# HOMEWORK 1

(1) Let $u$ be $C^2(D)$ function solving the following nonlinear equation $Lu = \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)$ in the punctured ball $D = B_1 \setminus \{0\} = \{0 < |x| < 1\}$. Under what condition $L$ is elliptic in the ball $B_1$?

(2) Let $u \in C^2(B_1) \cap C^1(\overline{B_1})$ be harmonic in $B_1$. Prove that

$$\sup_{B_1} |Du| \le \sup_{\partial B_1} |Du|.$$

(3) Let $u \in C^2(B_1) \cap C^1(\overline{B_1})$ be subharmonic in $B_1$. Prove that if $v$ is harmonic in $B_1$ and $v = u$ on $\partial B_1$ then

$$\int_{B_1} |Du|^2 \ge \int_{B_1} |Dv|^2.$$

(4) Let $u$ be harmonic in $B_r$ such that $u \ge 0$ in $B_1$. Let $z \in \partial B_r$ such that $u(z) = 0$ and $|\partial_\nu u(z)| \le K$. Then there is a dimensional constant $C$ such that

$$u(0) \le CKr.$$

Hint: Use the construction in the proof of Hopf's lemma.

(5) (The Liouville theorem) Let $u$ be harmonic in $\mathbb{R}^n$ and there is a constant $C > 0$ such that

$$|u(x)| \le C|x|.$$

Prove that $u$ is a linear function, i.e. $u(x) = a + \sum_{i=1}^n b^i x_i$ for some $a, b^i \in \mathbb{R}, i = 1, \ldots, n$.