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Hyperasymptotic solutions of higher order linear differential equations with a singularity of rank one

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A sequence of re-expansions is developed for the remainder terms in the well-known Poincaré series expansions of the solutions of homogeneous linear differential equations of higher order in the neighbourhood of an irregular singularity of rank one. These re-expansions are a series whose terms are a product of Stokes multipliers, coefficients of the original Poincaré series expansions, and certain multiple integrals, the so-called hyperterminants. Each step of the process reduces the estimate of the error term by an exponentially small factor.

The method of this paper is based on the Borel–Laplace transform, which makes it applicable to other problems. At the end of the paper the method is applied to integrals with saddles.

Also, a powerful new method is presented to compute the Stokes multipliers. A numerical example is included.

Keywords: asymptotic expansion; exponential improvement; Borel–Laplace transform; hyperaymptotics; differential equations; Stokes multiplier

1. Introduction

We shall investigate solutions of differential equations of the form

$$\frac{\mathrm{d}^n w}{\mathrm{d}z^n} + f_{n-1}(z)\frac{\mathrm{d}^{n-1} w}{\mathrm{d}z^{n-1}} + \cdots + f_0(z)w = 0, \tag{1.1}$$

in which the coefficients $f_m(z)$, $m = 0, 1, \ldots, n-1$ can be expanded in power series

$$f_m(z) = \sum_{s=0}^{\infty} \frac{f_{sm}}{z^s}, \tag{1.2}$$

that converge on an open annulus $|z| > a$, and the point at infinity is an irregular singularity of rank one. Formal series solutions in descending powers of z are given by

$$\mathrm{e}^{\lambda_j z} z^{\mu_j} \sum_{s=0}^{\infty} a_{sj} z^{-s}, \quad j = 1, 2, \ldots, n. \tag{1.3}$$

The constants λ_j, μ_j and a_{sj} are found by substituting into the differential equation and equating coefficients after setting $a_{0j} = 1$. In this way we obtain the characteristic equation

$$\sum_{m=0}^{n} \lambda_j^m f_{0m} = 0, \tag{1.4}$$

where we take $f_{0n}=1$, to compute λ_j. The constants μ_j are given by

$$\mu_j = -\left(\sum_{m=0}^{n-1} \lambda_j^m f_{1m}\right) \Bigg/ \sum_{m=1}^{n} m\lambda_j^{m-1} f_{0m}. \tag{1.5}$$

For the coefficients a_{sj} we obtain the recurrence relation

$$(s-1)a_{s-1,j}\sum_{m=1}^{n} m\lambda_j^{m-1} f_{0,m} = \sum_{t=2}^{s} a_{s-t,j}\sum_{p=0}^{t}(\mu_j+t-s)_{\mathrm{p}}\sum_{m=p}^{n}\binom{m}{p}\lambda_j^{m-p} f_{t-p,m}, \tag{1.6}$$

where Pochhammer's symbol $(\alpha)_{\mathrm{p}}$ is defined by $(\alpha)_{\mathrm{p}} = \Gamma(\alpha+p)/\Gamma(\alpha)$.

We shall impose the restriction

$$\lambda_j \neq \lambda_k, \quad j \neq k. \tag{1.7}$$

This restriction ensures that the left-hand side of (1.6) does not vanish. We shall also assume that the μ_j are non-integers. At the end of § 6 we will remove this restriction on the μ_j.

It is well known (see, for example Wasow 1976; Olver 1997) that for any ray $\mathcal{L}$ there is a set of n linearly independent solutions of (1.1) that are represented asymptotically by the formal series (1.3) as $z \to \infty$ on $\mathcal{L}$. In general, these solutions are not uniquely determined by their asymptotic expansion on $\mathcal{L}$. In this paper we will give hyperasymptotic expansions for solutions of (1.1). These hyperasymptotic expansions will determine the solutions uniquely on $\mathcal{L}$ and give more accurate approximations.

We follow existing terminology for these new types of asymptotic expansions. The original Poincaré expansions (1.3) are regarded as being level zero. We shall obtain a level one expansion by truncating the level zero expansion at, or beyond, its optimal stage and re-expanding the remainder term in generalized exponential integrals. These level one expansions are called exponentially improved expansions. Further re-expansions are the higher levels in the hyperasymptotic expansion.

One way to derive the hyperasymptotic expansions is to extend the method that was developed in Olde Daalhuis & Olver (1994, 1995*a*) for the case $n=2$. In this approach, we begin with the Poincaré expansions of the solutions (Wasow 1976; Olver 1997), and construct an integral representation of Stieltjes type for the remainder term. This integral is then re-expanded repeatedly in series of hyperterminants, followed by an optimization procedure. In the present paper, however, we shall obtain the hyperasymptotic expansions via the Borel–Laplace transform. The Borel transforms of formal series solutions (1.3) are convergent expansions. In § 2 we give the important properties of these Borel transforms, including the local behaviour near the singularities in the bounded Riemann plane, and their growth near infinity. This is in fact all the knowledge that we will use, and in this and succeeding sections it is not necessary that the Borel transforms originate from differential equations. In § 9 we will show that our method is also applicable to integrals with saddles.

The solutions of (1.1) that we shall analyse are the Borel–Laplace transforms of (1.3). In this way we obtain double integral representations for the remainder terms. By deforming the contours of integration into special contours that are defined in § 2, we are able to obtain good estimates in § 3 for the minimal remainders of the truncated level zero expansions. In § 2 we will define a sequence of increasing positive numbers $\alpha_k^{(l)}$, $l=0,1,2,\ldots$, $k=1,\ldots,n$. We shall show that the minimal remainder terms of the original Poincaré expansions are of the order $\exp(-\alpha_k^{(0)}|z|)$ times a power of z.

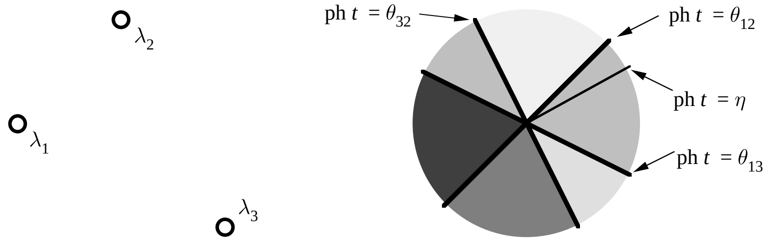

Figure 1. An example of admissible and non-admissible directions.

The method that we use to obtain the re-expansions is the substitution of a truncated Taylor series expansion for the Borel transforms. In § 4 we obtain the level one expansions, and we determine the optimal numbers of terms in these expansions.

In §§ 5 and 6 this method is continued to the second and higher levels. The minimal remainders at level l will be of the order $\exp(-\alpha_k^{(l)}|z|)$ times a power of z. We also determine the optimal numbers of terms of all branches of the hyperasymptotic expansion.

The hyperasymptotic expansions are a series whose terms are a product of Stokes multipliers, the original coefficients a_{sk}, and hyperterminants. At the end of § 2 we define the hyperterminants. In § 7 we give an efficient method to compute the required Stokes multipliers to sufficient precision. In § 8 we provide a numerical example.

In § 9 we sketch the application of the Borel–Laplace transform to integrals with saddles, and then give conclusions and generalizations in the final section § 10.

2. Definitions and lemmas

We define

$$\left.\begin{aligned} \theta_{kj} &= \mathrm{ph}(\lambda_j - \lambda_k), \\ \lambda_{kj} &= \lambda_k - \lambda_j, \\ \mu_{kj} &= \mu_k - \mu_j, \end{aligned}\right\} \quad j \neq k \quad \text{and} \quad \tilde{\mu} = \max\{\mathrm{Re}\,\mu_1, \ldots, \mathrm{Re}\,\mu_n\}. \tag{2.1}$$

We call

$$\eta \in \mathbb{R} \text{ is admissible } \iff \eta \neq \theta_{kj} (\mathrm{mod}\, 2\pi), \quad 1 \leqslant j, k \leqslant n, \quad j \neq k. \tag{2.2}$$

For an example with $n = 3$ see figure 1.

For fixed admissible η we consider a t-plane together with parallel cuts from each λ_k to ∞ along the ray $\mathrm{ph}(t - \lambda_k) = \eta$ (see figure 2). If we specify for $k = 1, 2, \ldots, n$,

$$\log(t - \lambda_k) = \log|t - \lambda_k| + i\eta, \tag{2.3}$$

for all t such that $\mathrm{ph}(t - \lambda_k) = \eta$, then we denote the t-plane with these cuts and choices of logarithms by $\mathcal{P}_\eta$. Thus $\log(t - \lambda_k)$ is continuous within $\mathcal{P}_\eta$, and is defined by (2.3) on $\mathrm{ph}(t - \lambda_k) = \eta$.

Let η be admissible. Then we define

$$\eta^- = \inf\{\check{\eta} < \eta | \tilde{\eta} \text{ is admissible for all } \tilde{\eta} \in (\check{\eta}, \eta]\}, \tag{2.4 a}$$

$$\eta^+ = \sup\{\hat{\eta} > \eta | \tilde{\eta} \text{ is admissible for all } \tilde{\eta} \in [\eta, \hat{\eta})\}, \tag{2.4 b}$$

$$\mathcal{I}_\eta = (\eta^-, \eta^+). \tag{2.4 c}$$

Note that $\eta^\pm$ are not admissible. In the example, η^- is the value θ_{13} (mod 2π) for which $|\eta - \theta_{13}|$ is least, and η^+ is the value θ_{12} (mod 2π) for which $|\theta_{12} - \eta|$ is least.

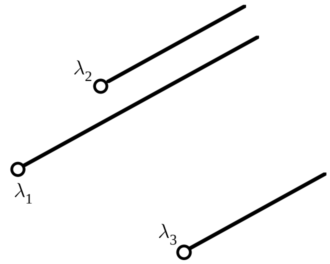

Figure 2. Cuts for $\mathcal{P}_\eta$.

With these definitions for $\eta^\pm$ we define the z sectors

$$\mathcal{S}(\eta) = \{z \,|\, \mathrm{Re}(z\mathrm{e}^{\mathrm{i}\eta}) < -a \quad \text{and} \quad \tfrac{1}{2}\pi - \eta^+ < \mathrm{ph}\, z < \tfrac{3}{2}\pi - \eta^-\}, \tag{2.5}$$

$$\overline{\mathcal{S}}(\eta) = \{z \,|\, \mathrm{Re}(z\mathrm{e}^{\mathrm{i}\eta}) < -a \quad \text{and} \quad \pi - \eta^+ \leqslant \mathrm{ph}\, z \leqslant \pi - \eta^-\}, \tag{2.6}$$

The main tools that we will use in this paper are theorems 1 and 2 of Balser *et al.* (1981). If we translate the results of these theorems to our notation we obtain the following.

Lemma 2.1. *The function $y_k(t)$ defined by*

$$y_k(t) = \sum_{p=0}^{\infty} a_{pk}\Gamma(\mu_k + 1 - p)(t-\lambda_k)^{p-\mu_k-1}, \quad |t-\lambda_k| < \min_{j\neq k} |\lambda_j - \lambda_k|, \tag{2.7}$$

is analytic in $\mathcal{P}_\eta$, satisfies

$$y_k(t) = \frac{K_{jk}}{1-\mathrm{e}^{-2\pi\mathrm{i}\mu_k}} y_j(t) + \mathrm{reg}(t-\lambda_j), \quad j \neq k, \tag{2.8}$$

where the K_{jk} are constants, and can be continued analytically along every path that does not intersect any of the points $\lambda_1, \ldots, \lambda_n$. Furthermore, if S is any sector in the t-plane of the form $S = \{|t| > R,\ \alpha < \mathrm{ph}\, t < \beta\}$ with $0 < \beta - \alpha < 2\pi$ and $R > \max|\lambda_j|$, then

$$\lim_{t\to\infty} \mathrm{e}^{-(a+\varepsilon)|t|} y_k(t) = 0, \quad t \in S, \tag{2.9}$$

for $\varepsilon > 0$ arbitrary.

In (2.8) $\mathrm{reg}(t-\lambda_j)$ denotes a function that is regular (or analytic) in a neighbourhood of $t = \lambda_j$.

Lemma 2.2. *Let $\eta \in \mathbb{R}$ be admissible. If we define*

$$w_k(z,\eta) = \frac{1}{2\pi\mathrm{i}} \int_{\gamma_k(\eta)} \mathrm{e}^{zt} y_k(t)\,\mathrm{d}t, \tag{2.10}$$

where $\gamma_k(\eta)$ is the contour in $\mathcal{P}_\eta$ from ∞ along the left-hand side of the cut $\mathrm{ph}(t-\lambda_k) = \eta$, around λ_k in the positive sense, and back to ∞ along the right-hand side of the cut, then $w_k(z,\eta)$ is a solution of (1.1), $w_k(z,\tilde{\eta}) = w_k(z,\eta)$ for all $\tilde{\eta} \in \mathcal{I}_\eta$, and

$$w_k(z,\eta) \sim \mathrm{e}^{\lambda_k z} z^{\mu_k} \sum_{s=0}^{\infty} a_{sk} z^{-s}, \tag{2.11}$$

as $z \to \infty$ in $\mathcal{S}(\eta)$.

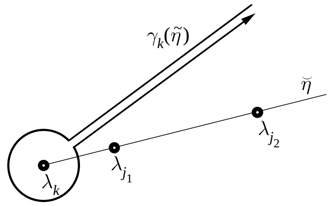

Figure 3. $\gamma_k(\tilde\eta)$ before the rotation.

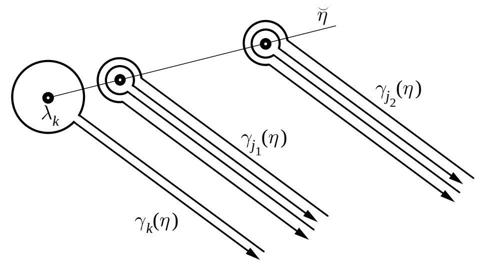

Figure 4. $\gamma_k(\tilde\eta)$ after the rotation.

For each admissible η we have n solutions $w_1(z,\eta),\ldots,w_n(z,\eta)$. Since (1.1) is a linear ordinary differential equation of order n, for each admissible $\tilde\eta$ and $k\in\{1,\ldots,n\}$ there are connection coefficients $C_{jk}(\tilde\eta,\eta)$ such that

$$w_k(z,\tilde\eta)=C_{1k}(\tilde\eta,\eta)w_1(z,\eta)+\cdots+C_{nk}(\tilde\eta,\eta)w_n(z,\eta). \tag{2.12}$$

If $\tilde\eta\in\mathcal{I}_\eta$, then $C_{jk}(\tilde\eta,\eta)=\delta_{jk}$. Hence, the connection coefficients can change only when we cross a non-admissible direction. The corresponding directions in the z-plane are generally known as Stokes lines. To compute all the connection coefficients it suffices to compute the connection coefficients of two neighbouring intervals $\mathcal{I}_\eta$.

Take $\eta<\tilde\eta$ in two neighbouring intervals $\mathcal{I}_\eta$ and $\mathcal{I}_{\tilde\eta}$, and let $\breve\eta$ be the non-admissible direction between η and $\tilde\eta$. Fix $k\in\{1,\ldots,n\}$ and let $j_1,\ldots,j_p$ be all the $j\neq k$ such that $\theta_{kj}=\breve\eta \pmod{2\pi}$ (see figure 3). If we rotate the contour $\gamma_k(\tilde\eta)$ across the Stokes line at $\mathrm{ph}(t-\lambda_k)=\breve\eta$ we obtain the contour $\gamma_k(\eta)$ plus for each j_l contours $\gamma_{j_l}(\eta)$ and $\tilde\gamma_{j_l}(\eta)$. The contour $\tilde\gamma_{j_l}(\eta)$ is the inner contour of the two contours encircling λ_{j_l} in figure 4, it is contour $\gamma_{j_l}(\eta)$ with the opposite direction of integration and it lies on the Riemann sheet $\log(t-\lambda_k)\in[\eta+2\pi,\eta+4\pi)$; furthermore $\log(t-\lambda_j)\in[\eta,\eta+2\pi)$, $j\neq k$. Hence, with (2.8) we obtain

$$\begin{aligned} w_k(z,\tilde\eta) &= w_k(z,\eta)+\sum_{l=1}^{p}\frac{1-\mathrm{e}^{-2\pi\mathrm{i}\mu_k}}{2\pi\mathrm{i}}\int_{\gamma_{j_l}(\eta)}\mathrm{e}^{zt}y_k(t)\,\mathrm{d}t\\ &= w_k(z,\eta)+\sum_{l=1}^{p}\frac{K_{j_lk}}{2\pi\mathrm{i}}\int_{\gamma_{j_l}(\eta)}\mathrm{e}^{zt}y_{j_l}(t)\,\mathrm{d}t\\ &= w_k(z,\eta)+\sum_{l=1}^{p}K_{j_lk}w_{j_l}(z,\eta). \end{aligned} \tag{2.13}$$

The constants K_{jk} are called the *Stokes multipliers*, and if we can compute the Stokes multipliers, then we can compute all the connection coefficients. In § 7 we give a numerical method to compute the Stokes multipliers.

The reason that we write the constant in front of $y_j(t)$ in (2.8) in such a complicated way is that with this choice the last line of (2.13) is in the simplest form. By trying to present (2.13) in its simplest form we show a slight preference for expressing $w_k(z,\tilde\eta)$ in terms of $w_j(z,\eta)$, $j=1,\ldots,n$, over expressing $w_k(z,\eta)$ in terms of $w_j(z,\tilde\eta)$, that is, we prefer 'right', the original η, over 'left'. This choice will be reflected further on in this section in the definition of special contours and hyperterminants.

The Stokes multipliers play an important role in the definitions of the following numbers. Let

$$\alpha_k^{(m)} = \min\{|\lambda_k - \lambda_{j_0}| + |\lambda_{j_0} - \lambda_{j_1}| + \cdots + |\lambda_{j_{m-1}} - \lambda_{j_m}| : \\ j_0 \neq k,\ K_{j_0 k} \neq 0,\ j_l \neq j_{l-1},\ K_{j_l j_{l-1}} \neq 0\}. \qquad (2.14)$$

If $G = (V, E)$ is a directed graph with vertices $V = \{\lambda_1, \ldots, \lambda_n\}$ and edges $E = \{(\lambda_p, \lambda_q) | 1 \leqslant p, q \leqslant n, p \neq q, K_{qp} \neq 0\}$, then $\alpha_k^{(m)}$ is the length of the shortest directed path of m steps starting at λ_k.

In the technical parts of the following sections we shall use the following two lemmas.

Lemma 2.3. *Let r be a positive constant and C be a contour that begins at $\lambda_j + r\mathrm{e}^{\mathrm{i}\theta_{kj}}$, encircles λ_j once in a positive sense, and returns to its starting point. Let t be a complex number outside C such that for all $\tau \in C$ we have $|t - \tau| \geqslant (\alpha/|z|)$, where α is a positive constant. Also, take $N = \beta|z| + \gamma$, where β is a positive constant, and γ is a real constant. Then for all $s \in \{1, 2, 3, \ldots\}$, $s < N + \operatorname{Re}\mu_{jk} - 1$ we have*

$$\int_C \frac{(\tau - \lambda_j)^{s-\mu_j-1}(\tau - \lambda_k)^{\mu_k+1-N}}{\tau - t}\,\mathrm{d}\tau = |z||\lambda_{jk}|^{s-N} \\ \times \frac{\Gamma(s - \operatorname{Re}\mu_j)\Gamma(N - s + \operatorname{Re}\mu_{jk} - 1)}{\Gamma(N - \operatorname{Re}\mu_k - 1)}\mathcal{O}(1), \qquad (2.15)$$

as $|z| \to \infty$.

Proof. If $\operatorname{Re}(s - \mu_j) > 0$, then we can collapse the contour of integration on to the join of λ_j and $\lambda_j + r\mathrm{e}^{\mathrm{i}\theta_{kj}}$. We obtain

$$\begin{aligned} &\int_C \frac{(\tau - \lambda_j)^{s-\mu_j-1}(\tau - \lambda_k)^{\mu_k+1-N}}{\tau - t}\,\mathrm{d}\tau \\ &\quad = (\mathrm{e}^{-2\pi\mathrm{i}\mu_j} - 1)\int_0^{r\mathrm{e}^{\mathrm{i}\theta_{kj}}} \frac{\tau^{s-\mu_j-1}(\tau + \lambda_{jk})^{\mu_k+1-N}}{\tau + \lambda_j - t}\,\mathrm{d}\tau \\ &\quad = (\mathrm{e}^{-2\pi\mathrm{i}\mu_j} - 1)(\lambda_{jk})^{s-N+\mu_{kj}+1}\int_0^{(r/|\lambda_{jk}|)} \frac{\tau^{s-\mu_j-1}(\tau + 1)^{\mu_k+1-N}}{\tau\lambda_{jk} + \lambda_j - t}\,\mathrm{d}\tau \\ &\quad = |z||\lambda_{jk}|^{s-N}\int_0^\infty \tau^{s-\operatorname{Re}\mu_j-1}(\tau + 1)^{\operatorname{Re}\mu_k+1-N}\,\mathrm{d}\tau\,\mathcal{O}(1) \\ &\quad = |z||\lambda_{jk}|^{s-N}\frac{\Gamma(s - \operatorname{Re}\mu_j)\Gamma(N - s + \operatorname{Re}\mu_{jk} - 1)}{\Gamma(N - \operatorname{Re}\mu_k - 1)}\mathcal{O}(1). \end{aligned} \qquad (2.16)$$

If $\operatorname{Re}(s - \mu_j) < 0$, then we take an integer P such that $0 < P + s - \operatorname{Re}\mu_j \leqslant 1$. Note that $P = \mathcal{O}(1)$ as $|z| \to \infty$. We obtain

$$\begin{aligned} &\int_C \frac{(\tau - \lambda_j)^{s-\mu_j-1}(\tau - \lambda_k)^{\mu_k+1-N}}{\tau - t}\,\mathrm{d}\tau \\ &\quad = \sum_{p=0}^{P-1} \frac{-1}{(t - \lambda_j)^{p+1}}\int_C (\tau - \lambda_j)^{s+p-\mu_j-1}(\tau - \lambda_k)^{\mu_k+1-N}\,\mathrm{d}\tau \\ &\qquad + (t - \lambda_j)^{-P}\int_C \frac{(\tau - \lambda_j)^{s+P-\mu_j-1}(\tau - \lambda_k)^{\mu_k+1-N}}{\tau - t}\,\mathrm{d}\tau. \end{aligned} \qquad (2.17)$$

If we apply $P-p$ integrations by parts, then we can estimate the pth term in the sum of (2.17) by

$$|z|^{p+1}|\lambda_{jk}|^{s+p-N}\frac{\Gamma(s+p-\operatorname{Re}\mu_j)\Gamma(N-s-p+\operatorname{Re}\mu_{jk}-1)}{\Gamma(N-\operatorname{Re}\mu_k-1)}\mathcal{O}(1), \quad \text{as } |z|\to\infty. \tag{2.18}$$

Furthermore, if we apply the first part of this proof to the final term of the right-hand side of (2.17), then we can estimate this term by means of (2.18) with $p=P$. Hence,

$$\begin{aligned}\int_C &\frac{(\tau-\lambda_j)^{s-\mu_j-1}(\tau-\lambda_k)^{\mu_k+1-N}}{\tau-t}\,\mathrm{d}\tau \\ &= \sum_{p=0}^{P}|z|^{p+1}|\lambda_{jk}|^{s+p-N}\frac{\Gamma(s+p-\operatorname{Re}\mu_j)\Gamma(N-s-p+\operatorname{Re}\mu_{jk}-1)}{\Gamma(N-\operatorname{Re}\mu_k-1)}\mathcal{O}(1) \\ &= P|z||\lambda_{jk}|^{s-N}\frac{\Gamma(s-\operatorname{Re}\mu_j)\Gamma(N-s+\operatorname{Re}\mu_{jk}-1)}{\Gamma(N-\operatorname{Re}\mu_k-1)}\mathcal{O}(1), \quad \text{as } |z|\to\infty. \end{aligned} \tag{2.19}$$

■

The following lemma is proved in a similar way.

Lemma 2.4. *Let U be an open set that contains $\{\lambda_j+x\mathrm{e}^{\mathrm{i}\theta_{kj}}|0\leqslant x\leqslant r\}$. Let $b(\tau)$ be analytic on U, and let C be a contour in U that begins at $\lambda_j+r\mathrm{e}^{\mathrm{i}\theta_{kj}}$, encircles λ_j once in a positive sense, and returns to its starting point. Then*

$$\begin{aligned}\int_C b(\tau)(\tau-\lambda_j)^{-\mu_j-1}(\tau-\lambda_k)^{\mu_k+1-N}\,\mathrm{d}\tau \\ = \max_{\tau\in C}|b(\tau)||\lambda_{jk}|^{-N}N^{\operatorname{Re}\mu_j}\mathcal{O}(1), \quad \text{as } N\to\infty.\end{aligned} \tag{2.20}$$

The main step needed to reach subsequent levels in the hyperasymptotic expansion for solutions of (1.1) is the following version of Taylor's theorem.

Lemma 2.5. *Let C be a closed contour encircling t and λ_k such that λ_j, $j\neq k$, is in the exterior of C. Then*

$$\begin{aligned}y_k(t) = \sum_{p=0}^{N-1}a_{pk}\Gamma(\mu_k+1-p)(t-\lambda_k)^{p-\mu_k-1} \\ +\frac{(t-\lambda_k)^{N-\mu_k-1}}{2\pi\mathrm{i}}\int_C\frac{y_k(\tau)(\tau-\lambda_k)^{\mu_k+1-N}}{\tau-t}\,\mathrm{d}\tau.\end{aligned} \tag{2.21}$$

In the next sections we take for the contour C of lemma 2.5 the following special contours. Let ε_0 be a fixed number such that $0<\varepsilon_0<\frac{1}{4}\min_{j\neq k}|\lambda_{jk}|$. Take $R_0>\max_{j\neq k}|\lambda_{jk}|+(\varepsilon_0/(a+1))$ and $R_n=(n+1)R_0$, $n=1,2,3,\ldots$.

In the case $|t-\lambda_k|<R_0-(\varepsilon_0/|z|)$ we define $C_k^{(0)}(t)$ to be the contour indicated in figure 5, with $m=0$. In the cases when $R_0-(\varepsilon_0/|z|)\leqslant|t-\lambda_k|\leqslant R_0+(\varepsilon_0/|z|)$ and $|t-\lambda_k|>R_0+(\varepsilon_0/|z|)$ we adjust $C_k^{(0)}(t)$ as indicated in the middle and right parts of figure 6.

Let $\varepsilon_0>\varepsilon_1>\varepsilon_2>\cdots>0$, and define $C_k^{(m)}$, $m=1,2,\ldots$, as the contours indicated in figure 5. Since $\max_{j\neq k}|\lambda_{jk}|<R_0<R_1<\cdots$, all of the λ_j are in the interior of the large circle of $C_k^{(m)}$.

The contour $C_{kj}^{(m)}$ will be the loop of $C_k^{(m)}$ that encircles λ_j, that is, the contour

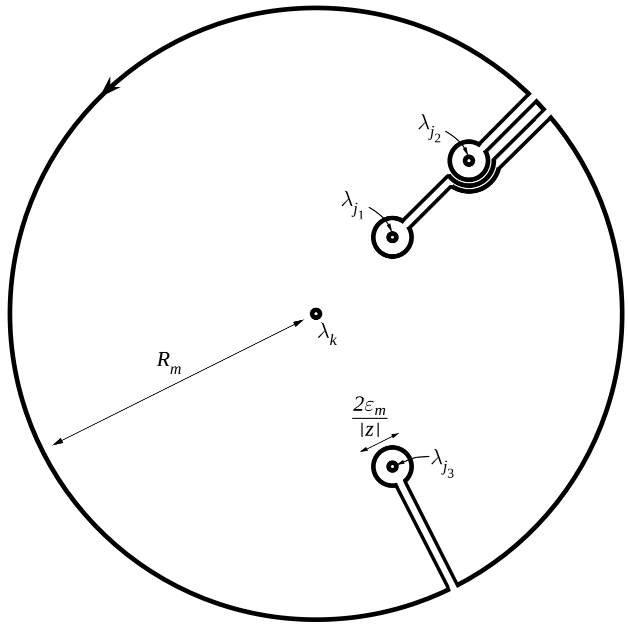

Figure 5. Contour $C_k^{(m)}$.

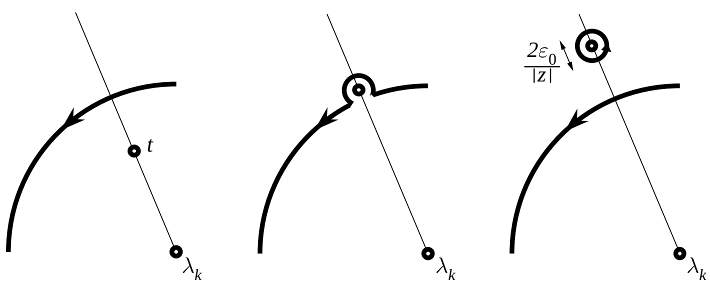

Figure 6. Contour $C_k^{(0)}(t)$ for $|t-\lambda_k| < R_0 - (\varepsilon_0/|z|)$ (left), $R_0 - (\varepsilon_0/|z|) \leqslant |t-\lambda_k| \leqslant R_0 + (\varepsilon_0/|z|)$ (middle) and $|t-\lambda_k| > R_0 + (\varepsilon_0/|z|)$ (right).

that starts at $\lambda_k + R_m e^{i\theta_{kj}}$, encircles λ_j once in a positive sense, and returns to its starting point. Notice that this loop encircles λ_j in the opposite direction to the loop of $C_k^{(m)}$. The choices for ε_m, $m = 1, 2, 3, \ldots$, are such that $C_{kj}^{(m)}$ is in the interior of $C_j^{(m+1)}$. Notice that again we prefer 'right' over 'left': in figure 5 the contour $C_{kj_1}^{(m)}$ passes $C_{kj_2}^{(m)}$ on the right-hand side.

Finally, let $C_{k-}^{(m)}$, $m = 0, 1, 2, \ldots$, denote $C_k^{(m)}$ minus the loops $C_{kj}^{(m)}$; thus

$$C_{k-}^{(m)} = C_k^{(m)} + \sum_{j\neq k} C_{kj}^{(m)}.$$

We finish this section with the definition of *hyperterminants*. In the definition we

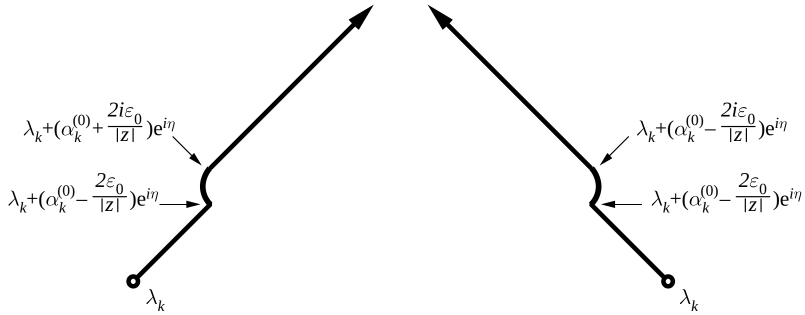

Figure 7. (Left) Contour $\mathcal{P}$ when $\eta^- \leqslant \eta < \frac{1}{2}(\eta^- + \eta^+)$. (Right) Contour $\mathcal{P}$ when $\frac{1}{2}(\eta^- + \eta^+) \leqslant \eta \leqslant \eta^+$.

shall use the notation

$$\int_\lambda^{[\eta]} = \int_\lambda^{\infty e^{i\eta}}, \quad \eta \in \mathbb{R}. \tag{2.22}$$

Let l be a non-negative integer, $\operatorname{Re} M_j > 1$, $\sigma_j \in \mathbb{C}$, $\sigma_j \neq 0$, $j = 0, \ldots, l$. Then

$$\left.\begin{aligned}
&F^{(0)}(z) = 1,\\
&F^{(1)}\left(z; \begin{matrix} M_0 \\ \sigma_0 \end{matrix}\right) = \int_0^{[\pi-\theta_0]} \frac{e^{\sigma_0 t_0} t_0^{M_0-1}}{z - t_0}\, dt_0,\\
&F^{(l+1)}\left(z; \begin{matrix} M_0, \cdots, M_l \\ \sigma_0, \cdots, \sigma_l \end{matrix}\right)\\
&\quad = \int_0^{[\pi-\theta_0]} \cdots \int_0^{[\pi-\theta_l]} \frac{e^{(\sigma_0 t_0 + \cdots + \sigma_l t_l)} t_0^{M_0-1} \cdots t_l^{M_l-1}}{(z - t_0)(t_0 - t_1)\cdots(t_{l-1} - t_l)}\, dt_l \cdots dt_0,
\end{aligned}\right\} \tag{2.23}$$

where $\theta_j = \operatorname{ph}\sigma_j$, $j = 0, 1, \ldots, l$. In the case $\operatorname{ph}\sigma_j = \operatorname{ph}\sigma_{j+1} \pmod{2\pi}$ we have to make the choice between the t_j contour being on the 'left' or 'right' of the t_{j+1} contour. We make the choice via the definition

$$F^{(l+1)}\left(z; \begin{matrix} M_0, \cdots, M_l \\ \sigma_0, \cdots, \sigma_l \end{matrix}\right) = \lim_{\varepsilon \downarrow 0} F^{(l+1)}\left(z; \begin{matrix} M_0, & M_1, & \cdots, & M_{l-1}, & M_l \\ \sigma_0 e^{-l\varepsilon i}, & \sigma_1 e^{-(l-1)\varepsilon i}, & \cdots, & \sigma_{l-1}e^{-\varepsilon i}, & \sigma_l \end{matrix}\right), \tag{2.24}$$

which means that once again we prefer 'right' over 'left'. The multiple integrals converge when $-\pi - \theta_0 < \operatorname{ph} z < \pi - \theta_0$.

3. Superasymptotics

In this section we show how the remainder

$$R_k^{(0)}(z, \eta; N_0) = w_k(z, \eta) - e^{\lambda_k z} z^{\mu_k} \sum_{s=0}^{N_0-1} a_{sk} z^{-s}, \tag{3.1}$$

depends on N_0. Let η be admissible. Throughout this section, we suppose that $z \in \overline{\mathcal{S}}(\eta)$. We permit N_0 to be a linear function of $|z|$. More precisely, we assume that

$$N_0 = \beta_k^{(0)}|z| + \gamma_k^{(0)}, \tag{3.2}$$

where $\beta_k^{(0)}$ is a positive constant at our disposal and $\gamma_k^{(0)}$ is bounded. If we apply lemma 2.5 with $N=N_0$ and $C=C_k^{(0)}(t)$ in (2.10), we obtain the integral representation

$$R_k^{(0)}(z,\eta;N_0)=\frac{1}{(2\pi\mathrm{i})^2}\int_{\gamma_k(\eta)}\int_{C_k^{(0)}(t)}\mathrm{e}^{zt}\left(\frac{t-\lambda_k}{\tau-\lambda_k}\right)^{N_0-\mu_k-1}\frac{y_k(\tau)}{\tau-t}\,\mathrm{d}\tau\mathrm{d}t. \tag{3.3}$$

We assume throughout that $|z|$ is sufficiently large to ensure that

$$N_0>\operatorname{Re}\mu_k. \tag{3.4}$$

We may then collapse $\gamma_k(\eta)$ in (3.3) on to $[\lambda_k,\infty\mathrm{e}^{\mathrm{i}\eta})$. We obtain

$$R_k^{(0)}(z,\eta;N_0)=\frac{\mathrm{e}^{-2\pi\mathrm{i}\mu_k}-1}{(2\pi\mathrm{i})^2}\int_{\lambda_k}^{[\eta]}\int_{C_k^{(0)}(t)}\mathrm{e}^{zt}\left(\frac{t-\lambda_k}{\tau-\lambda_k}\right)^{N_0-\mu_k-1}\frac{y_k(\tau)}{\tau-t}\,\mathrm{d}\tau\mathrm{d}t. \tag{3.5}$$

Split $C_k^{(0)}(t)$ into $C_{k-}^{(0)}$ and $C_{kj}^{(0)}$, $j\neq k$, and use (2.8). We then have

$$\begin{aligned}R_k^{(0)}(z,\eta;N_0)&=\sum_{j\neq k}\frac{1-\mathrm{e}^{-2\pi\mathrm{i}\mu_k}}{(2\pi\mathrm{i})^2}\int_{\lambda_k}^{[\eta]}\int_{C_{kj}^{(0)}}\mathrm{e}^{zt}\left(\frac{t-\lambda_k}{\tau-\lambda_k}\right)^{N_0-\mu_k-1}\frac{y_k(\tau)}{\tau-t}\,\mathrm{d}\tau\mathrm{d}t\\&\quad+S_k^{(0)}(z,\eta)\\&=\sum_{j\neq k}\frac{K_{jk}}{(2\pi\mathrm{i})^2}\int_{\lambda_k}^{[\eta]}\int_{C_{kj}^{(0)}}\mathrm{e}^{zt}\left(\frac{t-\lambda_k}{\tau-\lambda_k}\right)^{N_0-\mu_k-1}\frac{y_j(\tau)}{\tau-t}\,\mathrm{d}\tau\mathrm{d}t+S_k^{(0)}(z,\eta),\end{aligned} \tag{3.6}$$

where

$$S_k^{(0)}(z,\eta)=\frac{\mathrm{e}^{-2\pi\mathrm{i}\mu_k}-1}{(2\pi\mathrm{i})^2}\int_{\lambda_k}^{[\eta]}\int_{C_{k-}^{(0)}}\mathrm{e}^{zt}\left(\frac{t-\lambda_k}{\tau-\lambda_k}\right)^{N_0-\mu_k-1}\frac{y_k(\tau)}{\tau-t}\,\mathrm{d}\tau\mathrm{d}t. \tag{3.7}$$

To estimate $S_k^{(0)}(z,\eta)$ as $|z|\to\infty$, we take $\eta=\pi-\operatorname{ph}z$. When $|t-\lambda_k|\leqslant R_0+(\varepsilon_0/|z|)$ we take $\tilde{C}_{k-}=C_{k-}^{(0)}$, and when $|t-\lambda_k|>R_0+(\varepsilon_0/|z|)$ we take $\tilde{C}_{k-}$ to be $C_{k-}^{(0)}$ without the small circle encircling t (see figure 6). If we use (2.9) we obtain

$$\begin{aligned}S_k^{(0)}(z,\eta)&=\frac{\mathrm{e}^{-2\pi\mathrm{i}\mu_k}-1}{(2\pi\mathrm{i})^2}\int_{\lambda_k}^{[\eta]}\int_{\tilde{C}_{k-}}\mathrm{e}^{zt}\left(\frac{t-\lambda_k}{\tau-\lambda_k}\right)^{N_0-\mu_k-1}\frac{y_k(\tau)}{\tau-t}\,\mathrm{d}\tau\mathrm{d}t\\&\quad+\frac{\mathrm{e}^{-2\pi\mathrm{i}\mu_k}-1}{2\pi\mathrm{i}}\int_{\lambda_k+(R_0+(\varepsilon_0/|z|))\mathrm{e}^{\mathrm{i}\eta}}^{[\eta]}\mathrm{e}^{zt}y_k(t)\,\mathrm{d}t\\&=\mathrm{e}^{\lambda_k z}\int_0^\infty\mathrm{e}^{-|z|t}t^{N_0-\operatorname{Re}\mu_k-1}\,\mathrm{d}t\,R_0^{-N_0}|z|\mathcal{O}(1)\\&\quad+\mathrm{e}^{\lambda_k z}\int_{R_0+(\varepsilon_0/|z|)}^\infty\mathrm{e}^{-(|z|-a-\varepsilon)t}\,\mathrm{d}t\,o(1)\\&=\mathrm{e}^{\lambda_k z}z^{\operatorname{Re}\mu_k+1-N_0}R_0^{-N_0}\Gamma(N_0-\operatorname{Re}\mu_k)\mathcal{O}(1)+\mathrm{e}^{\lambda_k z}z^{-1}\mathrm{e}^{-|z|R_0}o(1),\end{aligned} \tag{3.8}$$

as $z\to\infty$ in $\overline{\mathcal{S}}(\eta)$. To obtain the final estimate for $S_k^{(0)}(z,\eta)$ we substitute into (3.8) by means of (3.2) and apply Stirling's formula for $\Gamma(N_0-\operatorname{Re}\mu_k)$. We obtain

$$S_k^{(0)}(z,\eta)=\mathrm{e}^{\lambda_k z}(\beta_k^{(0)}/R_0\mathrm{e})^{\beta_k^{(0)}|z|}z^{(1/2)}\mathcal{O}(1)+\mathrm{e}^{\lambda_k z-|z|R_0}z^{-1}o(1), \tag{3.9}$$

as $z\to\infty$ in $\overline{\mathcal{S}}(\eta)$.

To estimate the other terms on the right-hand side of (3.6) we adjust the t contour of integration. Again take $\eta = \pi - \operatorname{ph} z$, and let $\mathcal{P}$ be the contour indicated in figure 7.

We assume that $\frac{1}{2}(\eta^- + \eta^+) \leqslant \eta \leqslant \eta^+$. In the following derivation we shall use lemma 2.4 with $\max|b(\tau)| = |z|\mathcal{O}(1)$. We have

$$\begin{aligned}
&\frac{K_{jk}}{(2\pi \mathrm{i})^2}\int_{\mathcal{P}}\int_{C_{kj}^{(0)}} \mathrm{e}^{zt}\left(\frac{t-\lambda_k}{\tau-\lambda_k}\right)^{N_0-\mu_k-1}\frac{y_j(\tau)}{\tau-t}\,\mathrm{d}\tau\mathrm{d}t \\
&\quad= \mathrm{e}^{\lambda_k z}\frac{K_{jk}}{(2\pi \mathrm{i})^2}\int_0^{(\alpha_k^{(0)}-(2\varepsilon_0/|z|))\mathrm{e}^{\mathrm{i}\eta}} \mathrm{e}^{zt}t^{N_0-\mu_k-1}\int_{C_{kj}^{(0)}}\frac{y_j(\tau)(\tau-\lambda_k)^{\mu_k+1-N_0}}{\tau-\lambda_k-t}\,\mathrm{d}\tau\mathrm{d}t \\
&\qquad+\mathrm{e}^{\lambda_k z}\frac{K_{jk}}{(2\pi \mathrm{i})^2}\int_{\alpha_k^{(0)}\mathrm{e}^{\mathrm{i}\eta}}^{[\eta]} \mathrm{e}^{z(t-(2\mathrm{i}\varepsilon_0/|z|)\mathrm{e}^{\mathrm{i}\eta})}\left(t-\frac{2\mathrm{i}\varepsilon_0}{|z|}\mathrm{e}^{\mathrm{i}\eta}\right)^{N_0-\mu_k-1} \\
&\qquad\times\int_{C_{kj}^{(0)}}\frac{y_j(\tau)(\tau-\lambda_k)^{\mu_k+1-N_0}}{\tau-\lambda_k-(t-(2\mathrm{i}\varepsilon_0/|z|)\mathrm{e}^{\mathrm{i}\eta})}\,\mathrm{d}\tau\mathrm{d}t \\
&\qquad+\mathrm{e}^{\lambda_k z}\frac{K_{jk}}{(2\pi \mathrm{i})^2}\frac{2\mathrm{i}\varepsilon_0}{|z|}\int_{\eta-\pi}^{\eta-(\pi/2)} \mathrm{e}^{z(\alpha_k^{(0)}\mathrm{e}^{\mathrm{i}\eta}+(2\varepsilon_0/|z|)\mathrm{e}^{\mathrm{i}\theta})}\left(\alpha_k^{(0)}\mathrm{e}^{\mathrm{i}\eta}+\frac{2\varepsilon_0}{|z|}\mathrm{e}^{\mathrm{i}\theta}\right)^{N_0-\mu_k-1} \\
&\qquad\times\mathrm{e}^{\mathrm{i}\theta}\int_{C_{kj}^{(0)}}\frac{y_j(\tau)(\tau-\lambda_k)^{\mu_k+1-N_0}}{\tau-\lambda_k-\alpha_k^{(0)}\mathrm{e}^{\mathrm{i}\eta}-(2\varepsilon_0/|z|)\mathrm{e}^{\mathrm{i}\theta}}\,\mathrm{d}\tau\mathrm{d}\theta \\
&\quad= K_{jk}\mathrm{e}^{\lambda_k z}|\lambda_{jk}|^{-N_0}N_0^{\operatorname{Re}\mu_j}|z|\int_0^\infty \mathrm{e}^{-|z|t}t^{N_0-\operatorname{Re}\mu_k-1}\,\mathrm{d}t\,\mathcal{O}(1) \\
&\qquad+K_{jk}\mathrm{e}^{\lambda_k z-\alpha_k^{(0)}|z|}\left(\frac{\alpha_k^{(0)}}{|\lambda_{jk}|}\right)^{N_0}N_0^{\operatorname{Re}\mu_j}\mathcal{O}(1) \\
&\quad= K_{jk}\mathrm{e}^{\lambda_k z}|\lambda_{jk}|^{-N_0}N_0^{\operatorname{Re}\mu_j}|z|^{\operatorname{Re}\mu_k+1-N_0}\Gamma(N_0-\operatorname{Re}\mu_k)\mathcal{O}(1) \\
&\qquad+K_{jk}\mathrm{e}^{\lambda_k z-\alpha_k^{(0)}|z|}\left(\frac{\alpha_k^{(0)}}{|\lambda_{jk}|}\right)^{N_0}N_0^{\operatorname{Re}\mu_j}\mathcal{O}(1),
\end{aligned} \tag{3.10}$$

as $z \to \infty$. Again, we use (3.2) and Stirling's formula, and we obtain

$$\begin{aligned}
&\frac{K_{jk}}{(2\pi \mathrm{i})^2}\int_{\mathcal{P}}\int_{C_{kj}^{(0)}} \mathrm{e}^{zt}\left(\frac{t-\lambda_k}{\tau-\lambda_k}\right)^{N_0-\mu_k-1}\frac{y_j(\tau)}{\tau-t}\,\mathrm{d}\tau\mathrm{d}t \\
&\quad= K_{jk}\mathrm{e}^{\lambda_k z}\left(\frac{\beta_k^{(0)}}{|\lambda_{jk}|\mathrm{e}}\right)^{\beta_k^{(0)}|z|}|z|^{\operatorname{Re}\mu_j+(1/2)}\mathcal{O}(1) \\
&\qquad+K_{jk}\mathrm{e}^{\lambda_k z-\alpha_k^{(0)}|z|}\left(\frac{\alpha_k^{(0)}}{|\lambda_{jk}|}\right)^{\beta_k^{(0)}|z|}|z|^{\operatorname{Re}\mu_j}\mathcal{O}(1),
\end{aligned} \tag{3.11}$$

as $z \to \infty$. We notice that both terms on the right-hand side of (3.11) contain a factor $K_{jk}|\lambda_{jk}|^{-\beta_k^{(0)}|z|}$. Hence, the main contribution to the final sum in (3.6) comes from the value of j for which

$$|\lambda_{jk}| = \min\{|\lambda_{lk}| : l \neq k, K_{lk} \neq 0\} = \alpha_k^{(0)}.$$

It follows that

$$\sum_{j\neq k}\frac{K_{jk}}{(2\pi\mathrm{i})^2}\int_{\mathcal{P}}\int_{C_{kj}^{(0)}}\mathrm{e}^{zt}\left(\frac{t-\lambda_k}{\tau-\lambda_k}\right)^{N_0-\mu_k-1}\frac{y_j(\tau)}{\tau-t}\,\mathrm{d}\tau\mathrm{d}t$$
$$=\mathrm{e}^{\lambda_k z}\left(\frac{\beta_k^{(0)}}{\alpha_k^{(0)}\mathrm{e}}\right)^{\beta_k^{(0)}|z|}|z|^{\tilde{\mu}+(1/2)}\mathcal{O}(1)+\mathrm{e}^{\lambda_k z-\alpha_k^{(0)}|z|}|z|^{\tilde{\mu}}\mathcal{O}(1), \quad (3.12)$$

as $z\to\infty$. Since the right-hand side of (3.9) can be absorbed into the right-hand side of (3.12), we have

$$R_k^{(0)}(z,\eta;N_0)=\mathrm{e}^{\lambda_k z}\left(\frac{\beta_k^{(0)}}{\alpha_k^{(0)}\mathrm{e}}\right)^{\beta_k^{(0)}|z|}|z|^{\tilde{\mu}+(1/2)}\mathcal{O}(1)+\mathrm{e}^{\lambda_k z-\alpha_k^{(0)}|z|}|z|^{\tilde{\mu}}\mathcal{O}(1), \quad (3.13)$$

as $z\to\infty$, where $\eta=\pi-\mathrm{ph}\,z\in[\frac{1}{2}(\eta^-+\eta^+),\eta^+]$. In the same way we can show that (3.13) holds for $\eta\in[\eta^-,\frac{1}{2}(\eta^-+\eta^+)]$. Hence, (3.13) holds for $z\in\overline{\mathcal{S}}(\eta)$.

The estimate (3.13) applies with any value of the positive constant $\beta_k^{(0)}$, but it is minimal when $\beta_k^{(0)}=\alpha_k^{(0)}$. Then letting $z\to\infty$ in $\overline{\mathcal{S}}(\tau)$ we arrive at the main result of this section: *if* $z\to\infty$ *in* $\overline{\mathcal{S}}(\eta)$ *and* $N_0=\alpha_k^{(0)}|z|+\mathcal{O}(1)$*, then*

$$R_k^{(0)}(z,\eta;N_0)=\mathrm{e}^{\lambda_k z-\alpha_k^{(0)}|z|}|z|^{\tilde{\mu}+(1/2)}\mathcal{O}(1). \quad (3.14)$$

4. Level one

Throughout this section we shall assume that $z\in\overline{\mathcal{S}}(\eta)$. Since $S_k^{(0)}(z,\eta)$ is the subdominant term in (3.6) we concentrate our analysis on the other terms on the right-hand side of (3.6). The main step is the substitution of (2.21), with $k=j$, into the right-hand side of (3.6). We obtain

$$R_k^{(0)}(z,\eta;N_0)=\sum_{j\neq k}\frac{K_{jk}}{(2\pi\mathrm{i})^2}\sum_{s=0}^{N_j^{(1)}-1}a_{sj}\Gamma(\mu_j+1-s)$$
$$\times\int_{\lambda_k}^{[\eta]}\int_{\gamma_j(\theta_{kj})}\frac{\mathrm{e}^{zt_0}\left(\dfrac{t_0-\lambda_k}{t_1-\lambda_k}\right)^{N_0-\mu_k-1}(t_1-\lambda_j)^{s-\mu_j-1}}{t_1-t_0}\,\mathrm{d}t_1\mathrm{d}t_0+R_k^{(1)}(z,\eta), \quad (4.1)$$

where

$$R_k^{(1)}(z,\eta)=\sum_{j\neq k}\frac{K_{jk}}{(2\pi\mathrm{i})^3}$$
$$\times\int_{\lambda_k}^{[\eta]}\int_{C_{kj}^{(0)}}\int_{C_j^{(1)}}\frac{\mathrm{e}^{zt_0}\left(\dfrac{t_0-\lambda_k}{t_1-\lambda_k}\right)^{N_0-\mu_k-1}\left(\dfrac{t_1-\lambda_j}{t_2-\lambda_j}\right)^{N_j^{(1)}-\mu_j-1}y_j(t_2)}{(t_1-t_0)(t_2-t_1)}\,\mathrm{d}t_2\mathrm{d}t_1\mathrm{d}t_0$$
$$+\sum_{j\neq k}\frac{K_{jk}}{(2\pi\mathrm{i})^2}(1-\mathrm{e}^{-2\pi\mathrm{i}\mu_j})\sum_{s=0}^{N_j^{(1)}-1}a_{sj}\Gamma(\mu_j+1-s)$$
$$\times\int_{\lambda_k}^{[\eta]}\int_{R_0\mathrm{e}^{\mathrm{i}\theta_{kj}}+\lambda_k}^{[\theta_{kj}]}\frac{\mathrm{e}^{zt_0}\left(\dfrac{t_0-\lambda_k}{t_1-\lambda_k}\right)^{N_0-\mu_k-1}(t_1-\lambda_j)^{s-\mu_j-1}}{t_1-t_0}\,\mathrm{d}t_1\mathrm{d}t_0+S_k^{(0)}(z,\eta). \quad (4.2)$$

If we substitute into the double integral of (4.1) by means of the transformation

$$\tilde{t}_0 = z\left(\frac{t_0-\lambda_k}{t_1-\lambda_k}\right), \quad \tilde{t}_1 = \tilde{t}_0(t_1-\lambda_j), \tag{4.3}$$

and use Hankel's loop integral representation for the reciprocal Gamma function (see Temme 1996, § 3.2.6), we obtain

$$\begin{aligned} R_k^{(0)}(z,\eta;N_0) &= \sum_{j\neq k}\frac{K_{jk}}{2\pi \mathrm{i}}\sum_{s=0}^{N_j^{(1)}-1} a_{sj}\mathrm{e}^{\lambda_k z}z^{\mu_k+1-N_0} \\ &\quad\times\int_0^{[\pi-\theta_{kj}]}\frac{\mathrm{e}^{\lambda_{jk}\tilde{t}_0}\tilde{t}_0^{\,N_0-s+\mu_{jk}-1}}{z-\tilde{t}_0}\,\mathrm{d}\tilde{t}_0 + R_k^{(1)}(z,\eta) \\ &= \mathrm{e}^{\lambda_k z}z^{\mu_k+1-N_0}\sum_{j\neq k}\frac{K_{jk}}{2\pi\mathrm{i}}\sum_{s=0}^{N_j^{(1)}-1} a_{sj}F^{(1)}\left(z;\begin{matrix}N_0-s+\mu_{jk}\\ \lambda_{jk}\end{matrix}\right) + R_k^{(1)}(z,\eta). \end{aligned} \tag{4.4}$$

This is the desired re-expansion of $R_k^{(0)}(z,\eta;N_0)$. We now seek to optimize the new remainder term $R_k^{(1)}(z,\eta)$ by assuming that (3.2) applies and also that

$$N_j^{(1)} = \beta_j^{(1)}|z| + \gamma_j^{(1)}, \tag{4.5}$$

where $\beta_j^{(1)} \in (0,\beta_k^{(0)})$ is another constant at our disposal, and $\gamma_j^{(1)}$ is bounded. Again, we assume that (3.4) holds and that

$$N_j^{(1)} > \operatorname{Re}\mu_j, \quad N_0 - N_j^{(1)} > \operatorname{Re}\mu_{kj}+1. \tag{4.6}$$

We may then collapse the t_1 contours of integration of the triple integrals in (4.2) on to the join of λ_j and $R_0\mathrm{e}^{\mathrm{i}\theta_{kj}}+\lambda_k$. We will show that this sum with triple integrals is the dominant term in the right-hand side of (4.2). We split $C_j^{(1)}$ into $C_{j-}^{(1)}$ and $C_{jl}^{(1)}$, $l\neq j$, and use (2.8). We obtain

$$\begin{aligned} R_k^{(1)}(z,\eta) &= \sum_{j\neq k}\sum_{l\neq j}\frac{K_{jk}K_{lj}}{(2\pi\mathrm{i})^3}\int_{\lambda_k}^{[\eta]}\int_{\lambda_j}^{R_0\mathrm{e}^{\mathrm{i}\theta_{kj}}+\lambda_k}\int_{C_{jl}^{(1)}} \\ &\quad\times\frac{\mathrm{e}^{zt_0}\left(\dfrac{t_0-\lambda_k}{t_1-\lambda_k}\right)^{N_0-\mu_k-1}\left(\dfrac{t_1-\lambda_j}{t_2-\lambda_j}\right)^{N_j^{(1)}-\mu_j-1}y_l(t_2)}{(t_1-t_0)(t_2-t_1)}\,\mathrm{d}t_2\mathrm{d}t_1\mathrm{d}t_0 + S_k^{(1)}(z,\eta), \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} S_k^{(1)}(z,\eta) &= \sum_{j\neq k}\frac{K_{jk}(\mathrm{e}^{-2\pi\mathrm{i}\mu_j}-1)}{(2\pi\mathrm{i})^3}\int_{\lambda_k}^{[\eta]}\int_{\lambda_j}^{R_0\mathrm{e}^{\mathrm{i}\theta_{kj}}+\lambda_k}\int_{C_{j-}^{(1)}} \\ &\quad\times\frac{\mathrm{e}^{zt_0}\left(\dfrac{t_0-\lambda_k}{t_1-\lambda_k}\right)^{N_0-\mu_k-1}\left(\dfrac{t_1-\lambda_j}{t_2-\lambda_j}\right)^{N_j^{(1)}-\mu_j-1}y_j(t_2)}{(t_1-t_0)(t_2-t_1)}\,\mathrm{d}t_2\mathrm{d}t_1\mathrm{d}t_0 \\ &\quad+\sum_{j\neq k}\frac{K_{jk}}{(2\pi\mathrm{i})^2}(1-\mathrm{e}^{-2\pi\mathrm{i}\mu_j})\sum_{s=0}^{N_j^{(1)}-1}a_{sj}\Gamma(\mu_j+1-s)\int_{\lambda_k}^{[\eta]}\int_{R_0\mathrm{e}^{\mathrm{i}\theta_{kj}}+\lambda_k}^{[\theta_{kj}]} \end{aligned}$$

$$\times \frac{\mathrm{e}^{zt_0}\left(\dfrac{t_0-\lambda_k}{t_1-\lambda_k}\right)^{N_0-\mu_k-1}(t_1-\lambda_j)^{s-\mu_j-1}}{t_1-t_0}\,\mathrm{d}t_1\mathrm{d}t_0 + S_k^{(0)}(z,\eta). \tag{4.8}$$

In order to take $\eta = \pi - \mathrm{ph}\, z$, $z \in \overline{\mathcal{S}}(\eta)$, we have to adjust the t_0 contour of integration. If we take this contour to be $\mathcal{P}$, then the contribution of the straight lines of $\mathcal{P}$ can be estimated by $\mathrm{e}^{\lambda_k z}|z|^{\mathrm{Re}\,\mu_k+1-N_0}\Gamma(N_0-\mathrm{Re}\,\mu_k)$. And if we take $\beta_k^{(0)} \geqslant \alpha_k^{(0)}$, then the contribution of the quarter circle part of $\mathcal{P}$ is smaller than the estimate for the straight lines. To simplify the technical details in the following analysis, we omit this adjustment.

To estimate the triple integrals in (4.8) we use $(t_1-t_0)^{-1} = |z|\mathcal{O}(1)$ and $(t_2-t_1)^{-1} = \mathcal{O}(1)$. We obtain

$$\begin{aligned}
&\int_{\lambda_k}^{[\eta]}\int_{\lambda_j}^{R_0\mathrm{e}^{\mathrm{i}\theta_{kj}}+\lambda_k}\int_{C_{j-}^{(1)}} \frac{\mathrm{e}^{zt_0}\left(\dfrac{t_0-\lambda_k}{t_1-\lambda_k}\right)^{N_0-\mu_k-1}\left(\dfrac{t_1-\lambda_j}{t_2-\lambda_j}\right)^{N_j^{(1)}-\mu_j-1} y_j(t_2)}{(t_1-t_0)(t_2-t_1)}\,\mathrm{d}t_2\mathrm{d}t_1\mathrm{d}t_0\\
&\quad = \mathrm{e}^{\lambda_k z}|z|\int_0^\infty \mathrm{e}^{-|z|t_0} t_0^{N_0-\mathrm{Re}\,\mu_k-1}\,\mathrm{d}t_0\\
&\qquad \times \int_0^\infty \frac{t_1^{N_j^{(1)}-\mathrm{Re}\,\mu_j-1}}{(t_1+|\lambda_{jk}|)^{N_0-\mathrm{Re}\,\mu_k-1}}\,\mathrm{d}t_1 R_1^{-N_j^{(1)}}\mathcal{O}(1)\\
&\quad = \mathrm{e}^{\lambda_k z}|z|^{\mathrm{Re}\,\mu_k+2-N_0}|\lambda_{jk}|^{N_j^{(1)}-N_0}\\
&\qquad \times\Gamma(N_j^{(1)}-\mathrm{Re}\,\mu_j)\Gamma(N_0-N_j^{(1)}+\mathrm{Re}\,\mu_{jk}-1)R_1^{-N_j^{(1)}}\mathcal{O}(1)\\
&\quad = \mathrm{e}^{\lambda_k z-\beta_k^{(0)}|z|}\left(\frac{\beta_k^{(0)}-\beta_j^{(1)}}{|\lambda_{jk}|}\right)^{(\beta_k^{(0)}-\beta_j^{(1)})|z|}\left(\frac{\beta_j^{(1)}}{R_1}\right)^{\beta_j^{(1)}|z|}\mathcal{O}(1),
\end{aligned}\tag{4.9}$$

as $z \to \infty$ in $\overline{\mathcal{S}}(\eta)$. To estimate the sum of double integrals of (4.8) we use $0 \leqslant s < N_j^{(1)} < N_0$, and $(t_1-t_0)^{-1} = |z|\mathcal{O}(1)$. We obtain

$$\begin{aligned}
&a_{sj}r(\mu_j+1-s)\int_{\lambda_k}^{[\eta]}\int_{R_0\mathrm{e}^{\mathrm{i}\theta_{kj}}+\lambda_k}^{[\theta_{kj}]} \frac{\mathrm{e}^{zt_0}\left(\dfrac{t_0-\lambda_k}{t_1-\lambda_k}\right)^{N_0-\mu_k-1}(t_1-\lambda_j)^{s-\mu_j-1}}{t_1-t_0}\,\mathrm{d}t_1\mathrm{d}t_0\\
&\quad = \mathrm{e}^{\lambda_k z}|z|^{\mathrm{Re}\,\mu_k+1-N_0}\Gamma(N_0-\mathrm{Re}\,\mu_k)a_{sj}\Gamma(\mu_j+1-s)\\
&\qquad \times\int_{R_0}^\infty t_1^{s-N_0+\mathrm{Re}\,\mu_{kj}}\,\mathrm{d}t_1\mathcal{O}(1)\\
&\quad = \mathrm{e}^{\lambda_k z}|z|^{\mathrm{Re}\,\mu_k+1-N_0}\Gamma(N_0-\mathrm{Re}\,\mu_k)a_{sj}\Gamma(\mu_j+1-s)\\
&\qquad \times\frac{R_0^{s-N_0}}{N_0-s+\mathrm{Re}\,\mu_{jk}-1}\mathcal{O}(1)\\
&\quad = \mathrm{e}^{\lambda_k z}|z|^{\mathrm{Re}\,\mu_k-N_0}\Gamma(N_0-\mathrm{Re}\,\mu_k)R_0^{N_j^{(1)}-N_0}\\
&\qquad \times(\tfrac{1}{2}\alpha_j^{(0)})^{-N_j^{(1)}}a_{sj}\Gamma(\mu_j+1-s)(\tfrac{1}{2}\alpha_j^{(0)})^s\mathcal{O}(1),
\end{aligned}\tag{4.10}$$

as $z \to \infty$ in $\overline{\mathcal{S}}(\eta)$. From (2.7) we see that $\sum_{s=0}^\infty |a_{sj}\Gamma(\mu_j+1-s)(\frac{1}{2}\alpha_j^{(0)})^s|$ is bounded.

Hence

$$\sum_{s=0}^{N_j^{(1)}-1} a_{sj}\Gamma(\mu_j+1-s)\int_{\lambda_k}^{[\eta]}\int_{R_0 e^{i\theta_{kj}}+\lambda_k}^{[\theta_{kj}]} \frac{e^{zt_0}\left(\dfrac{t_0-\lambda_k}{t_1-\lambda_k}\right)^{N_0-\mu_k-1}(t_1-\lambda_j)^{s-\mu_j-1}}{t_1-t_0}\,dt_1 dt_0$$
$$= e^{\lambda_k z}|z|^{\operatorname{Re}\mu_k-N_0}\Gamma(N_0-\operatorname{Re}\mu_k)R_0^{N_j^{(1)}-N_0}(\tfrac{1}{2}\alpha_j^{(0)})^{-N_j^{(1)}}\mathcal{O}(1)$$
$$= e^{\lambda_k z-\beta_k^{(0)}|z|}|z|^{-(1/2)}\left(\frac{\beta_k^{(0)}}{R_0}\right)^{\beta_k^{(0)}|z|}\left(\frac{2R_0}{\alpha_j^{(0)}}\right)^{\beta_j^{(1)}|z|}\mathcal{O}(1), \tag{4.11}$$

as $z\to\infty$ in $\overline{\mathcal{S}}(\eta)$.

From (3.9), (4.9) and (4.11) we observe that for every positive constant κ, we can find a positive number $\tilde{R}_0$, such that if $R_0 \geqslant \tilde{R}_0$, then

$$S_k^{(1)}(z,\eta) = e^{\lambda_k z-\kappa|z|}\mathcal{O}(1), \tag{4.12}$$

as $z\to\infty$ in $\overline{\mathcal{S}}(\eta)$.

Finally, we estimate the triple integral in (4.7). We use $(t_1-t_0)^{-1} = |z|\mathcal{O}(1)$, $(t_2-t_1)^{-1} = |z|\mathcal{O}(1)$ and lemma 2.4. We obtain

$$\int_{\lambda_k}^{[\eta]}\int_{\lambda_j}^{R_0 e^{i\theta_{kj}}+\lambda_k}\int_{C_{jl}^{(1)}} \frac{e^{zt_0}\left(\dfrac{t_0-\lambda_k}{t_1-\lambda_k}\right)^{N_0-\mu_k-1}\left(\dfrac{t_1-\lambda_j}{t_2-\lambda_j}\right)^{N_j^{(1)}-\mu_j-1} y_l(t_2)}{(t_1-t_0)(t_2-t_1)}\,dt_2 dt_1 dt_0$$
$$= e^{\lambda_k z}|z|^2\int_0^\infty e^{-|z|t_0}t_0^{N_0-\operatorname{Re}\mu_k-1}\,dt_0$$
$$\times\int_0^\infty \frac{t_1^{N_j^{(1)}-\operatorname{Re}\mu_j-1}}{(t_1+|\lambda_{jk}|)^{N_0-\operatorname{Re}\mu_k-1}}\,dt_1|\lambda_{lj}|^{-N_j^{(1)}}(N_j^{(1)})^{\operatorname{Re}\mu_l}\mathcal{O}(1)$$
$$= e^{\lambda_k z}|z|^{\operatorname{Re}(\mu_k+\mu_l)+3-N_0}|\lambda_{jk}|^{N_j^{(1)}-N_0}|\lambda_{lj}|^{-N_j^{(1)}}$$
$$\times\Gamma(N_j^{(1)}-\operatorname{Re}\mu_j)\Gamma(N_0-N_j^{(1)}+\operatorname{Re}\mu_{jk}-1)\mathcal{O}(1)$$
$$= e^{\lambda_k z}|z|^{\operatorname{Re}\mu_l+1}\left(\frac{\beta_k^{(0)}-\beta_j^{(1)}}{|\lambda_{jk}|e}\right)^{(\beta_k^{(0)}-\beta_j^{(1)})|z|}\left(\frac{\beta_j^{(1)}}{|\lambda_{lj}|e}\right)^{\beta_j^{(1)}|z|}\mathcal{O}(1), \tag{4.13}$$

as $z\to\infty$ in $\overline{\mathcal{S}}(\eta)$.

Remark 4.1. *When λ_k, λ_j, λ_l are not collinear, or when λ_k is λ_l, then we can use $(t_2-t_1)^{-1} = \mathcal{O}(1)$. Hence, we can sharpen estimate (4.13) and all the succeeding estimates in this section by a factor $|z|^{-1}$. With a method that is similar to the analysis between (5.17) and (5.19) of Olde Daalhuis & Olver (1995a) we can sharpen these estimates by an extra factor $|z|^{-(1/2)}$.*

We combine (4.12) and (4.13) into the following result:

$$R_k^{(1)}(z,\eta) = e^{\lambda_k z}\sum_{j\neq k}\sum_{l\neq j}K_{jk}K_{lj}\left(\frac{\beta_k^{(0)}-\beta_j^{(1)}}{|\lambda_{jk}|e}\right)^{(\beta_k^{(0)}-\beta_j^{(1)})|z|}\left(\frac{\beta_j^{(1)}}{|\lambda_{lj}|e}\right)^{\beta_j^{(1)}|z|}|z|^{\operatorname{Re}\mu_l+1}\mathcal{O}(1), \tag{4.14}$$

as $z \to \infty$ in $\overline{\mathcal{S}}(\eta)$. Each term on the right-hand side of (4.14) is minimal for $\beta_k^{(0)} - \beta_j^{(1)} = |\lambda_{jk}|$ and $\beta_j^{(1)} = |\lambda_{lj}|$. Thus these minimal terms are

$$K_{jk}K_{lj}\exp[\lambda_k z - (|\lambda_{jk}| + |\lambda_{lj}|)|z|]|z|^{\operatorname{Re}\mu_l+1}\mathcal{O}(1).$$

Hence, we are interested in

$$\min\{|\lambda_{jk}| + |\lambda_{lj}| : j \neq k, l \neq j, K_{jk} \neq 0, K_{lj} \neq 0\} = \alpha_k^{(1)}, \tag{4.15}$$

which means that we have

$$R_k^{(1)}(z,\eta) = \mathrm{e}^{\lambda_k z - \alpha_k^{(1)}|z|}|z|^{\tilde{\mu}+1}\mathcal{O}(1), \tag{4.16}$$

as the final estimate of this section. We obtained (4.16) by taking $\beta_k^{(0)} = \alpha_k^{(1)}$, which fixes N_0 up to an additive term $\mathcal{O}(1)$. But we still have to find the optimal choices for $\beta_j^{(1)}$, that is, the minimal number of terms in the re-expansion (4.4), such that (4.16) holds. The optimal choice for $\beta_j^{(1)}$ follows from

$$\left(\frac{\beta_k^{(0)} - \beta_j^{(1)}}{|\lambda_{jk}|}\right)^{\beta_k^{(0)} - \beta_j^{(1)}} \left(\frac{\beta_j^{(1)}}{|\lambda_{lj}|}\right)^{\beta_j^{(1)}} \leqslant 1, \tag{4.17}$$

for all $l \neq j$, such that $K_{lj} \neq 0$. We can always solve this equation numerically. However, a reasonable choice is given by

$$\beta_j^{(1)} = \max(0, \beta_k^{(0)} - |\lambda_{jk}|). \tag{4.18}$$

5. Level two

Again, we assume that $z \in \overline{\mathcal{S}}(\eta)$. We want to re-expand $R_k^{(1)}(z,\eta)$ and we proceed in a manner analogous to the preceding section. Accordingly, we substitute into (4.7) by means of (2.21), with $k = l$, and obtain

$$\begin{aligned} R_k^{(1)}(z,\eta) = {} & \sum_{j\neq k}\sum_{l\neq j} \frac{K_{jk}K_{lj}}{(2\pi\mathrm{i})^3} \sum_{s=0}^{N_l^{(2)}-1} a_{sl}\Gamma(\mu_l+1-s) \int_{\lambda_k}^{[\eta]}\int_{\lambda_j}^{[\theta_{kj}]}\int_{\gamma_l(\theta_{jl})} \\ & \times \frac{\mathrm{e}^{zt_0}\left(\dfrac{t_0-\lambda_k}{t_1-\lambda_k}\right)^{N_0-\mu_k-1}\left(\dfrac{t_1-\lambda_j}{t_2-\lambda_j}\right)^{N_j^{(1)}-\mu_j-1}(t_2-\lambda_l)^{s-\mu_l-1}}{(t_1-t_0)(t_2-t_1)}\,\mathrm{d}t_2\mathrm{d}t_1\mathrm{d}t_0 \\ & + R_k^{(2)}(z,\eta), \end{aligned} \tag{5.1}$$

where

$$\begin{aligned} R_k^{(2)}(z,\eta) = {} & \sum_{j\neq k}\sum_{l\neq j} \frac{K_{jk}K_{lj}}{(2\pi\mathrm{i})^4} \int_{\lambda_k}^{[\eta]}\int_{\lambda_j}^{R_0\mathrm{e}^{\mathrm{i}\theta_{kj}}+\lambda_k}\int_{C_{jl}^{(1)}}\int_{C_l^{(2)}} \\ & \times \left[\mathrm{e}^{zt_0}\left(\frac{t_0-\lambda_k}{t_1-\lambda_k}\right)^{N_0-\mu_k-1}\left(\frac{t_1-\lambda_j}{t_2-\lambda_j}\right)^{N_j^{(1)}-\mu_j-1}\left(\frac{t_2-\lambda_l}{t_3-\lambda_l}\right)^{N_l^{(2)}-\mu_l-1} y_l(t_3)\right] \\ & \times \frac{\mathrm{d}t_3\mathrm{d}t_2\mathrm{d}t_1\mathrm{d}t_0}{(t_1-t_0)(t_2-t_1)(t_3-t_2)} \\ & + \sum_{j\neq k}\sum_{l\neq j} \frac{K_{jk}K_{lj}}{(2\pi\mathrm{i})^3} \sum_{s=0}^{N_l^{(2)}-1} a_{sl}\Gamma(\mu_l+1-s)\left[(1-\mathrm{e}^{-2\pi\mathrm{i}\mu_l})\int_{\lambda_k}^{[\eta]}\int_{\lambda_j}^{[\theta_{kj}]}\int_{R_1\mathrm{e}^{\mathrm{i}\theta_{jl}}+\lambda_j}^{[\theta_{jl}]}\right. \end{aligned}$$

$$\times \frac{\mathrm{e}^{zt_0}\left(\dfrac{t_0-\lambda_k}{t_1-\lambda_k}\right)^{N_0-\mu_k-1}\left(\dfrac{t_1-\lambda_j}{t_2-\lambda_j}\right)^{N_j^{(1)}-\mu_j-1}(t_2-\lambda_l)^{s-\mu_l-1}}{(t_1-t_0)(t_2-t_1)}\,\mathrm{d}t_2\mathrm{d}t_1\mathrm{d}t_0$$

$$-\int_{\lambda_k}^{[\eta]}\int_{R_0\mathrm{e}^{\mathrm{i}\theta_{kj}}+\lambda_k}^{[\theta_{kj}]}\int_{C_{jl}^{(1)}}$$

$$\left.\times \frac{\mathrm{e}^{zt_0}\left(\dfrac{t_0-\lambda_k}{t_1-\lambda_k}\right)^{N_0-\mu_k-1}\left(\dfrac{t_1-\lambda_j}{t_2-\lambda_j}\right)^{N_j^{(1)}-\mu_j-1}(t_2-\lambda_l)^{s-\mu_l-1}}{(t_1-t_0)(t_2-t_1)}\,\mathrm{d}t_2\,\mathrm{d}t_1\,\mathrm{d}t_0\right]$$

$$+S_k^{(1)}(z,\eta). \tag{5.2}$$

Again, we wish to express the re-expansion in terms of our hyperterminants. We substitute into the triple integral of (5.1) by means of the equations

$$\tilde{t}_0=z\left(\frac{t_0-\lambda_k}{t_1-\lambda_k}\right),\quad \tilde{t}_1=\tilde{t}_0\left(\frac{t_1-\lambda_j}{t_2-\lambda_j}\right),\quad \tilde{t}_2=\tilde{t}_1\,(t_2-\lambda_l)\,, \tag{5.3}$$

and obtain

$$R_k^{(1)}(z,\eta)=\mathrm{e}^{\lambda_k z}z^{\mu_k+1-N_0}\sum_{j\neq k}\sum_{l\neq j}\frac{K_{jk}K_{lj}}{(2\pi\mathrm{i})^2}$$

$$\times\sum_{s=0}^{N_l^{(2)}-1}a_{sl}F^{(2)}\left(z;\begin{matrix}N_0-N_j^{(1)}+\mu_{jk}+1, & N_j^{(1)}-s+\mu_{lj}\\ \lambda_{jk}, & \lambda_{lj}\end{matrix}\right)+R_k^{(2)}(z,\eta). \tag{5.4}$$

This is the desired re-expansion. We now seek to optimize the new remainder term by assuming that

$$N_0=\beta_k^{(0)}|z|+\gamma_k^{(0)},\quad N_j^{(1)}=\beta_j^{(1)}|z|+\gamma_j^{(1)},\quad N_l^{(2)}=\beta_l^{(2)}|z|+\gamma_l^{(2)}, \tag{5.5}$$

where $\beta_k^{(0)}$, $\beta_j^{(1)}$, $\beta_l^{(2)}$ are constants such that $0<\beta_l^{(2)}<\beta_j^{(1)}<\beta_k^{(0)}$, and $\gamma_k^{(0)}$, $\gamma_j^{(1)}$, $\gamma_l^{(2)}$ are bounded. We also assume that

$$\left.\begin{matrix}N_0>\operatorname{Re}\mu_k,\quad N_j^{(1)}>\operatorname{Re}\mu_j,\quad N_l^{(2)}>\operatorname{Re}\mu_l,\\ N_0-N_j^{(1)}>\operatorname{Re}\mu_{kj}+1,\quad N_j^{(1)}-N_l^{(2)}>\operatorname{Re}\mu_{jl}+1,\end{matrix}\right\} \tag{5.6}$$

so that we can collapse the t_2 contours of integration of the quadruple integrals in (5.2) onto the join of λ_l and $R_1\mathrm{e}^{\mathrm{i}\theta_{jl}}+\lambda_j$. We split $C_l^{(2)}$ into $C_{l-}^{(2)}$ and $C_{lq}^{(2)}$, $q\neq l$, and use (2.8). We obtain

$$R_k^{(2)}(z,\eta)=\sum_{j\neq k}\sum_{l\neq j}\sum_{q\neq l}\frac{K_{jk}K_{lj}K_{ql}}{(2\pi\mathrm{i})^4}\int_{\lambda_k}^{[\eta]}\int_{\lambda_j}^{R_0\mathrm{e}^{\mathrm{i}\theta_{kj}}+\lambda_k}\int_{\lambda_l}^{R_1e^{i\theta_{jl}}+\lambda_j}\int_{C_{lj}^{(2)}}$$

$$\times\left[\mathrm{e}^{zt_0}\left(\frac{t_0-\lambda_k}{t_1-\lambda_k}\right)^{N_0-\mu_k-1}\left(\frac{t_1-\lambda_j}{t_2-\lambda_j}\right)^{N_j^{(1)}-\mu_j-1}\left(\frac{t_2-\lambda_l}{t_3-\lambda_l}\right)^{N_l^{(2)}-\mu_l-1}\right]$$

$$\times\frac{y_q(t_3)\,\mathrm{d}t_3\mathrm{d}t_2\mathrm{d}t_1\mathrm{d}t_0}{(t_1-t_0)(t_2-t_1)(t_3-t_2)}$$

$$+S_k^{(2)}(z,\eta), \tag{5.7}$$

where

$$S_k^{(2)}(z,\eta) = \sum_{j\neq k}\sum_{l\neq j}(\mathrm{e}^{-2\pi\mathrm{i}\mu_l}-1)\frac{K_{jk}K_{lj}}{(2\pi\mathrm{i})^4}\int_{\lambda_k}^{[\eta]}\int_{\lambda_j}^{R_0\mathrm{e}^{\mathrm{i}\theta_{kj}}+\lambda_k}\int_{\lambda_l}^{R_1\mathrm{e}^{\mathrm{i}\theta_{jl}}+\lambda_j}\int_{C_{l-}^{(2)}}$$
$$\times\left[\mathrm{e}^{zt_0}\left(\frac{t_0-\lambda_k}{t_1-\lambda_k}\right)^{N_0-\mu_k-1}\left(\frac{t_1-\lambda_j}{t_2-\lambda_j}\right)^{N_j^{(1)}-\mu_j-1}\left(\frac{t_2-\lambda_l}{t_3-\lambda_l}\right)^{N_l^{(2)}-\mu_l-1}y_l(t_3)\right]$$
$$\times\frac{\mathrm{d}t_3\mathrm{d}t_2\mathrm{d}t_1\mathrm{d}t_0}{(t_1-t_0)(t_2-t_1)(t_3-t_2)}$$
$$+\sum_{j\neq k}\sum_{l\neq j}\frac{K_{jk}K_{lj}}{(2\pi\mathrm{i})^3}\sum_{s=0}^{N_l^{(2)}-1}a_{sl}\Gamma(\mu_l+1-s)\Bigg[(1-\mathrm{e}^{-2\pi\mathrm{i}\mu_l})$$
$$\times\int_{\lambda_k}^{[\eta]}\int_{\lambda_j}^{[\theta_{kj}]}\int_{R_1\mathrm{e}^{\mathrm{i}\theta_{jl}}+\lambda_j}^{[\theta_{jl}]}\mathrm{e}^{zt_0}\left(\frac{t_0-\lambda_k}{t_1-\lambda_k}\right)^{N_0-\mu_k-1}\left(\frac{t_1-\lambda_j}{t_2-\lambda_j}\right)^{N_j^{(1)}-\mu_j-1}$$
$$\times\frac{(t_2-\lambda_l)^{s-\mu_l-1}}{(t_1-t_0)(t_2-t_1)}\,\mathrm{d}t_2\mathrm{d}t_1\mathrm{d}t_0$$
$$-\int_{\lambda_k}^{[\eta]}\int_{R_0\mathrm{e}^{\mathrm{i}\theta_{kj}}+\lambda_k}^{[\theta_{kj}]}\int_{C_{jl}^{(1)}}\mathrm{e}^{zt_0}\left(\frac{t_0-\lambda_k}{t_1-\lambda_k}\right)^{N_0-\mu_k-1}\left(\frac{t_1-\lambda_j}{t_2-\lambda_j}\right)^{N_j^{(1)}-\mu_j-1}$$
$$\times\frac{(t_2-\lambda_l)^{s-\mu_l-1}}{(t_1-t_0)(t_2-t_1)}\mathrm{d}t_2\mathrm{d}t_1\mathrm{d}t_0\Bigg]$$
$$+S_k^{(1)}(z,\eta). \tag{5.8}$$

The details for obtaining estimates for the sums in (5.8) are very similar to those of the previous section. The only real difference is that we use lemma 2.3. We omit the details and give the estimates

$$S_k^{(2)}(z,\eta) = \mathrm{e}^{\lambda_k z-\beta_k^{(0)}|z|}|z|^{(3/2)}$$
$$\times\left(\frac{\beta_k^{(0)}-\beta_j^{(1)}}{|\lambda_{jk}|}\right)^{(\beta_k^{(0)}-\beta_j^{(1)})|z|}\left(\frac{\beta_j^{(1)}-\beta_l^{(2)}}{|\lambda_{lj}|}\right)^{(\beta_j^{(1)}-\beta_l^{(2)})|z|}\left(\frac{\beta_l^{(2)}}{R_2}\right)^{\beta_l^{(2)}|z|}\mathcal{O}(1)$$
$$+\mathrm{e}^{\lambda_k z-\beta_k^{(0)}|z|}\left(\frac{\beta_k^{(0)}-\beta_j^{(1)}}{|\lambda_{jk}|}\right)^{(\beta_k^{(0)}-\beta_j^{(1)})|z|}$$
$$\times(\beta_j^{(1)})^{\beta_j^{(1)}|z|}(\tfrac{1}{2}\alpha_l^{(0)})^{-\beta_l^{(2)}|z|}(R_1)^{(\beta_l^{(2)}-\beta_j^{(1)})|z|}\mathcal{O}(1)$$
$$+\mathrm{e}^{\lambda_k z-\beta_k^{(0)}|z|}|z|^{\mathrm{Re}\,\mu_l+(1/2)}(R_0)^{(\beta_j^{(1)}-\beta_k^{(0)})|z|}(\tfrac{1}{2}\alpha_l^{(0)})^{-\beta_j^{(1)}|z|}\mathcal{O}(1)+S_k^{(1)}(z,\eta), \tag{5.9}$$

as $z\to\infty$ in $\overline{\mathcal{S}}(\eta)$. Again, for each positive constant κ, we can find a positive number $\tilde{R}_0$, such that for $R_0\geqslant\tilde{R}_0$

$$S_k^{(2)}(z,\eta)=\mathrm{e}^{\lambda_k z-\kappa|z|}\mathcal{O}(1), \tag{5.10}$$

as $z\to\infty$ in $\overline{\mathcal{S}}(\eta)$.

Finally, we estimate the quadruple integrals in (5.7). We use the equations $(t_1-t_0)^{-1}=|z|\mathcal{O}(1)$, $(t_2-t_1)^{-1}=|z|\mathcal{O}(1)$, $(t_3-t_2)^{-1}=|z|\mathcal{O}(1)$ and lemma 2.4.

We obtain

$$\begin{aligned}
&\int_{\lambda_k}^{[\eta]}\int_{\lambda_j}^{R_0 \mathrm{e}^{\mathrm{i}\theta_{kj}}+\lambda_k}\int_{\lambda_l}^{R_1 \mathrm{e}^{\mathrm{i}\theta_{jl}}+\lambda_j}\int_{C_{lj}^{(2)}} \\
&\quad\times \mathrm{e}^{zt_0}\left(\frac{t_0-\lambda_k}{t_1-\lambda_k}\right)^{N_0-\mu_k-1}\left(\frac{t_1-\lambda_j}{t_2-\lambda_j}\right)^{N_j^{(1)}-\mu_j-1}\left(\frac{t_2-\lambda_l}{t_3-\lambda_l}\right)^{N_l^{(2)}-\mu_l-1} y_q(t_3) \\
&\quad\times\frac{\mathrm{d}t_3\mathrm{d}t_2\mathrm{d}t_1\mathrm{d}t_0}{(t_1-t_0)(t_2-t_1)(t_3-t_2)} \\
&= \mathrm{e}^{\lambda_k z}|z|^3\int_0^\infty \mathrm{e}^{-|z|t_0} t_0^{N_0-\operatorname{Re}\mu_k-1}\,\mathrm{d}t_0\int_0^\infty \frac{t_1^{N_j^{(1)}-\operatorname{Re}\mu_j-1}}{(t_1+|\lambda_{jk}|)^{N_0-\operatorname{Re}\mu_k-1}}\,\mathrm{d}t_1 \\
&\quad\times\int_0^\infty \frac{t_2^{N_l^{(2)}-\operatorname{Re}\mu_l-1}}{(t_2+|\lambda_{lj}|)^{N_j^{(1)}-\operatorname{Re}\mu_j-1}}\,\mathrm{d}t_2|\lambda_{ql}|^{-N_l^{(2)}}(N_l^{(2)})^{\operatorname{Re}\mu_q}\mathcal{O}(1) \\
&= \mathrm{e}^{\lambda_k z}|z|^{\operatorname{Re}(\mu_k+\mu_q)+3-N_0}|\lambda_{jk}|^{N_j^{(1)}-N_0}|\lambda_{lj}|^{N_l^{(2)}-N_j^{(1)}}|\lambda_{ql}|^{-N_l^{(2)}}\Gamma(N_0-\operatorname{Re}\mu_k) \\
&\quad\times\frac{\Gamma(N_j^{(1)}-\operatorname{Re}\mu_j)\Gamma(N_0-N_j^{(1)}+\operatorname{Re}\mu_{jk}-1)}{\Gamma(N_0-\operatorname{Re}\mu_k-1)} \\
&\quad\times\frac{\Gamma(N_l^{(2)}-\operatorname{Re}\mu_l)\Gamma(N_j^{(1)}-N_l^{(2)}+\operatorname{Re}\mu_{lj}-1)}{\Gamma(N_j^{(1)}-\operatorname{Re}\mu_j-1)}\mathcal{O}(1) \\
&= \mathrm{e}^{\lambda_k z}|z|^{\operatorname{Re}\mu_q+(3/2)}\left(\frac{\beta_k^{(0)}-\beta_j^{(1)}}{|\lambda_{jk}|\mathrm{e}}\right)^{(\beta_k^{(0)}-\beta_j^{(1)})|z|} \\
&\quad\times\left(\frac{\beta_j^{(1)}-\beta_l^{(2)}}{|\lambda_{lj}|\mathrm{e}}\right)^{(\beta_j^{(1)}-\beta_l^{(2)})|z|}\left(\frac{\beta_l^{(2)}}{|\lambda_{ql}|\mathrm{e}}\right)^{\beta_l^{(2)}|z|}\mathcal{O}(1),
\end{aligned} \tag{5.11}$$

as $z\to\infty$ in $\overline{\mathcal{S}}(\eta)$.

Remark 5.1. *When λ_k, λ_j, λ_l are not collinear, or when λ_k is λ_l, then we can use $(t_2-t_1)^{-1}=\mathcal{O}(1)$. Also, when λ_j, λ_l, λ_q are not collinear, or when λ_j is λ_q, then we can use $(t_3-t_2)^{-1}=\mathcal{O}(1)$. Hence, in the case that none of the sets $\{\lambda_{l_1},\lambda_{l_2},\lambda_{l_3}|\lambda_{l_1}\neq\lambda_{l_2}\neq\lambda_{l_3}\neq\lambda_{l_1}\}$ is collinear, we can sharpen the estimate (5.11) and all succeeding estimates in this section by a factor $|z|^{-2}$. And an extra factor $|z|^{-(1/2)}$ can be obtained with the method explained in Olde Daalhuis & Olver (1995a, § 4 (Remark)).*

We now combine (5.10) and (5.11) into the following result:

$$\begin{aligned}
R_k^{(2)}(z,\eta) &= \mathrm{e}^{\lambda_k z}\sum_{j\neq k}\sum_{l\neq j}\sum_{q\neq l}K_{jk}K_{lj}K_{ql}|z|^{\operatorname{Re}\mu_q+(3/2)} \\
&\quad\times\left(\frac{\beta_k^{(0)}-\beta_j^{(1)}}{|\lambda_{jk}|\mathrm{e}}\right)^{(\beta_k^{(0)}-\beta_j^{(1)})|z|}\left(\frac{\beta_j^{(1)}-\beta_l^{(2)}}{|\lambda_{lj}|\mathrm{e}}\right)^{(\beta_j^{(1)}-\beta_l^{(2)})|z|}\left(\frac{\beta_l^{(2)}}{|\lambda_{ql}|\mathrm{e}}\right)^{\beta_l^{(2)}|z|}\mathcal{O}(1),
\end{aligned} \tag{5.12}$$

as $z\to\infty$ in $\overline{\mathcal{S}}(\eta)$. Each term on the right-hand side of (5.12) is minimal for $\beta_k^{(0)}-\beta_j^{(1)}=|\lambda_{jk}|$, $\beta_j^{(1)}-\beta_l^{(2)}=|\lambda_{lj}|$ and $\beta_l^{(2)}=|\lambda_{ql}|$. Again, we see that the best choice for $\beta_k^{(0)} = \alpha_k^{(2)}$. Reasonable choices for the number of terms in the re-expansions are given by

$$\beta_j^{(1)} = \max(0, \beta_k^{(0)} - |\lambda_{jk}|), \quad \beta_l^{(2)} = \max(0, \beta_j^{(1)} - |\lambda_{lj}|). \tag{5.13}$$

With these choices, we obtain the final estimate for the remainder at level two, i.e.

$$R_k^{(2)}(z,\eta) = \mathrm{e}^{\lambda_k z - \alpha_k^{(2)}|z|}|z|^{\tilde{\mu}+(3/2)}\mathcal{O}(1), \tag{5.14}$$

as $z \to \infty$ in $\overline{\mathcal{S}}(\eta)$.

6. Main theorems

The pattern of resubstitution of (2.21) into successive remainder terms is now clear. The general results can be written down by inspection and verified by induction.

Theorem 6.1. *Let l be an arbitrary non-negative integer and $N_k^{(0)}, N_{k_1}^{(1)}, \ldots, N_{k_l}^{(l)}$ be integers such that*

$$N_k^{(0)} = \beta_k^{(0)}|z| + \gamma_k^{(0)}, \quad N_{k_j}^{(j)} = \beta_{k_j}^{(j)}|z| + \gamma_{k_j}^{(j)}, \quad j = 1, 2, \ldots, l, \tag{6.1}$$

in which the βs are constants that satisfy

$$0 < \beta_{k_l}^{(l)} < \beta_{k_{l-1}}^{(l-1)} < \cdots < \beta_{k_1}^{(1)} < \beta_k^{(0)}, \tag{6.2}$$

and the γs are bounded as $|z| \to \infty$. Then

$$\begin{aligned} w_k(z,\eta) = \mathrm{e}^{\lambda_k z} z^{\mu_k} \Bigg(& \sum_{s=0}^{N_k^{(0)}-1} a_{sk} z^{-s} \\ & + z^{1-N_k^{(0)}} \Bigg[\sum_{k_1 \neq k} \frac{K_{k_1 k}}{2\pi\mathrm{i}} \Bigg\{ \sum_{s=0}^{N_{k_1}^{(1)}-1} a_{sk_1} F^{(1)}\left(z; \begin{matrix} N_k^{(0)} - s + \mu_{k_1 k} \\ \lambda_{k_1 k} \end{matrix}\right) \\ & + \sum_{k_2 \neq k_1} \frac{K_{k_2 k_1}}{2\pi\mathrm{i}} \Bigg\{ \sum_{s=0}^{N_{k_2}^{(2)}-1} a_{sk_2} F^{(2)}\left(z; \begin{matrix} N_k^{(0)} - N_{k_1}^{(1)} + \mu_{k_1 k} + 1, & N_{k_1}^{(1)} - s + \mu_{k_2 k_1} \\ \lambda_{k_1 k}, & \lambda_{k_2 k_1} \end{matrix}\right) \\ & \ddots \\ & + \sum_{k_l \neq k_{l-1}} \frac{K_{k_l k_{l-1}}}{2\pi\mathrm{i}} \Bigg\{ \sum_{s=0}^{N_{k_l}^{(l)}-1} a_{sk_l} F^{(l)}\Bigg(z; \begin{matrix} N_k^{(0)} - N_{k_1}^{(1)} + \mu_{k_1 k} + 1, \cdots, \\ \lambda_{k_1 k}, \quad \cdots, \end{matrix} \\ & \qquad \begin{matrix} N_{k_{l-2}}^{(l-2)} - N_{k_{l-1}}^{(l-1)} + \mu_{k_{l-1} k_{l-2}} + 1, & N_{k_{l-1}}^{(l-1)} - s + \mu_{k_l k_{l-1}} \\ \lambda_{k_{l-1} k_{l-2}}, & \lambda_{k_l k_{l-1}} \end{matrix} \Bigg) \\ & \Bigg\} \cdots \Bigg\} \Bigg\} \Bigg] \Bigg) + R_k^{(l)}(z,\eta), \end{aligned} \tag{6.3}$$

and as $z \to \infty$ in $\overline{\mathcal{S}}(\eta)$,

$$\begin{aligned} R_k^{(l)}(z,\eta) = \mathrm{e}^{\lambda_k z} & \sum_{k_1 \neq k} \cdots \sum_{k_{l+1} \neq k_l} K_{k_1 k} \cdots K_{k_{l+1} k_l} |z|^{\mathrm{Re}\,\mu_{k_{l+1}} + ((l+1)/2)} \\ & \times \left(\frac{\beta_k^{(0)} - \beta_{k_1}^{(1)}}{|\lambda_{k_1 k}|\mathrm{e}} \right)^{(\beta_k^{(0)} - \beta_{k_1}^{(1)})|z|} \cdots \left(\frac{\beta_{k_{l-1}}^{(l-1)} - \beta_{k_l}^{(l)}}{|\lambda_{k_l k_{l-1}}|\mathrm{e}} \right)^{(\beta_{k_{l-1}}^{(l-1)} - \beta_{k_l}^{(l)})|z|} \left(\frac{\beta_{k_l}^{(l)}}{|\lambda_{k_{l+1} k_l}|\mathrm{e}} \right)^{\beta_{k_l}^{(l)}|z|} \mathcal{O}(1). \end{aligned} \tag{6.4}$$

In theorem 6.1 the remainder terms $R_k^{(l)}(z,\eta)$ are not optimized. Optimization yields are as follows.

Theorem 6.2. *Let*

$$\beta_k^{(0)}=\alpha_k^{(l)},\quad \beta_{k_1}^{(1)}=\max(0,\beta_k^{(0)}-|\lambda_{k_1k}|),\ldots,\quad \beta_{k_l}^{(l)}=\max(0,\beta_{k_{l-1}}^{(l-1)}-|\lambda_{k_lk_{l-1}}|), \tag{6.5}$$

where $\alpha_k^{(l)}$ is defined by (2.14). Then, as $z\to\infty$ in $\overline{\mathcal{S}}(\eta)$,

$$R_k^{(l)}(z,\eta)=\mathrm{e}^{\lambda_k z-\alpha_k^{(l)}|z|}|z|^{\tilde\mu+((l+1)/2)}\mathcal{O}(1). \tag{6.6}$$

If none of the sets $\{\lambda_{l_1},\lambda_{l_2},\lambda_{l_3}|\lambda_{l_1}\neq\lambda_{l_2}\neq\lambda_{l_3}\neq\lambda_{l_1}\}$ are collinear, then as $z\to\infty$ in $\overline{\mathcal{S}}(\eta)$,

$$R_k^{(l)}(z,\eta)=\mathrm{e}^{\lambda_k z-\alpha_k^{(l)}|z|}|z|^{\tilde\mu-(1/2)l}\mathcal{O}(1). \tag{6.7}$$

Remark 6.3. *In deriving these two theorems, we assumed that all the μ_j are non-integers. If this is not the case, then we can multiply all the solutions of (1.1) by z^α, where α is chosen so that all the $\mu_j+\alpha$ are non-integers. We observe that this multiplication does not affect the μ_{jk}. Hence, if we multiply the hyperasymptotic expansions of $\tilde w(z)=z^\alpha w(z)$ by $z^{-\alpha}$, then we obtain the hyperasymptotic expansions of $w(z)$. In this way, we remove the restriction on the μ_j.*

7. Computation of the Stokes multipliers

In order to make theorem 6.1 usable in numerical computation for the solutions of (1.1) we need to be able to compute the Stokes multipliers K_{jk}. In this section we give a method to compute these multipliers to sufficient precision.

To analyse what precision we require in the Stokes multiplier $K_{k_mk_{m-1}}$, write (6.3) as

$$w_k(z,\eta)=P_m+\sum_{k_m\neq k_{m-1}}K_{k_mk_{m-1}}Q_{k_m}+R_k^{(l)}(z,\eta), \tag{7.1}$$

and take

$$\left.\begin{aligned}\beta_k^{(0)}=\alpha_k^{(l)},\quad \beta_{k_1}^{(1)}=\max(0,\beta_k^{(0)}-|\lambda_{k_1k}|),\quad\cdots,\\ \beta_{k_{m-1}}^{(m-1)}=\max(0,\beta_{k_{m-2}}^{(m-2)}-|\lambda_{k_{m-1}k_{m-2}}|),\quad \beta_{k_m}^{(m)}=\cdots=\beta_{k_l}^{(l)}=0.\end{aligned}\right\} \tag{7.2}$$

Then from (6.4) we have

$$Q_{k_m}=\mathrm{e}^{\lambda_k z-\alpha_k^{(l)}|z|}|z|^{\tilde\mu+((l+1)/2)}\left(\frac{\beta_{k_{m-1}}^{(m-1)}}{|\lambda_{k_mk_{m-1}}|}\right)^{\beta_{k_{m-1}}^{(m-1)}|z|}\mathcal{O}(1). \tag{7.3}$$

Comparing this estimate with (6.6), we see that in the computation of $K_{k_mk_{m-1}}$ we can permit a relative error

$$\mathcal{O}(\beta_{k_{m-1}}^{(m-1)}/|\lambda_{k_mk_{m-1}}|)^{-\beta_{k_{m-1}}^{(m-1)}|z|}.$$

Notice that if $|\lambda_{k_mk_{m-1}}|\geqslant\beta_{k_{m-1}}^{(m-1)}$, then Q_{k_m} can be absorbed into $R_k^{(l)}(z,\eta)$, which means that this branch of the hyperasymptotic expansion does not contribute to the level l hyperasymptotic expansion. This is in agreement with our choice $\beta_{k_m}^{(m)}=\max(0,\beta_{k_{m-1}}^{(m-1)}-|\lambda_{k_mk_{m-1}}|)$ in (6.5).

From (6.3) and (2.23) we obtain

$$
\begin{aligned}
a_{N_k^{(0)}k} &= \mathrm{e}^{-\lambda_k z} z^{N_k^{(0)}-\mu_k}(R^{(0)}(z,\eta;N_k^{(0)}) - R^{(0)}(z,\eta;N_k^{(0)}+1)) \\
&= -\sum_{k_1\neq k} \frac{K_{k_1k}}{2\pi\mathrm{i}}\bigg\{\sum_{s=0}^{N_{k_1}^{(1)}-1} a_{sk_1}F^{(1)}\left(0;\begin{matrix}(N_k^{(0)}+1)-s+\mu_{k_1k}\\ \lambda_{k_1k}\end{matrix}\right) + \sum_{k_2\neq k_1}\frac{K_{k_2k_1}}{2\pi\mathrm{i}} \\
&\quad\times\bigg\{\sum_{s=0}^{N_{k_2}^{(2)}-1} a_{sk_2}F^{(2)}\left(0;\begin{matrix}(N_k^{(0)}+1)-N_{k_1}^{(1)}+\mu_{k_1k}+1, & N_{k_1}^{(1)}-s+\mu_{k_2k_1}\\ \lambda_{k_1k}, & \lambda_{k_2k_1}\end{matrix}\right) \\
&\quad\ddots \\
&\quad+\sum_{k_l\neq k_{l-1}}\frac{K_{k_lk_{l-1}}}{2\pi\mathrm{i}}\bigg\{\sum_{s=0}^{N_{k_l}^{(l)}-1} a_{sk_l}F^{(l)}\bigg(0;\begin{matrix}(N_k^{(0)}+1)-N_{k_1}^{(1)}+\mu_{k_1k}+1, & \cdots,\\ \lambda_{k_1k}, & \cdots,\end{matrix} \\
&\qquad\qquad\begin{matrix}N_{k_{l-2}}^{(l-2)}-N_{k_{l-1}}^{(l-1)}+\mu_{k_{l-1}k_{l-2}}+1, & N_{k_{l-1}}^{(l-1)}-s+\mu_{k_lk_{l-1}}\\ \lambda_{k_{l-1}k_{l-2}}, & \lambda_{k_lk_{l-1}}\end{matrix}\bigg) \\
&\quad\bigg\}\cdots\bigg\}\bigg\} + r_k^{(l)}(N_k^{(0)}).
\end{aligned}
\tag{7.4}
$$

This may be regarded as a hyperasymptotic expansion for the late coefficients as $N_k^{(0)}\to\infty$. In order to use this result for the computation of the Stokes multipliers in (6.3), we suppose that the numbers of coefficients are given by (6.1), and that (6.2) holds. Thus, instead of $N_k^{(0)}$, we take $|z|$ as the large parameter.

It follows from the previous section that if (6.5) holds, then

$$
r_k^{(l)}(N_k^{(0)}) = \mathrm{e}^{-\alpha_k^{(l)}|z|}|z|^{N_k^{(0)}-\mu_k+\tilde{\mu}+((l+1)/2)}\mathcal{O}(1), \tag{7.5}
$$

as $|z|\to\infty$.

The expansion (7.4) is of the form

$$
a_{N_k^{(0)}k} = \tilde{P}_m + \sum_{k_m\neq k_{m-1}} K_{k_mk_{m-1}}\tilde{Q}_{k_m} + r_k^{(l)}(N_k^{(0)}). \tag{7.6}
$$

Again, we can show that

$$
r_k^{(l)}(N_k^{(0)})/\tilde{Q}_{k_m} = \mathcal{O}(\beta_{k_{m-1}}^{(m-1)}/|\lambda_{k_mk_{m-1}}|)^{-\beta_{k_{m-1}}^{(m-1)}|z|},
$$

which means that we can use (7.4) to compute $K_{k_mk_{m-1}}$ to the required precision.

First we analyse the case $l=1$. Then (7.4) reads

$$
\begin{aligned}
a_{N_k^{(0)}k} = \sum_{k_1\neq k}\frac{K_{k_1k}}{2\pi\mathrm{i}}\sum_{s=0}^{N_{k_1}^{(1)}-1} a_{sk_1}(-1)^{N_k^{(0)}-s}\mathrm{e}^{\mu_{k_1k}\pi\mathrm{i}}\lambda_{k_1k}^{s-N_k^{(0)}-\mu_{k_1k}}\Gamma(N_k^{(0)}-s+\mu_{k_1k}) \\
+r_k^{(1)}(N_k^{(0)}).
\end{aligned}
\tag{7.7}
$$

This is an optimally truncated (Darboux) asymptotic expansion for the late coefficients. In the notation of (7.6) we have $\tilde{P}_1=0$. In (7.7) the factor of K_{k_1k} contributes a relative error $\mathcal{O}(\beta_k^{(0)}/|\lambda_{k_1k}|)^{-\beta_k^{(0)}|z|}$. We neglect the terms for which $|\lambda_{k_1k}|\geqslant\beta_k^{(0)}$, and we assume that there are n_1 terms such that $|\lambda_{k_1k}|<\beta_k^{(0)}$. On replacing $N_k^{(0)}$ by

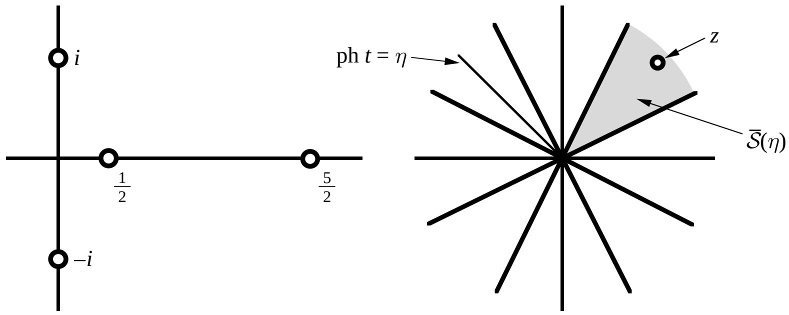

Figure 8. The admissible directions.

$N_k^{(0)}-1,\ N_k^{(0)}-2,\dots,N_k^{(0)}-n_1$, in turn, and ignoring the error terms, we arrive at a system of n_1 linear equations. In this way we can compute the K_{k_1k} to the required precision.

We now analyse the cases $l=2,3,\dots$. Let $m\in(1,\dots,l)$. We assume that we already know all the required Stokes multipliers of the form $K_{k_{\tilde m}k_{\tilde m-1}}$, $m<\tilde m\leqslant l$, to the required precision. This means that if we write the level $(l-m+1)$ hyperasymptotic expansion of

$$a_{N^{(m-1)}_{k_{m-1}}k_{m-1}}$$

as

$$a_{N^{(m-1)}_{k_{m-1}}k_{m-1}}=\sum_{k_m\neq k_{m-1}}K_{k_mk_{m-1}}\check Q_{k_m}+r^{(l-m+1)}_{k_{m-1}}(N^{(m-1)}_{k_{m-1}}) \tag{7.8}$$

(cf. (7.4) and (7.6)), then we can compute $\check Q_{k_m}$ to the required precision. Again, we neglect the terms for which

$$|\lambda_{k_mk_{m-1}}|\geqslant\beta^{(m-1)}_{k_{m-1}},$$

and we assume that there are n_m terms such that

$$|\lambda_{k_mk_{m-1}}|<\beta^{(m-1)}_{k_{m-1}}.$$

On replacing $N^{(m-1)}_{k_{m-1}}$ by $N^{(m-1)}_{k_{m-1}}-1,\ N^{(m-1)}_{k_{m-1}}-2,\dots,N^{(m-1)}_{k_{m-1}}-n_m$, in turn, and ignoring the error terms, we arrive at a system of n_m linear algebraic equations for the $K_{k_mk_{m-1}}$. In this way we are able to compute all the Stokes multipliers in the lth level hyperasymptotic expansion to the required precision.

The reason that we choose this method to compute the Stokes multipliers is so that we can use all the coefficients a_{sj} that we need in the lth level hyperasymptotic expansion.

8. An example

We use the example

$$w^{(4)}(z)-3w^{(3)}(z)+(\tfrac94+\tfrac12z^{-2})w^{(2)}(z)-(3+\tfrac34z^{-2})w'(z)+(\tfrac54+\tfrac{9}{16}z^{-2})w(z)=0. \tag{8.1}$$

In this case we have $n=4$ and

$$\lambda_1=\tfrac12,\quad \lambda_2=\tfrac52,\quad \lambda_3=\mathrm{i},\quad \lambda_4=-\mathrm{i},\quad \mu_j=0,\quad j=1,\dots,4. \tag{8.2}$$

We take $\eta=\frac34\pi$ and assume that we wish to compute $w_3(z,\eta)$ at $z=10\mathrm{e}^{(1/4)\pi\mathrm{i}}$.

Since,

$$\lambda_{13}=\tfrac{1}{2}-\mathrm{i},\quad \lambda_{23}=\tfrac{5}{2}-\mathrm{i},\quad \lambda_{43}=-2\mathrm{i}, \tag{8.3}$$

we obtain from (2.14) $\alpha_3^{(0)}=|\lambda_{13}|=\frac{1}{2}\sqrt{5}=1.118\ldots$. Hence, the optimal number of terms in the original Poincaré expansion is $N_3^{(0)}=11$.

In the level-one expansion we have $\beta_3^{(0)}=\alpha_3^{(1)}=|\lambda_{13}|+|\lambda_{31}|=|\lambda_{13}|+|\lambda_{41}|=\sqrt{5}=2.236\ldots$. Hence,

$$\beta_3^{(0)}=\sqrt{5}=2.236\ldots,\qquad \begin{cases}\beta_1^{(1)}=\max(0,\beta_3^{(0)}-|\lambda_{13}|)=\frac{1}{2}\sqrt{5}=1.118\ldots,\\ \beta_2^{(1)}=\max(0,\beta_3^{(0)}-|\lambda_{23}|)=0,\\ \beta_4^{(1)}=\max(0,\beta_3^{(0)}-|\lambda_{43}|)=\sqrt{5}-2=0.236\ldots.\end{cases} \tag{8.4}$$

Thus the optimal numbers of terms at the level-one hyperasymptotic expansion are $N_3^{(0)}=22$, $N_1^{(1)}=11$, $N_2^{(1)}=0$ and $N_4^{(1)}=2$.

To compute the Stokes multipliers K_{13} and K_{43} to the required precision, we first compute

$$a_{N_3^{(0)}-1,3}=(-2.6246115745148737538\cdots+\mathrm{i}7.7381487701887858960\cdots)\times10^{16},$$
$$a_{N_3^{(0)}-2,3}=(-3.2145346630254340602\cdots-\mathrm{i}3.2463846167062710532\cdots)\times10^{15}.$$

On replacing $N_k^{(0)}$ by $N_3^{(0)}-1$, and $N_3^{(0)}-2$ in turn in (7.7), and ignoring the error terms, we arrive at a system of two linear equations. On solving these equations we find that

$$\left.\begin{aligned}K_{13}&=-1.3167355550447009754+1.7502706941028591333\mathrm{i},\\ K_{43}&=0.3443668989089293253-2.1203898698716294832\mathrm{i}.\end{aligned}\right\} \tag{8.5}$$

Further on we shall see that in these approximations the value of K_{13} is correct to eight decimal places and that of K_{43} is correct to two decimal places. The required precision for K_{13} is approximately $(\beta_3^{(0)}/|\lambda_{13}|)^{-\beta_3^{(0)}|z|}=1.9\times10^{-7}$, and the required precision for K_{43} is approximately, $(\beta_3^{(0)}/|\lambda_{43}|)^{-\beta_3^{(0)}|z|}=8.3\times10^{-2}$.

In the level-two expansion we have $\beta_3^{(0)}=\alpha_3^{(2)}=\frac{3}{2}\sqrt{5}$. Hence,

$$\beta_3^{(0)}=3.354\ldots,\qquad \begin{cases}\beta_1^{(1)}=2.236\ldots, & \begin{cases}\beta_2^{(2)}=0.236\ldots,\\ \beta_3^{(2)}=1.118\ldots,\\ \beta_4^{(2)}=1.118\ldots,\end{cases}\\ \beta_2^{(1)}=0.661\ldots, & \beta_j^{(2)}=0,\quad j=1,3,4,\\ \beta_4^{(1)}=1.354\ldots, & \begin{cases}\beta_1^{(2)}=0.236\ldots,\\ \beta_j^{(2)}=0,\quad j=2,3.\end{cases}\end{cases} \tag{8.6}$$

Thus we take

$$N_3^{(0)}=33,\qquad \begin{cases}N_1^{(1)}=22, & \begin{cases}N_2^{(2)}=2,\\ N_3^{(2)}=11,\\ N_4^{(2)}=11,\end{cases}\\ N_2^{(1)}=6, & N_j^{(2)}=0,\quad j=1,3,4,\\ N_4^{(1)}=13, & \begin{cases}N_1^{(2)}=2,\\ N_j^{(2)}=0,\quad j=2,3.\end{cases}\end{cases} \tag{8.7}$$

Hence the level-two hyperasymptotic expansion of $w_3(z,\eta)$, with $z=10\mathrm{e}^{(1/4)\pi\mathrm{i}}$, is of the form

$$\begin{aligned}
\mathrm{e}^{-\mathrm{i}z}w_3(z,\eta) &= \sum_{s=0}^{32}\frac{a_{s3}}{z^s}+z^{-32}\frac{K_{13}}{2\pi\mathrm{i}}\sum_{s=0}^{21}a_{s1}F^{(1)}\left(z;\begin{matrix}33-s\\ \frac{1}{2}-\mathrm{i}\end{matrix}\right)\\
&+z^{-32}\frac{K_{23}}{2\pi\mathrm{i}}\sum_{s=0}^{5}a_{s2}F^{(1)}\left(z;\begin{matrix}33-s\\ \frac{5}{2}-\mathrm{i}\end{matrix}\right)\\
&+z^{-32}\frac{K_{43}}{2\pi\mathrm{i}}\sum_{s=0}^{12}a_{s4}F^{(1)}\left(z;\begin{matrix}33-s\\ -2\mathrm{i}\end{matrix}\right)+z^{-32}\frac{K_{13}}{2\pi\mathrm{i}}\left[\frac{K_{21}}{2\pi\mathrm{i}}\sum_{s=0}^{1}a_{s2}F^{(2)}\left(z;\begin{matrix}12, & 22-s\\ \frac{1}{2}-\mathrm{i}, & 2\end{matrix}\right)\right.\\
&\left.+\frac{K_{31}}{2\pi\mathrm{i}}\sum_{s=0}^{10}a_{s3}F^{(2)}\left(z;\begin{matrix}12, & 22-s\\ \frac{1}{2}-\mathrm{i}, & \mathrm{i}-\frac{1}{2}\end{matrix}\right)+\frac{K_{41}}{2\pi\mathrm{i}}\sum_{s=0}^{10}a_{s4}F^{(2)}\left(z;\begin{matrix}12, & 22-s\\ \frac{1}{2}-\mathrm{i}, & -\mathrm{i}-\frac{1}{2}\end{matrix}\right)\right]\\
&+z^{-32}\frac{K_{43}}{2\pi\mathrm{i}}\frac{K_{14}}{2\pi\mathrm{i}}\sum_{s=0}^{1}a_{s1}F^{(2)}\left(z;\begin{matrix}21, & 13-s\\ -2\mathrm{i}, & \mathrm{i}+\frac{1}{2}\end{matrix}\right)+R_3^{(2)}(z,\eta).
\end{aligned}\tag{8.8}$$

To compute the Stokes multipliers K_{21}, K_{31} and K_{41} we use the level-one hyperasymptotic expansions of $a_{N_1^{(1)}-1,1}$, $a_{N_1^{(1)}-2,1}$ and $a_{N_1^{(1)}-3,1}$. We obtain

$$\left.\begin{aligned}
K_{21} &= 0.32220037911218913862\mathrm{i},\\
K_{31} &= -0.33518471810856053233-0.17394369472610190908\mathrm{i},\\
K_{41} &= 0.33518471810856053233-0.17394369472610190908\mathrm{i}.
\end{aligned}\right\}\tag{8.9}$$

The level-one hyperasymptotic expansion of $a_{N_4^{(1)}-1,4}$ yields

$$K_{14}=1.3175812208411643253+1.7492444366777255110\mathrm{i}.\tag{8.10}$$

If we use (8.9) and (8.10) in the level-two hyperasymptotic expansions of $a_{N_3^{(0)}-1,3}$, $a_{N_3^{(0)}-2,3}$ and $a_{N_3^{(0)}-3,3}$, then we obtain

$$K_{13}=-1.3167355300409799821+1.7502707419178753228\mathrm{i},\tag{8.11}$$

$$K_{23}=-0.9600011769004704206-0.3257656311181001083\mathrm{i},\tag{8.12}$$

$$K_{43}=0.3553405998176582756-2.1172377431478990710\mathrm{i}.\tag{8.13}$$

To compute the level-two hyperasymptotic expansion of $w_3(z,\eta)$, we use these values for the Stokes multipliers in (8.8) and the methods described in Olde Daalhuis (1996, 1998) to compute the hyperterminants. To compute the 'exact' value of $w_3(z,\eta)$ at $z=10\mathrm{e}^{(1/2)\pi\mathrm{i}}$, we first use 47 terms of the asymptotic expansion of $w_3(z,\eta)$ to compute this function at the point $z+35\mathrm{i}$, and then use numerical integration of the differential equation (8.1) from $z+35\mathrm{i}$ to z. For more details on the stability of the numerical integration process see Olde Daalhuis & Olver (1998). The numerical results are given in table 1.

9. Integrals with saddles

In this section we show that theorems 6.1 and 6.2 also apply to integrals of the form

$$\int_C \mathrm{e}^{zf(t)}g(t)\,\mathrm{d}t.\tag{9.1}$$

Table 1. *Hyperasymptotic approximations to* $w_3(z,\eta)$ *for* $z = 10\mathrm{e}^{(1/4)\pi\mathrm{i}}$

level	approximation	|relative error|
0	0.0005987044270232453129 +0.0005928591225176298020i	1.9×10^{-6}
1	0.0005987042669559669510 +0.0005928607690584927942i	1.7×10^{-13}
2	0.0005987042669561103339 +0.0005928607690584721473i	1.9×10^{-19}
exact	0.0005987042669561103337 +0.0005928607690584721474i	0

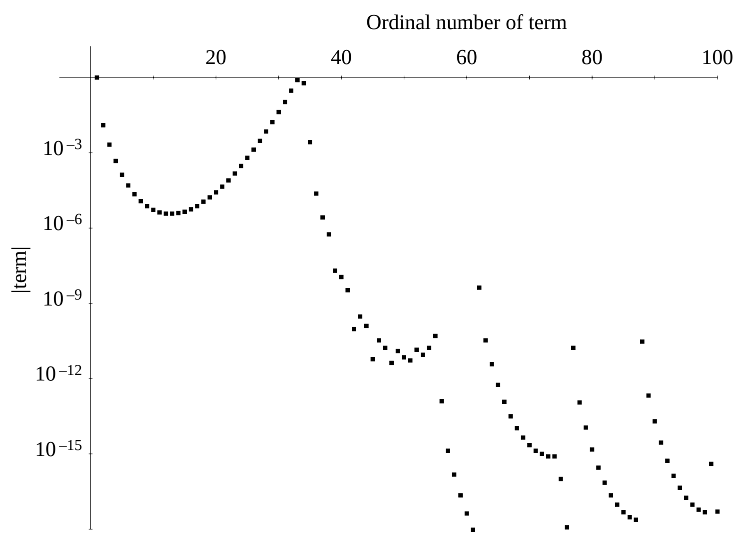

Figure 9. The terms in (8.8), the level two hyperasymptotic expansion of $w_3(z,\eta)$.

We assume that $f(t)$ has n simple saddle points $t_1,\dots,t_n$, and that $f(t)$ and $g(t)$ are analytic in certain regions in the complex plane. We omit the description of these regions, and refer the reader to Berry & Howls (1991) and Boyd (1993) and especially Howls (1997). We take $\lambda_j = f(t_j)$, $j=1,\dots,n$, and

$$a_{pj} = \frac{\Gamma(p+\frac{1}{2})}{2\pi\mathrm{i}}\oint_j \frac{g(t)}{(f(t)-f(t_j))^{p+(1/2)}}\,\mathrm{d}t, \quad j=1,\dots,n, \quad p=0,1,2,\dots, \tag{9.2}$$

where the subscript j indicates that the contour of integration is a small positive loop around t_j. From the results in Berry & Howls (1991) we see that the function $y_k(t)$, defined by

$$y_k(t) = \sum_{p=0}^{\infty} a_{pk}\Gamma(\tfrac{1}{2}-p)(t-\lambda_k)^{p-(1/2)}, \quad |t-\lambda_k| < \min_{j\neq k}|\lambda_j-\lambda_k|, \tag{9.3}$$

is analytic in $\mathcal{P}_\eta$, and satisfies

$$y_k(t) = \tfrac{1}{2} K_{jk} y_j(t) + \operatorname{reg}(t - \lambda_j), \quad j \neq k. \tag{9.4}$$

Note that in this case $\mu_j = -\frac{1}{2}$, $j = 1, \ldots, n$.

If η is admissible and $C_k(\eta)$ is the steepest descent path through t_k, given by $\operatorname{ph}(f(t) - f(t_k)) = \eta$, then

$$I_k(z, \eta) \stackrel{\text{def}}{=} \int_{C_k(\eta)} \mathrm{e}^{zf(t)} g(t)\,\mathrm{d}t = \frac{1}{2\pi\mathrm{i}} \int_{\gamma_k(\eta)} \mathrm{e}^{zt} y_k(t)\,\mathrm{d}t, \tag{9.5}$$

and

$$I_k(z, \eta) \sim \mathrm{e}^{\lambda_k z} z^{-(1/2)} \sum_{s=0}^{\infty} a_{sk} z^{-s}, \tag{9.6}$$

as $z \to \infty$ in $\mathcal{S}(\eta)$.

Since the proof of theorem 6.1 does not depend on $y_k(t)$ originating from (1.1), theorem 6.1 also applies to $I_k(z, \eta)$. In this way we re-obtain the results of Berry & Howls (1991). The only difference is that we specify the optimal number of terms at all levels.

There are two main differences between the results for integrals with saddles and solutions of linear differential equations. The first is that in the case of the linear differential equations we have the freedom to take $a_{0k} = 1$; the second is that in the case of integrals with saddles the constant K_{jk} has only three possible values, $\pm 2\pi\mathrm{i}$ and zero. If $K_{jk} = 0$, then saddle point j is not adjacent to k, and if $K_{jk} = \pm 2\pi\mathrm{i}$, then saddle point j is adjacent to k. The sign in front of $2\pi\mathrm{i}$ depends on the orientation of the steepest descent paths. Hence, in the case of integrals with saddles we are in a fortunate situation in that we need to approximate the Stokes multipliers K_{jk} only to a very low precision; we can then decide immediately the exact value of K_{jk}. For more details see Howls (1997).

10. Conclusions and generalizations

In this paper we have obtained hyperasymptotic expansions for solutions of nth order linear differential equations having a singularity of rank one at infinity. The hyperasymptotic expansion (6.3) is in its optimal form: a series whose terms are a product of Stokes multipliers, coefficients of the formal series solutions (1.3) and hyperterminants. These expansions can be seen as generalizations of the results in Berry & Howls (1990), Olde Daalhuis (1995) and Olde Daalhuis & Olver (1994, 1995*a*, *b*).

The method used was based on the properties of the Borel transforms of the formal series solutions. Once we have these Borel transforms and know their properties, we can forget that they originate from differential equations, and still obtain the hyperasymptotic expansions. The Borel transforms of formal series expansions of integrals with saddles are a special case of the Borel transforms studied. In § 9 we re-obtain the results of Berry & Howls (1991) for hyperasymptotic expansions of integrals with saddles. The case of multidimensional integrals with saddles is discussed in Howls (1997).

It is also not difficult to modify the methods of this paper to obtain the results of Murphy & Wood (1997), that is, hyperasymptotic expansions for solutions of second-order linear differential equations having a singularity of arbitrary rank. But the general case of higher order linear differential equations having a singularity of arbitrary rank is more complicated. Paris (1992) contains some preliminary results. In that paper the smoothing of the Stokes phenomenon is given for higher order linear differential equations.

The first part of the method that we use in this paper is similar to the method used in Lutz & Schäfke (1994). In that paper the authors obtain the optimal number of terms in the original Poincaré asymptotic expansions for systems of linear differential equations with an irregular singular point of rank one at infinity. But instead of re-expanding the minimal remainder in terms of hyperterminants, the authors re-expand the minimal remainder in powers of $1/z$. The coefficients in these re-expansions are complicated, and these re-expansions are valid only in sectors that do not contain Stokes lines.

There are several results in the literature (Braaksma 1991; Immink 1990; Jurkat *et al.* 1976*a*, *b*; Loday-Richaud 1990; Lutz & Schäfke, unpublished research; Olde Daalhuis & Olver 1995*b*) on the computation of Stokes multipliers. Our results can be seen as a direct generalization of those in Olde Daalhuis & Olver (1995*b*). Many of the other results are of the form (7.7), but with the right-hand side replaced by its dominant term. With additional terms available on the right-hand side we have a more powerful way of computing the Stokes multipliers. In general, it is not possible to compute all the Stokes multipliers from (7.7). In fact, with (7.7) we can compute only the Stokes multipliers that are required in the level-one hyperasymptotic expansions. Our result, (7.4), is a generalization of (7.7), and with this expansion we can compute the 'difficult' Stokes multipliers as well.

Other analytical methods for computing the difficult Stokes multipliers are based on conformal mappings in the t plane (Borel plane) (see, for example, Lutz & Schäfke, unpublished research). However, construction of the correct conformal mappings is still a difficult problem.

A numerical method for computing all the Stokes multipliers is discussed in Olde Daalhuis & Olver (1995*b*, 1998). This method is based on direct numerical integration of the differential equation.

The region of validity supplied by theorem 6.1 is the closed sector $\overline{\mathcal{S}}(\eta)$. Values of the analytic continuation of $w_k(z,\eta)$ to other sectors of the complex plane can be calculated by repeated application of connection formula (2.13). Furthermore, with a method that is similar to the method in § 10 of Olde Daalhuis & Olver (1995*a*), we can show that at all levels the region of validity of theorem 6.1 can be extended beyond $\overline{\mathcal{S}}(\eta)$, although at the cost of weakening the asymptotic estimates of the remainder terms. We omit further details.

As in Olde Daalhuis & Olver (1995*a*) the hyperasymptotic expansion, (6.3), for large l may be numerically unstable. This may be remedied in the same manner as in Olde Daalhuis & Olver (1995*a*). The optimal numerically stable scheme uses fewer terms than the corresponding optimal hyperasymptotic expansions, but yields less precision.

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