SCHAUDER ESTIMATES

The proof to follow is contained in the paper of Xu-Jia Wang [W].

Theorem 1. *Let $u \in C^2$ be a solution of*

$$\Delta u = f \quad \text{in } B_1(0) \tag{1}$$

where f is Dini continuous, i.e. $\int_0^1 \frac{\omega(t)}{t} dt < \infty$ where $\omega(t) = \sup_{|x-y|<t} |f(x) - f(y)|$. Then for all $x, y \in B_{\frac{1}{2}}(0)$ we have

$$|D^2u(x) - D^2u(y)| \leq C_n \left[d \sup_{B_1} |u| + \int_0^d \frac{\omega(t)}{t} dt + d \int_d^1 \frac{\omega(t)}{t^2} dt \right] \tag{2}$$

where $d = |x - y|$, $C_n > 0$ depends only on dimension n. It follows that if $f \in C^\alpha(B_1)$ then

$$\|u\|_{C^{2,\alpha}(B_{\frac{1}{2}})} \leq C_n \left[\sup_{B_1} |u| + \frac{\|f\|_{C^\alpha(B_1)}}{\alpha(1-\alpha)} \right], \quad \text{if } \alpha \in (0,1), \tag{3}$$

$$|D^2u(x) - D^2u(y)| \leq C_n d \left[\sup_{B_1} |u| + \|f\|_{C^\alpha(B_1)} |\log d| \right], \quad \text{if } \alpha = 1. \tag{4}$$

Proof. **Step 1**

From Bernstein's estimate we have that

$$|D^k w(0)| \leq C_{n,k} r^{-|k|} \sup_{B_r} |w| \tag{5}$$

for any harmonic function w in B_1. Here $C_{n,k}$ is a constant depending only on n and k.

Denote $B_k = B_{\rho^k}(0), \rho = \frac{1}{2}$. For $k = 0, 1, 2, \ldots$, let u_k be the solution of

$$\Delta u_k = f(0) \text{ in } B_k, \qquad u = u_k \text{ on } \partial B_k.$$

Recall the maximum principle $\sup_U |u| \leq \sup_{\partial U} + \frac{C}{\lambda} \sup_U |f|$, then we have after using the scaling properties of the equation $\Delta u = f$

$$\|u_k - u\|_{L^\infty(B_k)} \leq C\rho^{2k} \omega(\rho^k). \tag{6}$$

Hence

$$\|u_k - u_{k+1}\|_{L^\infty(B_{k+1})} \leq C\rho^{2k} \omega(\rho^k). \tag{7}$$

Since $u_{k+1} - u_k$ is harmonic, by (5) we have

$$\|D(u_k - u_{k+1})\|_{L^\infty(B_{k+2})} \leq C\rho^k \omega(\rho^k), \quad \|D^2(u_k - u_{k+1})\|_{L^\infty(B_{k+2})} \leq C\omega(\rho^k). \tag{8}$$

Since $u \in C^2$ then $(\Delta u)(0) = f(0)$. Let $Q(x) = u(0) + Du(0)x + \frac{1}{2}D^2u(0)xx$ then $\Delta(u_k - Q) = 0$ and $\|u_k - Q\|_{L^\infty(B_{k+1})} \leq o(\rho^{2k})$. Thus from (5)

$$Du(0) = \lim_{k\to\infty} Du_k(0), \quad D^2u(0) = \lim_{k\to\infty} D^2u_k(0). \tag{9}$$

Step 2

For any point z near the origin we have that

$$|D^2u(z) - D^2u(0)| \leq I_1 + I_2 + I_3 \tag{10}$$

where

$$\begin{aligned} I_1 &= |D^2u_k(z) - D^2u_k(0)|, \\ I_2 &= |D^2u_k(0) - D^2u(0)|, \\ I_3 &= |D^2u(z) - D^2u_k(z)|. \end{aligned}$$

Let $k \geq 1$ such that $\rho^{k+4} \leq |z| \leq \rho^{k+3}$. Then by (8) we have

$$I_2 \leq C\sum_{j=k}^{\infty} \omega(\rho^k) \leq C\int_0^{|z|} \frac{\omega(t)}{t}dt. \tag{11}$$

Similarly we can estimate I_3 through the solutions $\Delta v = f(z)$ in $B_j(z)$ and $v = u$ on $\partial B_j(z)$ for $j = k, k+1, \ldots$.

Step 3

To estimate I_1, denote $h_j = u_j - u_{j-1}$. By (5) and (7) we have

$$|D^2h_j(z) - D^2h_j(0)| \leq C\rho^{-j}\omega(\rho^j)|z| \tag{12}$$

where the last inequality follows from the mean value theorem, $D^2h_j(z) - D^2h_j(0) = D^3h_j(\tilde{z})z$. Hence

$$\begin{aligned} I_1 &\leq |D^2u_{k-1}(z) - D^2u_{k-1}(0)| + |D^2h_k(z) - D^2h_k(0)| \\ &\leq |D^2u_0(z) - D^2u_0(0)| + \sum_{j=1}^{k} |D^2h_j(z) - D^2h_j(0)| \\ &\leq C|z|\left(\|u_0\|_{L^\infty} + C\sum_{j=1}^{k} \rho^{-j}\omega(\rho^j)\right) \leq C|z|\left(\|u\|_{L^\infty} + C\int_{|z|}^{1} \frac{\omega(t)}{t^2}\right). \end{aligned} \tag{13}$$

Combining (10), (11), and (13) we obtain (2). This completes the proof. □

Similarly we have th estimates at the boundary by using the reflection across the flat portion of the boundary.

Theorem 2. *Let $u \in C^2(B_1 \cap \{x_n \geq 0\})$ be a solution of $\Delta u = f$ and $u = 0$ on $\{x_n = 0\}$. Suppose f is Dini continuous. Then $\forall x, y \in B_{\frac{1}{2}} \cap \{x_n \geq 0\}$, the estimate (2) holds.*

References

[W] Wang, Xu-Jia, Schauder estimates for elliptic and parabolic equations. Chinese Ann. Math. Ser. B 27 (2006), no. 6, 637–642.