HOMEWORK 2

(1) (Weyl's lemma) Let $u \in L^1(B_1)$ such that for all $\phi \in C_0^\infty(B_1)$ there holds

$$\int u\Delta\phi = 0.$$

Show that u is equivalent to a harmonic function $\bar{u}$, i.e. there is a harmonic function $\bar{u}$ agreeng with u a.e. in B_1.

Hint: Prove the mean value property by choosing a suitable radially symmetric test function.

(2) Let u be as in part (1) above and $\phi \in C_0^\infty(\mathbb{R}^n)$ be a radial symmetric function such that

$$1 = \int_{\mathbb{R}^n} \phi(x)dx = C(n)\int_0^\infty \phi(r)r^{n-1}dr$$

where $C(n)$ is the surface area of unit sphere in $\mathbb{R}^n$. Prove that

$$u(x) = \int_{\mathbb{R}^n} u(y)\phi(x-y)dy.$$

From here conclude that u is infinitely differentiable and

$$|D^k u(0)| \leq \begin{cases} C(k,n,\phi)\frac{\|u\|_{L^1(B_r)}}{r^{n+|k|}}, \\ C(k,n,\phi)\frac{\sup_{B_r}|u|}{r^{|k|}}. \end{cases}$$

(3) Let U be domain in $\mathbb{R}^n$. The total variation of a function $u \in L^1(U)$ is defined by

$$\int_U |Du| = \sup\left\{\int_U u\mathrm{div}\vec{v} \ : \ \vec{v} \in C_0^1(U), |\vec{v}| \leq 1\right\}.$$

Show that the space $BV(U)$ of functions of finite total variation is a Banach space with the norm

$$\|u\|_{BV(U)} = \|u\|_{L^1(U)} + \int_U |Du|,$$

and that $W^{1,1}(U)$ is a closed subspace.

(4) Let $u \in BV(U)$. By invoking the regularisation of u and appropriately modifying the proofs of approximation theorems show that there exists a sequence $\{u_k\} \subset C^\infty(U) \cap W^{1,1}(U)$ such that $u_k \to u$ in $L^1(U)$ and

$$\int_U |Du_k| \to \int_U |Du|.$$