(a) $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$ *for all* $a, b \in \bigcup_{n\geq 1} H_n$,

(b) $\mathrm{Tr}(1) = 1$, *and*

(c) $\mathrm{Tr}(x\delta_n) = zv\,\mathrm{Tr}(x)$ *for* $x \in H_n$.

Proof. First observe from (2.2) that an element of the standard basis of H_{n+1} is either of the form $x \in H_n$ or $y_1\delta_n y_2$ for some $y_1, y_2 \in H_n$. We define our trace inductively by specifying its values on these basis elements. We have $H_1 \cong \mathbf{Z}[v, v^{-1}]$ so we must set $\mathrm{Tr}(1) = 1$. For H_{n+1} we need only define the trace for basis elements of the second form above, and then properties (a) and (b) force $\mathrm{Tr}(y_1\delta_n y_2) = zv\,\mathrm{Tr}(y_1y_2)$. Note that this definition is determined by the properties in the statement of the theorem, so such a map is unique. By construction, Tr satisfies (b) and (c). It remains to check (a).

It suffices to check this for elements of the standard basis. First let b be a basis element of H_{n+1}. We show that for $i \leq n$ we have $\mathrm{Tr}(\delta_i b) = \mathrm{Tr}(b\delta_i)$. This certainly holds when $i < n$. Indeed, this follows by induction if $b \in H_n$, and otherwise we can remove the single δ_n factor by (c) and reduce to the $b \in H_n$ case. Similarly, if $b \in H_n$ then $\mathrm{Tr}(\delta_n b) = \mathrm{Tr}(b\delta_n)$. We are left with the case $b = y_1\delta_n y_2$ with $y_1, y_2 \in H_n$ and $i = n$. This further subdivides into four cases, depending on whether y_1 or y_2 are in H_{n-1}. For brevity, here is the case when $y_1 = x_1\delta_{n-1}x_2$ and $x_1, x_2, y_2 \in H_{n-1}$:

$$\begin{aligned}
\mathrm{Tr}(\delta_n b) = \mathrm{Tr}(\delta_n x_1\delta_{n-1}x_2\delta_n y_2) &= \mathrm{Tr}(x_1\delta_n\delta_{n-1}\delta_n x_2y_2)\\
&= \mathrm{Tr}(x_1\delta_{n-1}\delta_n\delta_{n-1}x_2y_2)\\
&= zv\,\mathrm{Tr}(x_1\delta_{n-1}^2x_2y_2)\\
&= zv(v^{-1} - v)\,\mathrm{Tr}(x_1\delta_{n-1}x_2y_2) + zv\,\mathrm{Tr}(x_1x_2y_2),
\end{aligned}$$

while

$$\begin{aligned}\operatorname{Tr}(b\delta_n) = \operatorname{Tr}(x_1\delta_{n-1}x_2\delta_n y_2\delta_n) &= \operatorname{Tr}(x_1\delta_{n-1}x_2y_2\delta_n^2)\\ &= (v^{-1}-v)\operatorname{Tr}(x_1\delta_{n-1}x_2y_2\delta_n) + \operatorname{Tr}(x_1\delta_{n-1}x_2y_2)\\ &= zv(v^{-1}-v)\operatorname{Tr}(x_1\delta_{n-1}x_2y_2) + zv\operatorname{Tr}(x_1x_2y_2).\end{aligned}$$

The other cases are similar. Now if $a = \delta_{i_1}\delta_{i_2}\ldots\delta_{i_k}$, we can pass the δ_{i_j} through b one at a time to get $\operatorname{Tr}(ab) = \operatorname{Tr}(ba)$. □

This trace is promising, but it is not quite invariant under (M2). Luckily, this can be solved if we scale the standard basis elements by a suitable factor. The idea is to choose an assignment $\rho : B_n \to H_n$ such that $\operatorname{Tr}\circ\rho$ changes the same way under both moves of type (M2), that is to say $\operatorname{Tr}(\rho(\alpha\sigma_n)) = \operatorname{Tr}(\rho(\alpha\sigma_n^{-1}))$ for $\alpha \in B_n$. A natural way of specifying ρ is setting $\rho(\sigma_i) = \theta\delta_i$ for some scalar θ and

$$\rho\left(\sigma_{i_1}^{\epsilon_1}\sigma_{i_2}^{\epsilon_2}\ldots\sigma_{i_k}^{\epsilon_k}\right) = \rho(\sigma_{i_1})^{\epsilon_1}\rho(\sigma_{i_2})^{\epsilon_2}\cdots\rho(\sigma_{i_k})^{\epsilon_k}.$$

This is well-defined because the δ_i satisfy all the relations that the σ_i satisfy. For such an assignment we would need $\operatorname{Tr}(\theta\delta_i) = \operatorname{Tr}((\theta\delta_i)^{-1})$, which gives

$$\theta^2 = \frac{\operatorname{Tr}(\delta_i^{-1})}{\operatorname{Tr}(\delta_i)} = \frac{\operatorname{Tr}(\delta_i + v - v^{-1})}{zv} = \frac{z+1-v^{-2}}{z}.$$

Evidently θ^2 does not lie in $\mathbf{Z}[v, v^{-1}, z]$, let alone θ. The answer is to specialise H_n using the ring homomorphism $\mathbf{Z}[v, v^{-1}] \to \mathbf{C}$ sending v to a nonzero complex number which we still denote v, and letting z be some other undetermined nonzero complex number. Our trace is then exactly that of [4, Theorem 5.1] after setting $q = v^{-2}$. We can then fix a square root θ of θ^2 and define ρ as above. Note that we can eliminate z, since $z = (1 - v^{-2})/(\theta^2 - 1)$ and then

$$\operatorname{Tr}(\rho(\sigma_i)) = \operatorname{Tr}(\rho(\sigma_i)^{-1}) = \frac{\theta(v-v^{-1})}{\theta^2-1} = \frac{v-v^{-1}}{\theta-\theta^{-1}}.$$

Proposition 3.2. *For a braid $\alpha \in B_n$, the two-variable rational function*

$$X_\alpha(\theta, v) = \left(\frac{\theta-\theta^{-1}}{v-v^{-1}}\right)^{n-1}\operatorname{Tr}(\rho(\alpha))$$

only depends on $\widehat{\alpha}$.

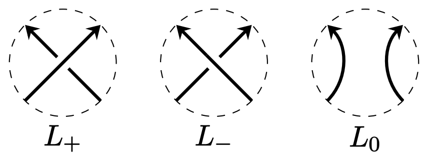
Proof. By Theorem 2.1 we only need to check that X_α is invariant under the Markov moves. It is invariant under (M1) because Tr is, and invariant under (M2) because

$$\begin{aligned}X_{\alpha\sigma_n^{\pm1}}(\theta, v) &= \left(\frac{\theta-\theta^{-1}}{v-v^{-1}}\right)^{n}\operatorname{Tr}(\rho(\alpha\sigma_n^{\pm1}))\\ &= \left(\frac{\theta-\theta^{-1}}{v-v^{-1}}\right)^{n}\cdot\frac{v-v^{-1}}{\theta-\theta^{-1}}\cdot\operatorname{Tr}(\rho(\alpha)) = X_\alpha(\theta, v).\end{aligned}$$

□

For any link L we can then define $X_L = X_\alpha$ where α is any braid such that $\widehat{\alpha} = L$. This invariant, or rather its reparametrisation in the next theorem [4, Proposition 6.2], is the *HOMFLY-PT polynomial.*

Theorem 3.3. *For each oriented link L there is a Laurent polynomial $P_L(t,x)$ in variables t and x uniquely defined by $P_\bigcirc = 1$ and the skein relation $t^{-1}P_{L_+} - tP_{L_-} = xP_{L_0}$. Here L_+, L_- and L_0 are oriented links with identical projections except at one crossing, where they are as shown here:*



Moreover, if $t = \theta$ and $x = v^{-1} - v$, then $P_L(t,x) = X_L(\theta, v)$.

Proof. Let L be an oriented link with a preferred crossing. If the crossing is positive, we set $L_+ = L$, otherwise we set $L_- = L$. Take a braid α such that $\widehat{\alpha} = L_-$. The crossing then corresponds to some σ_i^{-1} in the word α, and after a suitable (M1) move we may assume that α ends with σ_i^{-1}. Then $L_0 = \widehat{\alpha\sigma_i}$ and $L_+ = \widehat{\alpha\sigma_i^2}$, and

$$\begin{aligned}\operatorname{Tr}(\rho(\alpha\sigma_i^2)) = \theta^2 \operatorname{Tr}(\rho(\alpha)\delta_i^2) &= \theta^2(v^{-1} - v)\operatorname{Tr}(\rho(\alpha)\delta_i) + \theta^2 \operatorname{Tr}(\rho(\alpha)) \\ &= \theta(v^{-1} - v)\operatorname{Tr}(\rho(\alpha\sigma_i)) + \theta^2 \operatorname{Tr}(\rho(\alpha)).\end{aligned}$$

After multiplying by θ^{-1}, we see that $\theta^{-1}X_{L_+} - \theta X_{L_-} = (v^{-1} - v)X_{L_0}$, which is exactly the relation satisfied by P. Note also that $P_\bigcirc = 1 = X_\bigcirc$.

To show that invariance under isotopy and the skein relation is sufficient to compute P_L we induct on the number of crossings of a projection of L. A projection with no crossings represents the n-component unlink. By adding a kink to one of the circles in such a projection, the skein relation gives $P_L = ((t^{-1} - t)/x)^{n-1} = X_L$. For the inductive step, note that given a link projection one can always change a subset of the crossings from over- to under-crossing or vice versa in such a way that a diagram of the n-component unlink is left. Performing these changes in some order we get a sequence of link diagrams $L_0 = L, L_1, \ldots, L_k$, where L_k is the unlink. Assuming we have already computed P for all link projections with one fewer crossing and showed that it agrees with X, the skein relation gives an equation for P_{L_i} in terms of $P_{L_{i+1}}$. Since we already know P_{L_k} we can compute P_L. Moreover, since X satisfies the same relation we have $P_L = X_L$. As the latter is uniquely defined, so is P_L. In particular, if we had chosen a different way of changing the crossings in our projection to get the unlink we would still arrive at the same value for P_L. □

The existence of this polynomial invariant can be proved purely combinatorially with a careful induction argument [5]. Our algebraic method greatly simplifies the proof that the skein relation uniquely defines the polynomial. It also gives a method of finding the HOMFLY-PT polynomial from a braid word that is easy for a computer to perform.

4 Soergel bimodules

Our path towards a categorification of the HOMFLY-PT polynomial has the theory of Soergel bimodules at its foundation. These are certain graded bimodules over a polynomial algebra. They were first studied by Soergel with the goal of giving an algebraic proof of the Kazhdan–Lusztig conjecture, which relates certain quantities important to Lie theory to the value at 1 of the Kazhdan–Lusztig polynomials. This section gives an introduction to the Soergel bimodules of the Coxeter group S_n. For an extensive treatment of the general case see [17]. The reader unfamiliar with graded rings and modules should find Appendix A useful at this point.

4.1 Bott–Samelson bimodules

We begin by defining Bott–Samelson bimodules, which are the first examples of Soergel bimodules. To introduce our base ring, fix a positive integer n. Let k be a field of characteristic zero and let S_n act on k^n by permuting the standard basis vectors $x_1, \dots, x_n$. The subspace U of k^n consisting of the vectors whose coordinates add up to zero is preserved by this action. It has a basis $y_1, \dots, y_{n-1}$ where $y_i = x_i - x_{i+1}$. Let T be the graded polynomial ring $k[x_1, \dots, x_n]$ with the x_i placed in degree 2, and R be the graded subring $k[y_1, \dots, y_{n-1}]$. The action of S_n extends to T: we know how a permutation acts on the variables $x_1, \dots, x_n$ and we extend this so that $f \mapsto \sigma(f)$ is a k-algebra endomorphism for every $\sigma \in S_n$. Since U was S_n-invariant, so is R.

Write $s_1, \dots, s_{n-1}$ for the adjacent transpositions in S_n. For explicitness, we note that $s_i(y_i) = -y_i$, $s_i(y_{i+1}) = y_i + y_{i+1}$, $s_i(y_{i-1}) = y_i + y_{i-1}$ and $s_i(y_j) = y_j$ for all $j \notin \{i-1, i, i+1\}$[2]. For each i, we write T^{s_i} for the subring of s_i-invariants of T, i.e.

$$T^{s_i} = \{f \in T \mid s_i(f) = f\}.$$

Similarly, we write R^{s_i} for the ring of s_i-invariants of R. Explicitly, we have

$$\begin{aligned} T^{s_i} &= k[x_1, \dots, x_{i-1}, x_i + x_{i+1}, x_i x_{i+1}, x_{i+2}, \dots, x_n], \\ R^{s_i} &= k[y_1, \dots, y_{i-2}, y_i + 2y_{i-1}, y_i^2, y_i + 2y_{i+1}, y_{i+2}, \dots, y_{n-1}] \end{aligned}$$

[see 17, Example 4.12].

Proposition 4.1. *We have $R \cong R^{s_i} \oplus R^{s_i}(-2)$ as graded R^{s_i}-bimodules.*

Proof. Note that for any $f \in R$ we can write

$$f = \frac{f + s_i(f)}{2} + \frac{f - s_i(f)}{2}.$$

The first term is easily seen to be in R^{s_i}. The second is an s_i-antiinvariant, i.e. it satisfies $s_i(g) = -g$. The ideal (y_i) of R is invariant under the action of s_i, so $\{e, s_i\}$ acts on the quotient ring $R/(y_i)$. What is more, examining the explicit action of s_i above we see that its action on $R/(y_i)$ is trivial. In particular, if g is an s_i-antiinvariant then $g + (y_i) = s_i(g + (y_i)) = -g + (y_i)$ which implies that $g \in (y_i)$. This shows that $g = y_i h$ for some unique $h \in R$. In fact, $-y_i h = s_i(y_i h) = -y_i s_i(h)$ implies that

[2] In particular, this shows that U is isomorphic as an S_n-module to the geometric representation of S_n seen as the Coxeter group of type A_{n-1}.

$h \in R^{s_i}$. In other words, the R^{s_i}-submodule of s_i-antiinvariants is exactly $R^{s_i}y_i$. The decomposition of f above and the fact that $R^{s_i} \cap R^{s_i}y_i = 0$ show that we have a direct sum decomposition

$$R = R^{s_i} \oplus R^{s_i}y_i \cong R^{s_i} \oplus R^{s_i}(-2),$$

where the isomorphism is of graded R^{s_i}-bimodules (recall that y_i has degree 2). □

Definition 4.2. Let $B_{s_i} = R \otimes_{R^{s_i}} R(1)$, where R is seen as an R-bimodule. This is the simplest Soergel bimodule, other than R itself.

Proposition 4.3. *We have $B_{s_i} \cong R(1) \oplus R(-1)$ as left (resp. right) R-modules and as R^{s_i}-bimodules.*

Proof. Using Proposition 4.1 and distributivity of the tensor product and direct sum, we can rewrite this as

$$\begin{aligned} B_{s_i} = (R \otimes_{R^{s_i}} R)(1) &\cong (R \otimes_{R^{s_i}} [R^{s_i} \oplus R^{s_i}(-2)])(1) \\ &\cong (R \otimes_{R^{s_i}} R^{s_i} \oplus R \otimes_{R^{s_i}} R^{s_i}(-2))(1) \\ &\cong (R \oplus R(-2))(1) \\ &= R(1) \oplus R(-1). \end{aligned}$$

Because the isomorphism in the first line is of graded R^{s_i}-bimodules, we can only assert that $B_{s_i} \cong R(1) \oplus R(-1)$ as graded (R, R^{s_i})-bimodules and in particular as graded left R-modules. We could just as well have used Proposition 4.1 on the left factor to show that we also have an isomorphism of graded right R-modules. □

We will see in the next subsection, that this cannot be an isomorphism of R-bimodules. In either case, we conclude that B_{s_i} is graded-free of rank $v + v^{-1}$ as a left (resp. right) R-module.

For the rest of this text, we abbreviate $M \otimes_R N$ as MN.

Definition 4.4. The *Bott–Samelson bimodule* of a word $\underline{w} = (s_{i_1}, s_{i_2}, \ldots, s_{i_m})$ with $i_j \in \{1, 2, \ldots, n-1\}$ is the graded R-bimodule

$$\mathbf{BS}(\underline{w}) := B_{s_{i_1}} \otimes_R B_{s_{i_2}} \otimes_R \cdots \otimes_R B_{s_{i_m}} = B_{s_{i_1}} B_{s_{i_2}} \ldots B_{s_{i_m}}.$$

By convention, if $\underline{w}$ is the empty word then $\mathbf{BS}(\underline{w}) = R$.

Because each B_{s_i} is graded-free as a left (resp. right) R-module, so are all Bott–Samelson bimodules. By induction on m we have a canonical isomorphism

$$\mathbf{BS}(\underline{w}) \cong R \otimes_{R^{s_{i_1}}} R \otimes_{R^{s_{i_2}}} \cdots \otimes_{R^{s_{i_m}}} R(m). \tag{4.1}$$

We see that Bott–Samelson bimodules are not closed under grading shift. They are, however, closed under tensor product over R by definition. Bott–Samelson bimodules are our first examples of Soergel bimodules, and play a central role in the categorification of the HOMFLY-PT polynomial.

4.2 Soergel bimodules

Definition 4.5. A *Soergel bimodule* is a direct summand of a finite direct sum of grading shifts of Bott–Samelson bimodules. The *category of Soergel bimodules*, denoted $\mathbf{SBim}_n$, has objects the Soergel bimodules and morphisms the graded R-bimodule homogeneous homomorphisms of degree zero.

Since tensor product distributes over direct sums, $\mathbf{SBim}_n$ is closed under tensor product over R. It is also closed, by definition, under grading shifts, finite direct sums and direct summands. Recall that Bott–Samelson bimodules are graded-free of finite rank, and then so are their direct sums and grading shifts. Proposition A.4 then implies that all Soergel bimodules are graded-free of finite rank as left and right R-modules.

We will be particularly interested in the indecomposable Soergel bimodules. A nonzero bimodule is *indecomposable* if it cannot be written as the direct sum of two nonzero graded submodules. Since $\mathbf{SBim}_n$ is closed under direct summands, a bimodule in $\mathbf{SBim}_n$ is indecomposable if and only if it is indecomposable in the category of graded R-bimodules. It is easy to see that any grading shift of an indecomposable is also indecomposable. The following lemma allows us to identify the first indecomposable Soergel bimodules.

Lemma 4.6. *If a graded R-bimodule M is generated by a single homogeneous element m as an R-bimodule, i.e. $RmR = M$, then M is indecomposable.*

Proof. Set $d = \deg m$ and suppose $M = N \oplus P$. We must have $M^d = R^0 m R^0 = km$. Since M^d is isomorphic to $N^d \oplus P^d$ as k-vector spaces, we may assume without loss of generality that $m \in N^d$ and $L^d = 0$. But then $M = RmR \subseteq N$ so $P = 0$. □

We immediately see that R is indecomposable. For each i, B_{s_i} is easily seen to be generated as an R-bimodule by the degree -1 homogeneous element $1 \otimes 1$, so it is also indecomposable. In particular this confirms that the isomorphism in Proposition 4.3 could not have been of R-bimodules. Any Soergel bimodule can be written as a direct sum of indecomposable ones. In fact, this decomposition is unique up to reordering and isomorphism because $\mathbf{SBim}_n$ satisfies the Krull-Schmidt property [see 17, Appendix 1]. For example, we have the following:

Proposition 4.7. *We have $B_{s_i} B_{s_i} \cong B_{s_i}(1) \oplus B_{s_i}(-1)$ in $\mathbf{SBim}_n$.*

Proof. Consider the chain of isomorphisms

$$\begin{aligned}
B_{s_i} B_{s_i} &\cong R \otimes_{R^{s_i}} R \otimes_{R^{s_i}} R(2) && \text{by (4.1)} \\
&\cong R \otimes_{R^{s_i}} (R^{s_i} \oplus R^{s_i}(-2)) \otimes_{R^{s_i}} R(2) && \text{by Proposition 4.1} \\
&\cong R \otimes_{R^{s_i}} R^{s_i} \otimes_{R^{s_i}} R(2) \oplus R \otimes_{R^{s_i}} R^{s_i}(-2) \otimes_{R^{s_i}} R(2) \\
&\cong R \otimes_{R^{s_i}} R(2) \oplus R \otimes_{R^{s_i}} R \\
&= B_{s_i}(1) \oplus B_{s_i}(-1).
\end{aligned}$$

Note crucially that the middle R in the first line acts only as an R^{s_i}-bimodule, which means that unlike in Proposition 4.3 this is genuinely an isomorphism of R-bimodules. □

This gives a decomposition of $B_{s_i}B_{s_i}$ as a direct sum of indecomposable Soergel bimodules. If we view grading shift by 1 as multiplying by v, this isomorphism reminds us of (2.3). In fact, something much stronger is true: $\mathbf{SBim}_n$ is a categorification of the Hecke algebra in a sense that we will now make precise.

Definition 4.8. Given an essentially small[3] additive category $\mathcal{A}$, its *split Grothendieck group*, denoted $[\mathcal{A}]_\oplus$, is the abelian group generated by the symbols $[A]$ for each object A of $\mathcal{A}$ and subject to the relations

$$\begin{aligned} [A] &= [A'] && \text{whenever } A \cong A', \\ [A] &= [B] + [C] && \text{whenever } A \cong B \oplus C. \end{aligned}$$

In our case, $\mathbf{SBim}_n$ is not only additive but also monoidal (since it is closed under tensor products), so we can define a ring structure on $[\mathbf{SBim}_n]_\oplus$ by setting $[A][B] = [AB]$. We can also encode the grading shift by giving $[\mathbf{SBim}_n]_\oplus$ a $\mathbf{Z}[v, v^{-1}]$-algebra structure via $v[A] = [A(1)]$. The following theorem of Soergel [6] identifies this algebra as H_n.

Theorem 4.9. *The assignment $b_{s_i} \mapsto [B_{s_i}]$ for each i gives a $\mathbf{Z}[v, v^{-1}]$-algebra isomorphism $H_n \cong [\mathbf{SBim}_n]_\oplus$. Moreover, there is a bijection between S_n and the set of indecomposable Soergel bimodules up to shift and isomorphism denoted $w \mapsto B_w$. The indecomposable bimodule B_w is a direct summand of $\mathbf{BS}(\underline{w})$ for a reduced word for w and all other summands are shifts of B_x for $x < w$ in the Bruhat order.*

The first half of [17] is devoted to proving this theorem using diagrammatic techniques. This result is the starting point of our effort to categorify the HOMFLY-PT polynomial. There is one more part of this theorem which we state separately here. In order to state it, let us write $\mathrm{ch} : [\mathbf{SBim}_n]_\oplus \to H_n$ for the inverse of the isomorphism in Theorem 4.9.

Theorem 4.10 (Soergel Hom formula)**.** *For any two Soergel bimodules B and B' the space of graded R-bimodule homomorphisms $\mathrm{Hom}_R^\bullet(B, B')$ is graded-free as a left (resp. right) graded R-module. Its graded rank is $(\mathrm{ch}(B), \mathrm{ch}(B'))$, where $(-,-)$ denotes the standard form on H_n of Definition 2.7.*

Everything we have done so far was for a fixed n. Writing T_n and R_n for $k[x_1, \dots, x_n]$ and $k[y_1, \dots, y_{n-1}]$, respectively, we have an inclusion of rings $T_n \hookrightarrow T_{n+1}$, which restricts to an inclusion $R_n \hookrightarrow R_{n+1}$ (note that $R_1 = k$). We can use this to define an inclusion of categories $\mathbf{SBim}_n \hookrightarrow \mathbf{SBim}_{n+1}$, which is determined by setting $B_{s_i} \mapsto B_{s_i}$, or more explicitly

$$R_n \otimes_{R_n^{s_i}} R_n(1) \longmapsto R_{n+1} \otimes_{R_{n+1}^{s_i}} R_{n+1}(1).$$

This is the categorified equivalent of the inclusions $H_n \hookrightarrow H_{n+1}$ of Section 3.

5 Triply graded link homology

In this section, we use the theory of Soergel bimodules to construct an invariant of links that categorifies the HOMFLY-PT polynomial. This is done in way that is analogous

[3] A category is *essentially small* if the isomorphism classes of its objects form a set.

to the construction in Section 3. First we find a categorification of the standard basis of the Hecke algebra. This is the role of Rouquier complexes, which are complexes of Soergel bimodules. Next, we study the categorified equivalent of the Ocneanu trace, which turns out to be a combination of Hochschild homology and homology of Rouquier complexes. We then define Khovanov's triply graded link homology and sketch a proof of its invariance under the Markov moves. Last, we demonstrate how to do calculations of this homology by computing the homology of the $(2,n)$-torus link in great detail.

This construction first appeared in [12], where it was built it terms of complexes of matrix factorisations, which are certain 2-periodic sequences of maps of modules. This was later rephrased in terms of Soergel bimodules in [11]. However, the latter proves invariance under link isotopy by showing the equivalence of the two theories. In this essay, we avoid matrix factorisations altogether and we sketch a proof of invariance directly from the properties of Soergel bimodules and Hochschild homology.

5.1 Rouquier complexes

We have so far seen how Soergel bimodules categorify the Kazhdan–Lusztig basis of a Hecke algebra. However, in Section 3 the Ocneanu trace was defined on the standard basis instead. In this section we introduce a categorification of this basis after Rouquier [10].

Recall that we have $b_{s_i} = \delta_i + v$ and then $\delta_i = b_{s_i} - v$. In $\mathbf{SBim}_n$, b_{s_i} corresponds to B_{s_i} and v to $R(1)$, but the element of the split Grothendieck group $[B_{s_i}] - [R(1)]$ corresponding to δ_i is not the isomorphism class of any Soergel bimodule. This is because, while $[A]+[B]$ is the isomorphism class of $A\oplus B$, there is no concrete analogue for $[A]-[B]$. Rouquier showed that a concrete realisation of subtraction can be achieved by passing instead to the homotopy category of bounded (cochain) complexes of Soergel bimodules. We refer the reader to Appendix B for a brief exposition of complexes and the homotopy category. In particular, we will find Lemma B.4 very useful throughout the remainder of this essay. We will refer to it as *Gaussian elimination.*

If $\mathcal{A}$ is an additive category, we write $C^b(\mathcal{A})$ and $K^b(\mathcal{A})$ for the category of bounded complexes in A and its homotopy category, respectively. If $\mathcal{A}$ is essentially small, we can define the *Euler characteristic* of a bounded complex A by

$$\chi(A) \coloneqq \sum_{i\in\mathbf{Z}} (-1)^i [A^i] \in [\mathcal{A}]_\oplus.$$

It is easy to see that this descends to a group homomorphism $\chi : [C^b(\mathcal{A})]_\oplus \to [\mathcal{A}]_\oplus$. Crucially, for a chain map $f : A \to B$ we have $\chi(\mathrm{Cone}(f)) = \chi(B) - \chi(A)$, where $\mathrm{Cone}(f)$ is the mapping cone of f (see Definition B.1). This gives a way of categorifying subtraction in the split Grothendieck group.

Definition 5.1. The *triangulated Grothendieck group* of $K^b(\mathcal{A})$, denoted $[K^b(\mathcal{A})]_\triangle$, is the quotient of $[K^b(\mathcal{A})]_\oplus$ by the relations $[\mathrm{Cone}(f)] = [B] - [A]$ for any morphism $f : A \to B$.

Then χ and the inclusion functor $\phi : \mathcal{A} \to K^b(\mathcal{A})$ descend to inverse group isomorphisms [see 17, §19.2.3]:

$$[K^b(\mathcal{A})]_\triangle \underset{\phi}{\overset{\chi}{\rightleftarrows}} [\mathcal{A}]_\oplus$$

Therefore, we can categorify $[B]-[A]$ as the cone of some chain map $f : A \to B$. Note that the triangulated Grothendieck group structure is oblivious of which specific map we choose; choosing a suitable map is the job of the 'categorifier'.

We now turn to $\mathbf{SBim}_n$ and work in $C^b(\mathbf{SBim}_n)$. Note that a complex of Soergel bimodules is doubly graded: by its cohomological degree and by the internal degree of each graded bimodule. Recall that we want to categorify $\delta_i = b_{s_i} - v$. From our discussion of the triangulated Grothendieck group, we could take the cone in $K^b(\mathbf{SBim}_n)$ of a morphism $f : R(1) \to B_{s_i}$. However, the right thing do to instead is to take the cohomological shift of the cone of a morphism $B_{s_i} \to R(1)$. We denote this morphism μ_i and set $\mu_i(a \otimes b) = ab$. Recall that $B_{s_i} = (R \otimes_{R^{s_i}} R)(1)$, so this map is indeed homogeneous of degree zero. Then we define the complex F_i to be $\mathrm{Cone}(\mu_i)[-1]$. Explicitly,

$$F_i := \quad \cdots \longrightarrow 0 \longrightarrow \underline{B_{s_i}} \xrightarrow{\mu_i} R(1) \longrightarrow 0 \longrightarrow \cdots .$$

We specify which module sits in cohomological degree zero by underlining it. The following result, which is a special case of [10, Proposition 3.2], shows that $F_1, \dots, F_n$ are promising as possible categorifications of $\delta_1, \dots, \delta_n$. Recall that we omit the symbol $\otimes_R$.

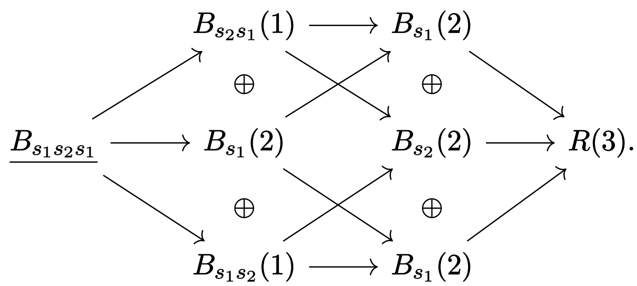
Proposition 5.2. *The complexes $F_1, \dots, F_n$ satisfy the braid relations up to homotopy equivalence, i.e.*

$$\begin{aligned} F_iF_j &\simeq F_jF_i && \text{for } |i-j| \geq 2, \\ F_iF_{i+1}F_i &\simeq F_{i+1}F_iF_{i+1} && \text{for } i \in \{1, \dots, n-2\}. \end{aligned}$$

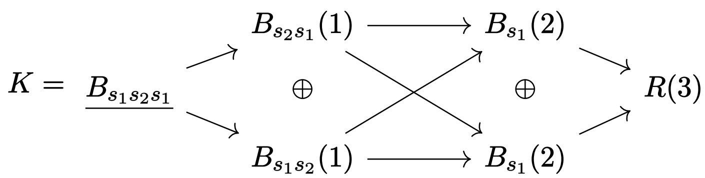
Proof. For brevity, we only sketch the proof of the second homotopy equivalence; the first one is similar. Moreover, for ease of notation we will only consider the $i = 1$ case. The complex $F_1F_2F_1$ is given by

$$\underline{B_{s_1}B_{s_2}B_{s_1}} \xrightarrow{\begin{pmatrix} \mu_1\otimes 1\otimes 1 \\ 1\otimes\mu_2\otimes 1 \\ 1\otimes 1\otimes\mu_1 \end{pmatrix}} \begin{matrix} B_{s_2}B_{s_1}(1) \\ \oplus \\ B_{s_1}B_{s_1}(1) \\ \oplus \\ B_{s_1}B_{s_2}(1) \end{matrix} \xrightarrow{\begin{pmatrix} -\mu_2\otimes 1 & \mu_1\otimes 1 & 0 \\ -1\otimes\mu_1 & 0 & \mu_1\otimes 1 \\ 0 & -1\otimes\mu_1 & 1\otimes\mu_2 \end{pmatrix}} \begin{matrix} B_{s_1}(2) \\ \oplus \\ B_{s_2}(2) \\ \oplus \\ B_{s_1}(2) \end{matrix} \xrightarrow{\begin{pmatrix} \mu_1 & -\mu_2 & \mu_1 \end{pmatrix}} R(3).$$

Appealing to Soergel's categorification theorem (Theorem 4.9), we have isomorphisms $B_{s_1}B_{s_2}B_{s_1} \cong B_{s_1s_2s_1} \oplus B_{s_1}$, $B_{s_2}B_{s_1} \cong B_{s_2s_1}$, $B_{s_1}B_{s_2} \cong B_{s_1s_2}$ and $B_{s_1}B_{s_1} \cong B_s(-1) \oplus B_s(1)$ (this last one is Proposition 4.7). Translating the complex through these isomorphisms results in a complex where the map from B_{s_1} in degree 0 to that in degree 1 (the first direct summand of $B_{s_1}B_{s_1}(1)$) is the identity. Hence, we can apply Gaussian elimination to find that $F_1F_2F_1$ is homotopy equivalent to



Moreover, in this complex the map that used to be $\mu_1 \otimes 1$ is now the identity on $B_{s_1}(2)$, so we can perform one more step of Gaussian elimination to give a complex



Since we have $s_1s_2s_1 = s_2s_1s_2$, this symmetric in s_1 and s_2 (of course one has to check that the actual maps are also symmetric). Because of this symmetry, if we repeat this process with $F_2F_1F_2$ instead, we will equally arrive at K. Therefore, $F_1F_2F_1 \simeq K \simeq F_2F_1F_2$. □

In fact, more familiar relations hold by Gaussian elimination:

$$F_iF_i \simeq \; \cdots \longrightarrow 0 \longrightarrow \underline{B_{s_i}(-1)} \longrightarrow B_{s_i}(1) \longrightarrow R(2) \longrightarrow 0 \longrightarrow \cdots, \tag{5.1}$$

where the Euler characteristic of the right-hand side corresponds under Theorem 4.9 to

$$v^{-1}b_{s_i} - vb_{s_i} + v^2 = (v^{-1} - v)\delta_i + 1.$$

We can also define a complex that categorifies $\delta_i^{-1} = \delta_i + v - v^{-1} = b_{s_i} - v^{-1}$. In this case we do take the cone of a map $\eta_i : R(-1) \to B_s$ given by $\eta_i(1) = (1 \otimes y_i + y_i \otimes 1)/2$. Explicitly,

$$F_i^{-1} := \; \cdots \longrightarrow 0 \longrightarrow R(-1) \xrightarrow{\eta_i} \underline{B_{s_i}} \longrightarrow 0 \longrightarrow \cdots.$$

One can show that $F_i^{-1}F_i \simeq R \simeq F_iF_i^{-1}$, again using Gaussian elimination. We then have the following:

Theorem 5.3. *The function $F : B_n \to K^b(\mathbf{SBim}_n)$ given by*

$$\sigma_{i_1}^{\epsilon_1}\sigma_{i_2}^{\epsilon_2}\dots\sigma_{i_k}^{\epsilon_k} \mapsto F_{i_1}^{\epsilon_1}F_{i_2}^{\epsilon_2}\dots F_{i_k}^{\epsilon_k},$$

for $k \geq 0$, $i_j \in \{1, \dots, n-1\}$ and $\epsilon_j \in \{\pm 1\}$ for each i, is well-defined and induces a group homomorphism from B_n to the group of isomorphism classes of invertible objects in $K^b(\mathbf{SBim}_n)$.

It follows from (5.1) that F factors through the quotient map $B_n \to H_n$, so that in fact we have a categorification of the standard basis of the Hecke algebra. We call the objects in the image of F *Rouquier complexes.*

5.2 Hochschild homology

We now introduce Hochschild homology of algebras over a field. We will see in the next section that taking Hochschild homology partly categorifies the Ocneanu trace on the Hecke algebra.

Let A be a k-algebra. The *opposite ring* of A, denoted A^{op}, has the same elements and addition as A but multiplication written in reverse order, i.e. if $a, b \in A$ then the element $ba \in A^{\mathrm{op}}$ corresponds to the element $ab \in A$. For instance, with this notation, the anti-involution ω used in Definition 2.7 is a ring homomorphism $H_n \to H_n^{\mathrm{op}}$. The *enveloping algebra* of A is $A^e := A \otimes_k A^{\mathrm{op}}$. We consider A-bimodules M for which we have $(am)b = a(mb)$ and $km = mk$ for all $m \in M$, $a, b \in A$ and $k \in k$. Such a bimodule is essentially the same thing as a left A^e-module, or a right A^e module, where the actions are related by

$$(a \otimes b)m = amb = m(b \otimes a).^4$$

This allows us to transfers concepts from the theory of modules to the theory of bimodules seamlessly. For example, a projective resolution of A-bimodules is simply a projective resolution of A^e-modules.

Given such an A-bimodule M, the *space of A-invariants* is the subset $\{m \in M \mid am = ma \text{ for all } a \in A\}$. It is a k-vector subspace of M, and it is isomorphic as such to $\mathrm{Hom}_{A^e}(A, M)$ via the k-linear isomorphism sending m to the A^e-linear map $a \mapsto am$. Similarly, we define

$$[A, M] := \{am - ma \mid a \in A, m \in M\}$$

which is again a k-vector subspace. The *space of A-coinvariants* is $M/[A, M]$, which is isomorphic to $A \otimes_{A^e} M$ (where we take A to be a right A^e-module) via the k-linear isomorphism sending $m + [A, M]$ to $1 \otimes m$. We then have two covariant additive functors from the category of A^e-modules (equivalently, A-bimodules) to the category of k-vector spaces: $A \otimes_{A^e} -$ and $\mathrm{Hom}_{A^e}(A, -)$. Because of the tensor-hom adjunction, we know that $A \otimes_{A^e} -$ is right exact and $\mathrm{Hom}_{A^e}(A, -)$ is left exact.

Definition 5.4. The *i-th Hochschild homology of M* is $\mathrm{HH}_i(A, M) := \mathrm{Tor}_i^{A^e}(A, M)$, i.e. the i-th left derived functor of $A \otimes_{A^e} -$. Similarly, the *i-th Hochschild cohomology of M* is $\mathrm{HH}^i(A, M) := \mathrm{Ext}^i_{A^e}(A, M)$, i.e. the i-th right derived functor of $\mathrm{Hom}_{A^e}(A, -)$. In both cases, we assemble these objects into graded[5] k-vector spaces which we call the *Hochschild homology* and *cohomology* of M:

$$\mathrm{HH}_*(A, M) := \bigoplus_{i \geq 0} \mathrm{HH}_i(A, M) \qquad \text{and} \qquad \mathrm{HH}^*(A, M) := \bigoplus_{i \geq 0} \mathrm{HH}^i(A, M).$$

Since $A \otimes_{A^e} -$ is already right exact, we have $\mathrm{HH}_0(A, M) \cong A \otimes_{A^e} M \cong M/[A, M]$. Similarly, $\mathrm{HH}^0(A, M) = \mathrm{Hom}_{A^e}(A, M)$ is isomorphic to the space of A-invariants.

In order to calculate Hochschild homology of M, we begin with a projective resolution for A in the category of right A^e-modules, i.e. an exact sequence of the form

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

[4] We need the assumption that the left and right actions of k on M commute, since $k \otimes 1 = 1 \otimes k$ in A^e.

[5] We see k as a graded ring concentrated in degree zero.

where P_i is projective for each i. We then apply $- \otimes_{A^e} M$ to each term and truncate the last one to get a complex

$$\cdots \longrightarrow P_2 \otimes_{A^e} M \longrightarrow P_1 \otimes_{A^e} M \longrightarrow P_0 \otimes_{A^e} M \longrightarrow 0$$

whose homology is by definition $\mathrm{HH}_*(A, M)$. Taking an injective resolution of A by left A^e-modules and applying $\mathrm{Hom}_{A^e}(-, M)$ will give a complex with cohomology $\mathrm{HH}^*(A, M)$.

If A is a graded ring with $k \subseteq A^0$ and M is a graded A-bimodule, then we can see A as a graded (k, A^e)-bimodule and M as a graded (A^e, k)-bimodule, so that $A \otimes_{A^e} M$ is already a graded k-vector space. In this case $\mathrm{HH}_*(A, -)$ is a functor from the category of graded A-bimodules to the category of doubly graded k-vector spaces. Similarly, $\mathrm{Hom}_{A^e}(A, M)$ becomes a graded k-vector space and $\mathrm{HH}^*(A, M)$ is doubly graded.

The Koszul complex

For R a commutative ring, the theory of Koszul complexes gives a convenient way to find free resolutions of R-modules of the form R/I for some special ideals I. This will prove useful when computing the Hochschild homology of Bot–Samelson bimodules.

Definition 5.5. If $x \in R$, we define the *Koszul complex* of x to be

$$K(x) = \ 0 \longrightarrow R \xrightarrow{\ x\ } R \longrightarrow 0$$

in homological degrees 1 and 0. If R is graded and x is homogeneous of degree d, we make the middle map homogeneous of degree 0 by instead writing

$$K(x) = \ 0 \longrightarrow R(-d) \xrightarrow{\ x\ } R \longrightarrow 0.$$

If $\boldsymbol{x} = (x_1, \ldots, x_n)$ is a sequence of elements in R, then we define its Koszul complex $K(\boldsymbol{x})$ to be the tensor product

$$K(x_1) \otimes_R K(x_2) \otimes_R \cdots \otimes_R K(x_n).$$

We say that a sequence $\boldsymbol{x} = (x_1, \ldots, x_n)$ of elements of R is *regular* if x_i is not a zero divisor for $R/(x_1, \ldots, x_{i-1})$ for each $1 \le i \le n$. It can be shown [see 7, Corollary 4.5.5] that if $\boldsymbol{x}$ is regular, then $K(\boldsymbol{x})$ is a free resolution of $R/(x_1, \ldots, x_n)$.

If we now set $A = k[x_1, \ldots, x_n]$ and $R = A^e$, it is an easy exercise to show that A^e is a polynomial ring in the $2n$ variables $x_1 \otimes 1, \ldots, x_n \otimes 1, 1 \otimes x_1, \ldots, 1 \otimes x_n$. Then $\boldsymbol{x} = (x_1 \otimes 1 - 1 \otimes x_1, \ldots, x_n \otimes 1 - 1 \otimes x_n)$ is easily seen to be a regular sequence, and $R/\boldsymbol{x} \cong A$. Hence, the Koszul complex $K(\boldsymbol{x})$ is a free resolution for A. Explicitly, if we write V for a k-vector space of dimension n and $\Lambda^i V$ for its i-th exterior power[6], then the resulting resolution can be written explicitly as

$$\Lambda^n V \otimes_k A^e \longrightarrow \cdots \longrightarrow \Lambda^2 V \otimes_k A^e \longrightarrow V \otimes_k A^e \longrightarrow A^e \xrightarrow{\ \mu\ } A \longrightarrow 0, \tag{5.2}$$

where $\mu(a \otimes b) = ab$. In particular, we see that $\mathrm{HH}_i(A, M)$ is trivial for i not in the range $0 \le i \le n$. This resolution satisfies a certain 'self-duality' that stems out of the

[6]For an introduction to exterior powers and the exterior algebra see for example [9, Chapter XIX].

k-vector space isomorphisms $(\Lambda^i V)^* \cong \Lambda^{d-i} V$, which can be used to prove a 'Poincaré duality' result for polynomial algebras:

$$\mathrm{HH}^i(A, M) \cong \mathrm{HH}_{n-i}(A, M).$$

It then follows that $\mathrm{HH}^*(A, M)$ is isomorphic to $\mathrm{HH}_*(A, M)$ up to a reflection of the outermost grading.

Example 5.6. If $A = k[x]$ is graded so that $\deg x = 2$, then we have the resolution

$$0 \longrightarrow k[x]^e(-2) \xrightarrow{x\otimes 1 - 1\otimes x} k[x]^e \xrightarrow{\mu} k[x] \longrightarrow 0,$$

where the degree shift is added so that all maps are homogeneous of degree 0. Applying $- \otimes_{k[x]^e} k[x]$ we get the complex

$$0 \longrightarrow k[x](-2) \xrightarrow{x^l - x^r} k[x] \longrightarrow 0,$$

where $x^l, x^r : k[x](-2) \to k[x]$ are given by multiplication by x on the left and right, respectively. Of course, since $k[x]$ is commutative, $x^l = x^r$ and hence the homology of this complex is just $k[x](-2)$ in degree 1 and $k[x]$ in degree zero. Its graded rank as a doubly graded k-module will be a Laurent series in two variables: if we write q for the inner grading of each module and a for the Hochschild grading then

$$\underline{\mathrm{rk}}\, \mathrm{HH}_*(k[x], k[x]) = (1 + aq^{-2}) \cdot \underline{\mathrm{rk}}\, k[x] = (1 + aq^{-2})(1 + q^2 + q^4 + \cdots) = \frac{1 + aq^{-2}}{1 - q^2}$$

Example 5.7. We can also use a Koszul complex to compute the Hochschild homology of Bot–Samelson bimodules. The most basic case after the polynomial ring itself is the bimodule B_{s_1} over $R = k[y_1]$. Recall that $B_{s_1} = R \otimes_{R^{s_1}} R(1)$ by definition. Temporarily ignoring the grading shift, $R \otimes_{R^{s_1}} R$ can be seen as a quotient of $R^e = R \otimes_k R$ by the ideal generated by the homogeneous element $y_1^2 \otimes 1 - 1 \otimes y_1^2$. This element is not a zero divisor for R^e, so we have a free resolution

$$0 \longrightarrow R^e(-3) \xrightarrow{y_1^2\otimes 1 - 1\otimes y_1^2} R^e(1) \longrightarrow B_{s_1} \longrightarrow 0.$$

Applying the functor $R \otimes_{R^e} -$ identifies the variables $y_1 \otimes 1$ and $1 \otimes y_1$, which results in a complex

$$0 \longrightarrow R(-3) \xrightarrow{0} R(1) \longrightarrow 0.$$

Its homology, which we can simply read off the complex, is the Hochschild homology of B_{s_1}, and its graded rank is

$$\underline{\mathrm{rk}}\, \mathrm{HH}_*(R, B_{s_1}) = (q + aq^{-3}) \cdot \underline{\mathrm{rk}}\, R = \frac{q + aq^3}{1 - q^2}.$$

5.3 Link homology

We are now ready to define the homology theory of links introduced in [11]. For this last step, we specialise our field k to the rationals $\mathbf{Q}$. Given a braid word $\alpha \in B_n$ of length m, the corresponding Rouquier complex $F(\alpha)$ has $m + 1$ nontrivial terms:

$$F(\alpha) = \cdots \xrightarrow{d} F^j(\alpha) \xrightarrow{d} F^{j+1}(\alpha) \xrightarrow{d} \cdots.$$

Now we apply Hochschild homology (or equivalently, cohomology) to each term to get a complex of doubly graded $\mathbf{Q}$-vector spaces

$$\mathrm{HH}_*(R, F(\alpha)) = \cdots \longrightarrow \mathrm{HH}_*(R, F^j(\alpha)) \longrightarrow \mathrm{HH}_*(R, F^{j+1}(\alpha)) \longrightarrow \cdots .$$

Taking the homology of this complex then yields a triply graded $\mathbf{Q}$-vector space which we call $\mathrm{HHH}(\alpha)$. The inner grading is the one introduced by each $F^j(\alpha)$, the middle is the Hochschild homology grading, and the outer one is the homology of the complex $\mathrm{HH}_*(R, F(\alpha))$.

By construction, the resulting vector space is graded-free. Its graded rank is then a Laurent series with positive coefficients in three variables, which we denote q, a and t for the inner, middle and outer gradings, respectively. The *Euler characteristic of* $\mathrm{HHH}(\alpha)$ is then the Laurent series in q and a resulting from setting $t = -1$. Khovanov then proved the following in [11]:

Theorem 5.8. *Up to an overall grading shift,* $\mathrm{HHH}(\alpha)$ *only depends on* $\widehat{\alpha}$*, i.e. it is an invariant of oriented links. Taking the Euler characteristic of* $\mathrm{HHH}(\alpha)$ *gives the HOMFLY-PT polynomial of Section 3 after some renormalisation.*

We call $\mathrm{HHH}(\alpha)$ the HOMFLY-PT homology of the link $\widehat{\alpha}$. In order to show that this only depends on $\widehat{\alpha}$, we proceed as in Section 3: we check that it is invariant under Markov moves of boh types. For moves of the first type, it suffices to show that $\mathrm{HH}_*(R, MN)$ is isomorphic to $\mathrm{HH}_*(R, NM)$ for any graded R-bimodules M and N.

Lemma 5.9. *The doubly graded* $\mathbf{Q}$*-vector spaces* $\mathrm{HH}_*(R, MN)$ *and* $\mathrm{HH}_*(R, NM)$ *are isomorphic.*

Proof. It suffices to show that $\mathrm{HH}_i(R, MN) \cong \mathrm{HH}_i(R, NM)$ for each i. Note from (5.2) that each term in the complex used to commute $\mathrm{HH}_i(R, MN)$ is of the form $\Lambda^i V \otimes_{\mathbf{Q}} (R \otimes_{R^e} MN)$. Hence, it would be enough to show that $R \otimes_{R^e} MN$ and $R \otimes_{R^e} NM$ are naturally isomorphic. To begin with, we have four actions of R on M and N: the left and right action on M and the left and right actions on M. This gives for actions on $M \otimes_{\mathbf{Q}} N$, and passing to the quotient $M \otimes_R N$ identifies the right action on M and the left action on N. When we apply $R \otimes_{R^e} -$ we identify the remaining two actions. Meanwhile, $N \otimes_R M$ identifies the right action on N and the left action on M, and then $R \otimes_{R^e} NM$ identifies the remaining two actions. But this is exactly what we did with $R \otimes_{R^e} MN$ in the opposite order. This shows that the bijection $m \otimes n \mapsto n \otimes m$ induces an isomorphism $\mathrm{HH}_*(R, MN) \to \mathrm{HH}_*(R, NM)$. $\square$

Invariance under (M2) is harder to see. Since this involves a change in the number of strands, we adopt the notation at the end of Section 4.2, and write I for the inclusion of categories $\mathbf{SBim}_n \hookrightarrow \mathbf{SBim}_{n+1}$. Given $\alpha \in B_n$, we need to relate the complexes $\mathrm{HH}_*(R_n, F(\alpha))$ and $\mathrm{HH}_*(R_{n+1}, IF(\alpha)F_n^{\pm 1})$.

For simplicity, we only consider the case where α is the identity in B_1. In this case, $R_1 = \mathbf{Q}$ and $F(\alpha) = 0 \longrightarrow \underline{R_1} \longrightarrow 0$. The Hochschild homology of R_1 is just R_1 in degree 0, so $\mathrm{HHH}(\alpha) = \mathbf{Q}$ with graded rank 1. Next, with $R = R_2 = \mathbf{Q}[y_1]$, we have $F_1 = 0 \longrightarrow \underline{B_{s_1}} \xrightarrow{\mu_1} R(1) \longrightarrow 0$. From Example 5.7, we know what the Hochschild homology of each of these terms looks like. To find the maps induced by μ_1 we write:

$$\begin{CD} R^e(-3) @>{y_1\otimes 1+1\otimes y_1}>> R^e(-1)\\ @V{y_1^2\otimes 1-1\otimes y_1^2}VV @VV{y_1\otimes 1-1\otimes y_1}V\\ R^e(1) @>{1}>> R^e(1)\\ @V{p}VV @VV{\mu}V\\ \underline{B_{s_1}} @>{\mu_1}>> R(1) \end{CD} \quad \rightsquigarrow \quad \begin{CD} R(-3) @>{2y_1}>> R(-1)\\ @V{0}VV @VV{0}V\\ \underline{R(1)} @>{1}>> R(1) \end{CD} \tag{5.3}$$

The lower complex has trivial homology. The upper one has trivial homology in degree 0 and $\mathbf{Q}$ in degree 1, so $\mathrm{HHH}(\sigma_1) = \mathbf{Q}$ with graded rank taq^{-1}. We indeed see that $\mathrm{HHH}(\alpha) \cong \mathrm{HHH}(\sigma_1)$ up to a shift in the grading. Lastly, we have $F_1^{-1} = 0 \longrightarrow R(-1) \xrightarrow{\eta} \underline{B_{s_1}} \longrightarrow 0$. In this case we have:

$$\begin{CD} R^e(-3) @>{1/2}>> R^e(-3)\\ @V{y_1\otimes 1-1\otimes y_1}VV @VV{y_1^2\otimes 1-1\otimes y_1^2}V\\ R^e(-1) @>{(1\otimes y_1+y_1\otimes 1)/2}>> R^e(1)\\ @V{\mu}VV @VV{p}V\\ R(-1) @>{(1\otimes y_1+y_1\otimes 1)/2}>> \underline{B_{s_1}} \end{CD} \quad \rightsquigarrow \quad \begin{CD} R(-3) @>{1/2}>> R(-3)\\ @V{0}VV @VV{0}V\\ R(-1) @>{y_1}>> \underline{R(1)} \end{CD}$$

The upper complex has trivial homology. The lower complex has trivial homology in degree -1 and $\mathbf{Q}$ in degree 0. Hence, $\mathrm{HHH}(\sigma_1^{-1}) = \mathbf{Q}$ with graded rank 1. The general case for $\alpha \in B_n$ involves n variables, but only the n-th produces a change to the Hochschild homology, which causes the same effect to HHH as in this simple case. We conclude that applying (M2) with positive exponent incurs in a degree shift of taq^{-1}, whereas a negative exponent does not affect the grading.

To see that taking the Euler characteristic recovers the HOMFLY-PT polynomial, recall from Section 5.1 that the Euler characteristic of $F(\alpha)$ represents the image of α in H_n. It is then enough to argue that taking Hochschild homology and then the homology of the resulting complex categorifies the Ocneanu trace on the Hecke algebra. But this exactly what we have shown, since after taking $t = -1$, the degree shift of taq^{-1} corresponds to the multiplication by zv in Theorem 3.1. Therefore, going through a renormalisation analogous to the one in Section 3 gives the same invariant of links. Explicitly, we introduce an artificial grading shift variable w which is a square root of taq^{-1} and for $\alpha \in B_n$ we set

$$H(\alpha) = w^{n-1-e(\alpha)}\mathrm{HHH}(\alpha),$$

where e is the number of positive crossings minus the number of negative crossings in α. This is invariant under both Markov moves, and hence is an invariant of links. Moreover, after setting $t = -1$, it satisfies the familiar skein relation $w^{-1}H(L_+) - wH(L_-) = (q^{-1} - q)H(L_0)$ of Theorem 3.3, confirming its relation with the HOMFLY-PT polynomial.

5.4 HOMFLY-PT homology of the $(2, n)$-torus link

As an example, we will demonstrate how to compute the link homology of $\sigma_1^n \in B_2$ for $n > 0$, which represents the $(2, n)$-torus link. In general, computing HHH explicitly is hard, but this case is small enough that it can be done using only elementary techniques, such as Gaussian elimination. These results are succinctly stated at the end of [11], which uses a different convention for Rouquier complexes. The detailed treatment here will surely be worthwhile for the reader following our definition. To simplify notation, we will write $s := s_1$, $B := B_s$ and $R := \mathbf{Q}[y]$. Given the isomorphism $R \cong R^s \oplus R^s(-2)$ of Proposition 4.1, we write π and τ for the projections onto the first and second direct summand, respectively. Explicitly, this is

$$\pi(f) = \frac{f + s(f)}{2} \qquad \text{and} \qquad \tau(f) = \frac{f - s(f)}{2y}.$$

We note that $\pi(fy) = y^2\tau(f)$ and $\tau(fy) = \pi(f)$. The injection $R^s \hookrightarrow R$ is the inclusion, and $R^s(-2) \hookrightarrow R$ is multiplication by y. With this notation, we can write the isomorphism $BB \cong B(1) \oplus B(-1)$ of Proposition 4.7 explicitly: for $a \otimes b \otimes c \in R \otimes_{R^s} R \otimes_{R^s} R(2) \cong BB$, the projections onto the first and second factors give $a\pi(b) \otimes c = a \otimes \pi(b)c$ and $a\tau(b) \otimes c = a \otimes \tau(b)c$, respectively. The injection $B(1) \hookrightarrow BB$ sends $1 \otimes 1$ to $1 \otimes 1 \otimes 1$, and $B(-1) \hookrightarrow BB$ sends $1 \otimes 1$ to $1 \otimes y \otimes 1$.

Proposition 5.10. *The complex $F(\sigma_1^n)$ is homotopy equivalent to G_n given by*

$$\underline{B(1-n)} \xrightarrow{d_{n-1}} B(3-n) \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_2} B(n-3) \xrightarrow{d_1} B(n-1) \xrightarrow{\mu} R(n),$$

where $\mu(a \otimes b) = ab$ and $d_i = y^r + (-1)^i y^l$, with y^l and y^l being multiplication by y on the left and right, respectively.

Proof. We proceed by induction on n. The $n = 1$ case is true by definition. Now $F(\sigma_1^{n+1}) = F(\sigma_1^n)F_1$, which by the induction hypothesis is homotopy equivalent G_nF_1:

$$\begin{array}{ccccccccccccc}
BB(1-n) & \xrightarrow{d_{n-1}\otimes 1} & BB(3-n) & \xrightarrow{d_{n-2}\otimes 1} & \cdots & \xrightarrow{d_1\otimes 1} & BB(n-1) & \xrightarrow{\mu\otimes 1} & B(n) & & \\
 & \searrow^{1\otimes\mu} & \oplus & \searrow^{-1\otimes\mu} & & & & \searrow^{(-1)^{n-1}\otimes\mu} & \oplus & \searrow^{(-1)^n\mu} & \\
 & & B(2-n) & \xrightarrow{d_{n-1}} & B(4-n) & \xrightarrow{d_{n-2}} & \cdots & \xrightarrow{d_1} & B(n) & \xrightarrow{\mu} & R(n+1)
\end{array}$$

We focus on the first three terms of the complex. Using the isomorphism of Proposition 4.7, we can rewrite this as

$$B(2-n) \oplus B(-n) \xrightarrow{D} BB(3-n) \oplus B(2-n) \longrightarrow BB(5-n) \oplus B(4-n),$$

where D is given by

$$\begin{pmatrix} (1 \otimes y + (-1)^{n-1} y \otimes 1) \otimes 1 & (1 \otimes y^2 + (-1)^{n-1} y \otimes y) \otimes 1 \\ 1 & y^r \end{pmatrix}.$$

Since the bottom-left entry is the identity, we can use simplify this complex using Gaussian elimination. The remaining map $B(-n) \to BB(3-n)$ sends $1 \otimes 1$ to

$$f := 1 \otimes y^2 \otimes 1 + (-1)^{n-1} y \otimes y \otimes 1 - 1 \otimes y \otimes y - (-1)^{n-1} y \otimes 1 \otimes y.$$

The beginning of our complex is now simplified to

$$\begin{array}{ccccccc} B(-n) & \xrightarrow{f} & BB(3-n) & \xrightarrow{d_{n-2}\otimes 1} & BB(5-n) & \xrightarrow{d_{n-3}\otimes 1} & \cdots \\ & & & \searrow^{-1\otimes\mu} & \oplus & & \\ & & & & B(4-n) & \xrightarrow{d_{n-2}} & \cdots . \end{array}$$

Ignoring the first term, we have exactly the complex $G_{n-1}(-F_1)$, where $-F_1$ denotes the complex F_1 with all differentials negated. Of course, $-F_1 \simeq F_1$ and then $G_{n-1}(-F_1) \simeq G_{n-1}F_1$. By induction, this complex can be simplified using Gaussian elimination to produce G_n. We must however study how f is changed after the next step of Gaussian elimination. For this, it suffices to write f as a map $B(-n) \to B(4-n) \oplus B(2-n)$ using the projection formulas above:

$$\begin{pmatrix} 1 \otimes y^2 - (-1)^{n-1} y \otimes y \\ (-1)^{n-1} y \otimes 1 - 1 \otimes y \end{pmatrix}.$$

After eliminating the $B(4-n)$ component, we are left with the map given by the second row, which is precisely $(-1)^{n-1}y^l - y^r$. Since multiplying a differential by an invertible scalar does not change the homotopy class of a complex, we can change this to $y^r - (-1)^n y^l$, which is indeed the leftmost map in G_{n+1}. □

Since Hochschild homology is an additive functor, it preserves chain homotopies and homotopy equivalences. As homotopy equivalent complexes have the same homology, this allows us to compute $\mathrm{HHH}(\sigma_1^n)$ using, instead of $F(\sigma_1^n)$, the much simpler complex G_n. In fact, the Hochschild homology of G_n is easy to compute. The last differential (or a grading shift of it) was computed in (5.3). The remaining ones are the subject of the following proposition.

Proposition 5.11. *The Hochschild homology of* $B(m) \xrightarrow{y^r \pm y^l} B(m+2)$ *is*

$$\begin{array}{ccc} R(m-3) & \xrightarrow{y \pm y} & R(m-1) \\ \oplus & & \oplus \\ R(m+1) & \xrightarrow{y \pm y} & R(m+3). \end{array}$$

Proof. We proceed as with the proof of invariance under (M2), using the Koszul resolution of B:

$$\begin{array}{ccc} R^e(m-3) & \overset{1\otimes y \pm y \otimes 1}{\dashrightarrow} & R^e(m-1) \\ {\scriptstyle y^2\otimes 1 - 1 \otimes y^2}\downarrow & & \downarrow{\scriptstyle y^2\otimes 1 - 1 \otimes y^2} \\ R^e(m+1) & \overset{1\otimes y \pm y \otimes 1}{\dashrightarrow} & R^e(m+3) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ B(m) & \xrightarrow{y^r \pm y^l} & B(m+2) \end{array}$$

Applying $R \otimes_{R^e} -$ to the four upper terms gives the claimed result. □

It is now easy to compute $\mathrm{HHH}(\sigma_1^n)$. The Hochschild homology complex splits into the direct sum of two complexes in Hochschild degree 0 and 1. The degree 0 complex is

$$\underline{R(2-n)} \xrightarrow{y+(-1)^{n-1}y} \cdots \xrightarrow{0} R(n-4) \xrightarrow{2y} R(n-2) \xrightarrow{0} R(n) \xrightarrow{1} R(n).$$

If n is even, this has homology $\mathbf{Q}[y]/(2y) \cong \mathbf{Q}$ in even degrees between 1 and $n-1$, and $\mathbf{Q}[y]$ in degree 0. If n is odd, then it has homology $\mathbf{Q}$ in odd degrees between 1 and $n-1$. Similarly, the degree 1 complex is

$$\underline{R(-2-n)} \xrightarrow{y+(-1)^{n-1}y} \cdots \xrightarrow{2y} R(n-6) \xrightarrow{0} R(n-4) \xrightarrow{2y} R(n-2).$$

If n is even, it has homology $\mathbf{Q}$ in even degrees between 1 and n, and $\mathbf{Q}[y]$ in degree 0. If n is odd, then it has homology $\mathbf{Q}$ in odd degrees between 0 and n. We summarise our results in the following theorem.

Theorem 5.12. *For $n > 0$, $\mathrm{HHH}(\sigma_1^n)$ has graded-rank*

$$atq^{-n} + (t + at^3)(q^{4-n} + t^2q^{8-n} + \cdots + t^{n-5}q^{n-6} + t^{n-3}q^{n-2})$$

if n is odd, and

$$\frac{aq^{-2-n} + q^{2-n}}{1-q^2} + at^2q^{2-n} + (t^2 + at^4)(q^{6-n} + t^2q^{10-n} + \cdots + t^{n-6}q^{n-6} + t^{n-4}q^{n-2})$$

if n is even. In particular, for odd n, the $(2, n)$-torus knot has ungraded homology $\mathbf{Q}^n$.

6 Conclusion

As outlined earlier, we have seen how the HOMFLY-PT polynomial can be defined as a trace on the Hecke algebra, and how each of these elements can be categorified to give a homology theory for links. In order to do this, we employed the theory of Soergel bimodules, which were first introduced to answer a question relating Lie theory to the Kazhdan–Lusztig polynomials. We saw how these bimodules categorify the Kazhdan–Lusztig basis of the Hecke algebra. However, since the Ocneanu trace was defined on the standard basis instead, we needed to pass to the homotopy category of Rouquier complexes, which categorifies the latter basis. Lastly, we saw how Hochschild homology followed by homology of complexes plays the role of the trace on the Hecke algebra. We concluded with a detailed computation of the homology of the $(2, n)$-torus link.

The interested reader is directed to [15], where more sophisticated techniques are developed to compute the HOMFLY-PT homology of a larger class of torus links. In [14], HHH is related to earlier link homology theories that categorify quantum link invariants through a spectral sequence. This is then used to prove that, for certain knots, HHH is determined by the HOMFLY-PT polynomial and the signature, which in particular allows the computation of HHH for all knots with 9 crossings or fewer. Much remains to be understood about HOMFLY-PT homology. It is conjectured [18] that HHH is related to the algebraic geometry of certain schemes. Work in this direction can be found in [16], where a recursive formula is given to compute the HOMFLY-PT homology of the (m, n)-torus link for $m, n \geq 0$.

A Overview of graded objects

We give some brief definitions of graded objects, mostly to settle the notation used. The words module, ideal, etc. are used in their left-handed variant. Of course everything is still true for the right-handed variants with the obvious modifications. For generality, we let I be an ordered commutative monoid.

An *I-graded ring* R is a ring with a decomposition $R = \bigoplus_{i\in I} R^i$ as abelian groups, such that $R^iR^j \subseteq R^{i+j}$ (i.e. for all $x \in R^i$ and $y \in R^j$ one has $xy \in R^{i+j}$). We call an element of R^i *homogeneous of degree i*. For example, if k is any commutative ring then $k[x_1, \dots, x_n]$ is $\mathbf{N}$-graded, where the homogeneous elements of degree i are the monomials in $x_1, \dots, x_n$ of degree i.

For R an I-graded ring, a *graded R-module* M is an R-module with a decomposition $M = \bigoplus_{i\in I} M^i$ as abelian groups, such that $R^iM^j \subseteq M^{i+j}$. As with rings, we call the elements of M^i *homogeneous of degree i*. If M is a graded R-module and $n \in I$, we define $M(n)$ to be the graded R-module where $M(n)^i = M^{n+i}$. For example, if $x \in M$ is homogeneous of degree n, then $x \in M(n)$ is homogeneous of degree 0. A submodule $N \leq M$ is a *graded submodule* if it is generated by its homogeneous elements, equivalently if it is a graded R-module in its own right and $N^i = N \cap M^i$. An ideal of R is *homogeneous* if it is a graded submodule of R over it self. If N is a graded submodule, then M/N can be given a grading by setting $(M/N)^i = (M^i + N)/N \cong M^i/N^i$.

If S is another I-graded ring then a *graded (R, S)-bimodule* is an (R, S)-bimodule that is an R-graded module on the left and an S-graded module on the right, i.e. $R^iM^j \subseteq M^{i+j}$ and $M^iS^j \subseteq M^{i+j}$. We require that the left an right actions commute, i.e. $(rm)s = r(ms)$ for all $r \in R$, $s \in S$ and $m \in M$.

If $\{M_\alpha \mid \alpha \in A\}$ is an indexed family of graded R-modules, we give the direct sum $M' = \bigoplus_{\alpha\in A} M_\alpha$ a grading by setting $(M')^i = \bigoplus_{\alpha\in A} M_\alpha^i$. Given a function $p : I \to \mathbf{N}$ we define

$$M^{\oplus p} = \bigoplus_{i\in I} M(i)^{\oplus p(i)}.$$

We say that N is a *direct summand* of M if there exists another graded R-module P such that $M \cong N \oplus R$ as graded R-modules. A *graded-free R-module* M is one with a basis consisting of homogeneous elements. For any m_i in this basis, we have an isomorphism $R(-\deg m_i) \to Rm_i \subseteq M$ given by $r \mapsto rm$. If the basis is finite, then we have an isomorphism $M \cong R^{\oplus p}$ where $p(k)$ is the number of m_i of degree $-k$. In this case, we say that p is the *graded rank* of M and write $\underline{\mathrm{rk}}\, M = p$.

If M and N are graded R-modules, an R-linear map $M \to N$ is *homogeneous of degree n* if it sends M^i to N^{i+n} for each i. Note that a homogeneous map $M \to N$ of degree n is the same as a homogeneous map $M \to N(n)$ (or $M(-n) \to N$) of degree 0. The homogeneous maps of degree 0 form an R^0-module which we denote $\mathrm{Hom}_R(M, N)$. The space of homogeneous maps of any degree is a graded R-module via

$$\mathrm{Hom}^\bullet_R(M, N) = \bigoplus_{i\in I} \mathrm{Hom}_R(M, N(i)).$$

If M is a graded (R, S)-bimodule and N is a graded (S, T)-bimodule, their tensor product over S is a graded (R, T)-bimodule with grading

$$(M \otimes_S N)^k = \left\langle m \otimes n \;\middle|\; m \in M^i, n \in N^j, i + j = k \right\rangle_{\mathbf{Z}}$$

where $\langle S \rangle_{\mathbf{Z}}$ denotes the abelian subgroup generated by S. For example, if $m \in M^i$ and $n \in N^j$ then $m \otimes n$ is homogeneous of degree $i + j$ in $M \otimes_R N$. It follows from this definition that $M(n) \otimes_R N$, $M \otimes_R N(n)$ and $(M \otimes_R N)(n)$ have the same grading, so they are equal as graded (R, T)-bimodules.

If $I = \mathbf{Z}^n$ for some positive integer n, then the rank of a graded-free module is a Laurent series in n variables with positive coefficients. We will be interested in graded rings and modules with $I = \mathbf{Z}, \mathbf{Z}^2$ and $\mathbf{Z}^3$. From now on, we will simply say 'graded' to mean $\mathbf{Z}$-graded, 'doubly graded' to mean $\mathbf{Z}^2$-graded and 'triply graded' to mean $\mathbf{Z}^3$-graded. We will build triply graded modules by adding one $\mathbf{Z}$-grading at time, so that we talk about an 'inner', 'middle' and 'outer' grading.

The following results are $\mathbf{Z}$-graded versions of Nakayama's lemma for commutative rings that we shall find useful. If R is a graded ring (not necessarily commutative), then the set $R^+ = \bigoplus_{i>0} R^i$ is a two-sided homogeneous ideal. For the remainder of this section, we assume M is a finitely generated graded R-module. In fact, this is equivalent to saying that M is generated by a finite set of homogeneous elements.

Lemma A.1. *If $\mathfrak{a} \subseteq R^+$ is a homogeneous left ideal of R such that $\mathfrak{a}M = M$, then $M = 0$.*

Proof. Since $\mathfrak{a} \subseteq R^+$, any homogeneous element of $\mathfrak{a}$ must have degree at least 1. If $M \neq 0$, since it is finitely generated, it must have some homogeneous element m of minimal degree. But then all homogeneous elements of $\mathfrak{a}M$ have degree at least $\deg m + 1$, so $m \notin \mathfrak{a}M$. □

Corollary A.2. *With $\mathfrak{a}$ as above and N a graded submodule of M, $M = N + \mathfrak{a}M$ implies $N = M$.*

Proof. We have $\mathfrak{a}(M/N) = (N + \mathfrak{a}M)/N = M/N$, so $M/N = 0$ by the previous lemma. □

Corollary A.3. *If R^0 is a field, and $m_1, \ldots, m_n$ are homogeneous elements of M such that $\overline{m_1}, \ldots, \overline{m_n}$ form a basis of the R^0-vector space M/R^+M, then $m_1, \ldots, m_n$ is a minimal generating set for M.*

Proof. Let N be the graded submodule of M generated by $m_1, \ldots, m_n$ and let π be the quotient map $M \to M/R^+M$. By assumption $\pi(N) = M$, so $M = N + R^+M$ and by the previous corollary $N = M$. A generating set with fewer elements would give a spanning set for M/R^+M with fewer than n elements, which contradicts the assumption that M/R^+M is n-dimensional. □

Proposition A.4. *If R^0 is a field and N is a direct summand of a graded-free R-module M of finite graded rank, then N is graded-free of finite graded rank.*

Proof. Let $\pi : M \to N$ and $\iota : N \to M$ be the projection and injection, so that $\pi\iota = \mathrm{id}_N$. By the previous corollary, we may take $n_1, \ldots, n_k$ to be a minimal generating set for N consisting of homogeneous elements. Let $F = \bigoplus_{i=1}^k R(-\deg n_i)$ and consider consider the degree-0 homogeneous surjection $p : F \to N$ given by $e_i \mapsto m_i$. Since M is free (and hence projective) we have a map $q : M \to F$ such that $pq = \pi$. Then $q\iota : N \to F$ splits p because $pq\iota = \pi\iota = \mathrm{id}_N$. This shows that we have a split exact sequence

$$0 \longrightarrow \ker p \hookrightarrow F \xrightarrow{p} N \longrightarrow 0,$$

so $F \cong N \oplus \ker p$. Then p descends to a map $\overline{p} : F/R^+F \to N/R^+N$ that sends $\overline{e_i} \mapsto \overline{m_i}$. This is an isomorphism of R^0-vector spaces so $0 = \ker \overline{p} \supseteq \ker p + R^+F$ and then $\ker p \subseteq R^+F$. Therefore

$$F = q\iota(N) + \ker p \subseteq q\iota(N) + R^+F \subseteq F,$$

so $F = q\iota(N) \cong N$ by Corollary A.2. □

B Introduction to complexes and the homotopy category

Given an additive category $\mathcal{A}$, we write $C(\mathcal{A})$ for *category of cochain complexes in* $\mathcal{A}$. An object in $C(\mathcal{A})$ is a sequence $(A^i, d^i)_{i\in\mathbf{Z}}$ where A^i is an object of $\mathcal{A}$ and $d^i : A^i \to A^{i+1}$ are such that $d^{i+1}d^i = 0$ for all $i \in \mathbf{Z}$. We will often abbreviate such an object as (A, d_A) or A. A morphism in $C(\mathcal{A})$ from (A, d_A) to (B, d_B) is a sequence $(f^i : A^i \to B^i)_{i\in\mathbf{Z}}$ such that $d^i_B f^i = f^{i+1} d^i_A$ for each $i \in \mathbf{Z}$. The category $C(\mathcal{A})$ has a *cohomological shift automorphism* [1] given by

$$(A[1])^i = A^{i+1} \qquad \text{and} \qquad d^i_{A[1]} = -d^{i+1}_A,$$

for an object A and $(f[1])^i = f^{i+1}$ for a morphism $f : A \to B$.

For each pair of objects A and B in $C(\mathcal{A})$, we define an equivalence relation on the set of morphisms $A \to B$ by $f \simeq g$ if there exists a sequence of morphisms $(h^i : A^i \to B^{i-1})_{i\in\mathbf{Z}}$ such that $f^i - g^i = h^{i+1}d^i_A + d^{i-1}_B h^i$. We say f and g are *homotopic* if $f \simeq g$ and call such a h a *homotopy*. One can show that $f \simeq g$ implies $hf \simeq hg$ and $fk \simeq gk$ whenever the composites are defined, so that we can form the quotient category $K(\mathcal{A}) := C(\mathcal{A})/\simeq$. Its objects are complexes in $\mathcal{A}$ and its morphisms are homotopy classes of morphisms in $C(\mathcal{A})$. We say that two complexes A and B are *homotopy equivalent*, denoted $A \simeq B$, if they are isomorphic in $K(\mathcal{A})$. Explicitly, $A \simeq B$ if there are morphisms $f : A \to B$ and $g : B \to A$ in $C(\mathcal{A})$ such that $gf \simeq \mathrm{id}_A$ and $fg \simeq \mathrm{id}_B$. If two complexes are homotopy equivalent then they have the same homology. We say A is *contractible* if $A \simeq 0$.

Definition B.1. For a morphism $f : A \to B$ in $C(\mathcal{A})$, its *mapping cone* is the complex $\mathrm{Cone}(f)$ with

$$\mathrm{Cone}(f)^i = A^{i+1} \oplus B^i \qquad \text{and} \qquad d^i_{\mathrm{Cone}(f)} = \begin{pmatrix} d^i_{A[1]} & 0 \\ f[1]^i & d^i_B \end{pmatrix} = \begin{pmatrix} -d^{i+1}_A & 0 \\ f^{i+1} & d^i_B \end{pmatrix}.$$

It is an easy exercise to show that $f \simeq 0$ if and only if $\mathrm{Cone}(f)$ is contractible.

Denote by $C^b(\mathcal{A})$ (resp. $K^b(\mathcal{A})$) the full subcategory of $C(\mathcal{A})$ (resp. $K(\mathcal{A})$) on *bounded complexes*, i.e. A such that $A^i = 0$ for all but finitely many $i \in \mathbf{Z}$. If $\mathcal{A}$ is monoidal, then so are $C^b(\mathcal{A})$ and $K^b(\mathcal{A})$:

Definition B.2. If $\mathcal{A}$ is an additive monoidal category, we define the tensor product of two complexes $A, B \in C^b(\mathcal{A})$ by

$$(A \otimes B)^n = \bigoplus_{i+j=n} A^i \otimes B^j \qquad \text{and} \qquad d^n(a \otimes b) = d^i_A(a) \otimes b + (-1)^i a \otimes d^j_B(b),$$

where $a \in A^i$ and $b \in B^j$ with $i + j = n$. Given maps $f : A \to C$ and $g : B \to D$, we have a map $f \otimes g : A \otimes B \to C \otimes D$ given by the components $f^i \otimes g^j$, i.e. $(f \otimes g)^n(a \otimes b) = f^i(a) \otimes g^j(b)$ for $i + j = n$, $a \in A^i$ and $b \in B^j$.

Note that the $(-1)^i$ term is needed so that the resulting sequence is still a complex. An alternative way of viewing this construction is through the following double complex:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle 1\otimes d_B} & & \downarrow{\scriptstyle -1\otimes d_B} & & \downarrow{\scriptstyle 1\otimes d_B} & & \\
\cdots & \xrightarrow{d_A\otimes 1} & A^0\otimes B^0 & \xrightarrow{d_A\otimes 1} & A^1\otimes B^0 & \xrightarrow{d_A\otimes 1} & A^2\otimes B^0 & \xrightarrow{d_A\otimes 1} & \cdots \\
 & & \downarrow{\scriptstyle 1\otimes d_B} & & \downarrow{\scriptstyle -1\otimes d_B} & & \downarrow{\scriptstyle 1\otimes d_B} & & \\
\cdots & \xrightarrow{d_A\otimes 1} & A^0\otimes B^1 & \xrightarrow{d_A\otimes 1} & A^1\otimes B^1 & \xrightarrow{d_A\otimes 1} & A^2\otimes B^1 & \xrightarrow{d_A\otimes 1} & \cdots \\
 & & \downarrow{\scriptstyle 1\otimes d_B} & & \downarrow{\scriptstyle -1\otimes d_B} & & \downarrow{\scriptstyle 1\otimes d_B} & & \\
\cdots & \xrightarrow{d_A\otimes 1} & A^0\otimes B^2 & \xrightarrow{d_A\otimes 1} & A^1\otimes B^2 & \xrightarrow{d_A\otimes 1} & A^2\otimes B^2 & \xrightarrow{d_A\otimes 1} & \cdots \\
 & & \downarrow{\scriptstyle 1\otimes d_B} & & \downarrow{\scriptstyle -1\otimes d_B} & & \downarrow{\scriptstyle 1\otimes d_B} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Then $A \otimes B$ is given by taking direct sums along the diagonals $A^i \otimes B^j$ with $i + j$ constant. Looking at tensor products of complexes this way can be enlightening: for example taking the tensor product of m complexes of the form $0 \to A \to B \to 0$ has the structure of an m-dimensional hypercube.

Proposition B.3. *If h is a homotopy between two maps $f, g : B \to C$, then $\tilde{h}$ is a homotopy between $1 \otimes f, 1 \otimes g : A \otimes B \to A \otimes C$, where*

$$\tilde{h}^n(a \otimes b) = (-1)^i a \otimes h^n(b)$$

if $i+j = n$, $a \in A^i$ and $b \in B^j$. Similarly, $f \otimes 1, g \otimes 1 : B \otimes A \to C \otimes A$ are homotopic.

Proof. With n, i, j, a and b as above, we have

$$
\begin{aligned}
&\tilde{h}^{n+1}d^n(a \otimes b) + d^{n-1}\tilde{h}^n(a \otimes b) \\
&\quad = \tilde{h}^{n+1}[d^i a \otimes b + (-1)^i a \otimes d^j b] + (-1)^i d^{n-1}(a \otimes h^j b) \\
&\quad = (-1)^{i+1} d^i a \otimes h^j b + (-1)^{2i} a \otimes h^{j+1} d^j b + (-1)^i d^i a \otimes h^j b + (-1)^{2i} a \otimes d^{j-1} h^j b \\
&\quad = a \otimes (h^{j+1} d^j b + d^{h-1} h^j b) \\
&\quad = a \otimes (fb - gb).
\end{aligned}
$$

The second statement is similar, with homotopy $\widehat{h}$ given by $\widehat{h}^n(b \otimes a) = h^j(b) \otimes a$. □

It follows that if $f_1, g_1 : A_1 \to B_1$ and $f_2, g_2 : A_2 \to B_2$ are homotopic, then

$$f_1 \otimes f_2 = (f_1 \otimes 1)(1 \otimes f_2) \simeq (g_1 \otimes 1)(1 \otimes g_2) = g_1 \otimes g_2.$$

Moreover, if f and g are inverse homotopy equivalences between A and C, and f' and g' are inverse homotopy equivalences between B and D, then

$$gf \otimes g'f' \simeq \mathrm{id}_A \otimes \mathrm{id}_B = \mathrm{id}_{A\otimes B} \quad \text{and} \quad fg \otimes f'g' \simeq \mathrm{id}_C \otimes \mathrm{id}_D = \mathrm{id}_{C\otimes D}.$$

It follows that $A \otimes B \simeq C \otimes D$ whenever $A \simeq C$ and $B \otimes D$, so that the tensor product of complexes is well-defined in $K^b(\mathcal{A})$, making it a monoidal category.

The following lemma is often useful for reducing a complex into a simpler, homotopy equivalent one.

Lemma B.4 (Gaussian elimination). *A complex of the form*

$$A = \quad \cdots \longrightarrow A^{i-1} \xrightarrow{\binom{a}{b}} \tilde{A}^i \oplus X \xrightarrow{\left(\begin{smallmatrix} c & d \\ e & \phi \end{smallmatrix}\right)} \tilde{A}^{i+1} \oplus Y \xrightarrow{(f \ g)} A^{i+2} \longrightarrow \cdots,$$

where $\phi : X \to Y$ is an isomorphism, is homotopy equivalent to the complex

$$\tilde{A} = \quad \cdots \longrightarrow A^{i-1} \xrightarrow{\ a\ } \tilde{A}^i \xrightarrow{c - d\phi^{-1}e} \tilde{A}^{i+1} \xrightarrow{\ f\ } A^{i+2} \longrightarrow \cdots.$$

Proof. Let B be the complex $X \xrightarrow{\phi} Y$, with X in degree i. Since ϕ is an isomorphism, the map ϕ^{-1} gives a homotopy $\mathrm{id}_B \cong 0$, showing that B is contractible. It is then enough to show that $A \cong \tilde{A} \oplus B$, since $\tilde{A} \oplus B \simeq \tilde{A}$. We have:

$$\begin{array}{ccccccc}
A^{i-1} & \xrightarrow{\binom{a}{b}} & \tilde{A}^i \oplus X & \xrightarrow{\left(\begin{smallmatrix} c & d \\ e & \phi \end{smallmatrix}\right)} & \tilde{A}^{i+1} \oplus Y & \xrightarrow{(f \ g)} & A^{i+2} \\
1 \big\updownarrow & & \left(\begin{smallmatrix} 1 & 0 \\ -\phi^{-1}e & 1 \end{smallmatrix}\right) \big\uparrow \big\downarrow \left(\begin{smallmatrix} 1 & 0 \\ \phi^{-1}e & 1 \end{smallmatrix}\right) & & \left(\begin{smallmatrix} 1 & d\phi^{-1} \\ 0 & 1 \end{smallmatrix}\right) \big\uparrow \big\downarrow \left(\begin{smallmatrix} 1 & -d\phi^{-1} \\ 0 & 1 \end{smallmatrix}\right) & & \big\updownarrow 1 \\
A^{i-1} & \xrightarrow[\binom{a}{b + \phi^{-1}ea}]{} & \tilde{A}^i \oplus X & \xrightarrow[\left(\begin{smallmatrix} c - d\phi^{-1}e & 0 \\ 0 & \phi \end{smallmatrix}\right)]{} & \tilde{A}^{i+1} \oplus Y & \xrightarrow[(f \ \ g + fd\phi^{-1})]{} & A^{i+2}
\end{array}$$

Since A is a complex, we have $\phi b + ea = 0$ and $g\phi + fd = 0$, and then $b + \phi^{-1}ea = 0$ and $g + fd\phi^{-1} = 0$, so that the bottom complex is exactly $\tilde{A} \oplus B$. □

In fact, one can prove that if all idempotents split in $\mathcal{A}$ then a complex in $C^b(\mathcal{A})$ is contractible if and only if it is a finite direct sum of complexes of the form $0 \to A \to B \to 0$, where the middle map is an isomorphism [17, Lemma 19.13]. We say that a complex is *minimal* if it has no contractible summands, i.e. it cannot be simplified using Gaussian elimination.

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