Advanced Partial Differential Equations 1

Based on Thomas Bäckdahl's notes

The University of Edinburgh

October 19, 2015

Lecture 4

http://www.maths.ed.ac.uk/~aram/advancedpde.html

Sobolev spaces

The most useful function spaces for the study of PDEs are the so called Sobolev spaces. To introduce them we first need to introduce the notation of test functions and weak derivatives.

Definition

Given an open set $U \subset \mathbb{R}^n$. A function $\phi \in C_c^\infty(U)$, i.e. $\phi \in C^\infty(U)$, $\phi : U \to \mathbb{R}$ with $\operatorname{spt} \phi = \overline{\{x : \phi(x) \neq 0\}} \subset U$ compact is called a *test function*. (Alternative notation $C_0^\infty(U)$ and supp.)

Remark

$\phi = 0$ outside a closed and bounded subset of U. This means that $\phi = 0$ near ∂U.

Example (Standard mollifier)

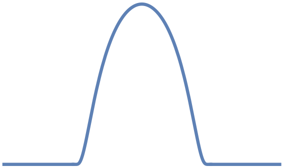
Define $\eta \in C^\infty(\mathbb{R}^n)$

$$\eta(x) = \begin{cases} C\exp\left(\frac{1}{|x|^2-1}\right) & \text{if} \quad |x| < 1 \\ 0 & \text{if} \quad |x| \geq 1 \end{cases},$$

with C such that $\int_{\mathbb{R}^n} \eta dx = 1$. For each $\epsilon > 0$ let

$$\eta_\epsilon(x) = \frac{1}{\epsilon^n}\eta\left(\frac{x}{\epsilon}\right),$$

Then we have $\int_{\mathbb{R}^n} \eta_\epsilon dx = 1$ and $\operatorname{spt} \eta_\epsilon \subset B(0,\epsilon)$.
For all $\epsilon > 0$, we have $\eta_{1-\epsilon} \in C_c^\infty(B^0(0,1))$.



Weak derivative

If $u \in C^1(U)$, $\phi \in C_c^\infty(U)$. Integration by parts gives

$$\int_U u\phi_{x_i} dx = -\int_U u_{x_i} \phi dx.$$

We have no boundary term because $\phi = 0$ near ∂U.
In general if $u \in C^k(U)$,

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U D^\alpha u \phi dx, \qquad |\alpha| \leq k$$

by applying the previous formula repeatedly.

Definition (Weak derivative)

Let $u \in L^1_{\text{loc}}(U)$ and α a multi-index. We say that u has a weak αth partial derivative v if there is a $v \in L^1_{\text{loc}}(U)$ such that

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v \phi dx, \qquad \forall \phi \in C_c^\infty(U).$$

We then write $v = D^\alpha u$.

Recall $u \in L^1_{\text{loc}}(U)$ if $u \in L^1(V)$ for every $V \Subset U$.

Lemma

If a weak αth partial derivative of u, exists, it is uniquely defined up to a set of measure zero.

Proof.

Let v and $\tilde{v}$ be weak αth partial derivatives of u. Then

$$(-1)^{|\alpha|} \int_U v\phi dx = \int_U uD^\alpha \phi dx = (-1)^{|\alpha|} \int_U \tilde{v}\phi dx, \quad \forall \phi \in C_c^\infty(U).$$

Hence,

$$\int_U (v - \tilde{v})\phi dx = 0, \qquad \forall \phi \in C_c^\infty(U).$$

This gives $v - \tilde{v} = 0$ a.e. $\square$

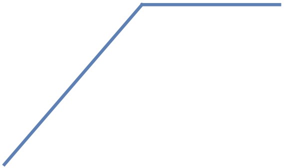
Example

Let $n = 1$, $U = (0, 2)$, and

$$u(x) = \begin{cases} x & \text{if} \quad 0 < x \leq 1 \\ 1 & \text{if} \quad 1 < x < 2, \end{cases} \qquad v(x) = \begin{cases} 1 & \text{if} \quad 0 < x \leq 1 \\ 0 & \text{if} \quad 1 < x < 2. \end{cases}$$

Then $u' = v$ in the weak sense because for any $\phi \in C_c^\infty(U)$ we have

$$\begin{aligned} \int_0^2 u\phi' dx &= \int_0^1 x\phi' dx + \int_1^2 \phi' dx \\ &= -\int_0^1 \phi dx + \phi(1) - \phi(1) = -\int_0^2 v\phi dx. \end{aligned}$$



Example

Let U and v be as in in the previous example. Then v' does not exist in the weak sense. Assume that $w = v' \in L^1_{\text{loc}}(U)$, i.e.

$$-\int_0^2 w\phi dx = \int_0^2 v\phi' dx = \int_1^2 \phi' dx = -\phi(1) \quad \forall \phi \in C_c^\infty(U).$$

Define the sequence $\{\phi_m\}_{m=2}^\infty$ in $C_c^\infty(U)$ by

$$\phi_m(x) = \begin{cases} \exp\left(1 + \frac{1}{m^2|x-1|^2-1}\right) & \text{if} \quad |x-1| < \frac{1}{m} \\ 0 & \text{if} \quad |x-1| \geq \frac{1}{m} \end{cases}.$$

Then $\phi_m(1) = 1$, $0 \leq \phi_m(x) \leq 1$, $\operatorname{spt} \phi_m \subset B(1, 1/m)$. This gives $w\phi_m \to 0$ a.e. and by dominated convergence

$$1 = \lim_{m\to\infty} \phi_m(1) = \lim_{m\to\infty} \int_0^2 w\phi_m dx = 0.$$

A contradiction. Hence, v' does not exist in the weak sense.

Sobolev spaces

Definition

The Sobolev space $W^{k,p}(U)$ consists of all functions $u \in L^1_{\mathrm{loc}}$ such that for each multi-index α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$.

Remark

1. *For $p = 2$, we use the notation $H^k(U) = W^{k,2}(U)$.*
2. *We will identify functions in $W^{k,p}(U)$ which agree a.e.*

Definition

The *norm* of $u \in W^{k,p}(U)$ is defined as

$$\|u\|_{W^{k,p}(U)} = \begin{cases} \left(\sum_{|\alpha|\le k} \int_U |D^\alpha u|^p dx\right)^{1/p} & \text{if } (1 \le p < \infty) \\ \sum_{|\alpha|\le k} \operatorname{ess\,sup}_U |D^\alpha u| & \text{if } (p = \infty). \end{cases}$$

Remark

1. *By convergence in $W^{k,p}(U)$ we mean convergence in this norm.*
2. *By convergence in $W^{k,p}_{loc}(U)$ we mean convergence in $W^{k,p}(V)$ for each $V \Subset U$.*

Definition

We let $W_0^{k,p}(U)$ denote the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$.

Remark

Loosely speaking the functions $u \in W_0^{k,p}(U)$ are the functions $u \in W^{k,p}(U)$ such that $D^\alpha u = 0$ on ∂U for all $|\alpha| \leq k-1$.
(This statement only makes sense in terms of traces – will be presented later.)

Example

Let $U = B^0(0,1) \in \mathbb{R}^n$, and

$$u(x) = |x|^{-\alpha} \qquad (x \in U, x \neq 0).$$

When does this belong to $W^{1,p}(U)$? We have

$$u_{x_i}(x) = \frac{-\alpha x_i}{|x|^{\alpha+2}} \quad (x \neq 0),$$
$$|Du(x)| = \frac{|\alpha|}{|x|^{\alpha+1}} \quad (x \neq 0).$$

$|Du(x)| \in L^1(U)$ if and only if $\alpha < n-1$.
Let $\phi \in C_c^\infty(U)$ and fix $\epsilon > 0$. Then

$$\int_{U-B(0,\epsilon)} u\phi_{x_i} dx = -\int_{U-B(0,\epsilon)} u_{x_i}\phi dx + \int_{\partial B(0,\epsilon)} u\phi\nu^i dS,$$

where $\nu = (\nu^1, \ldots, \nu^n)$ is the inward pointing normal on $\partial B(0,\epsilon)$.

If $\alpha < n-1$, we get

$$\left|\int_{\partial B(0,\epsilon)} u\phi\nu^i dS\right| \leq \|\phi\|_{L^\infty(U)} \int_{\partial B(0,\epsilon)} \epsilon^{-\alpha} dS \leq C\epsilon^{n-1-\alpha} \to 0.$$

Hence, as long as $\alpha < n-1$ we get

$$\int_U u\phi_{x_i} dx = -\int_U u_{x_i}\phi dx \qquad \text{for all } \phi \in C_c^\infty(U).$$

We also see that $|Du(x)| = \frac{|\alpha|}{|x|^{\alpha+1}} \in L^p(U)$ if and only if $(\alpha+1)p < n$ (which implies $\alpha < n-1$).
Consequently $u \in W^{1,p}(U)$ if and only if $\alpha < \frac{n-p}{n}$.

Example

Let $\{r_k\}_{k=1}^{\infty}$ be a countable, dense subset of $U = B^0(0,1) \in \mathbb{R}^n$ (for instance an enumeration of the points with rational coefficients). Define

$$u(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} |x - r_k|^{-\alpha} \qquad (x \in U). \tag{1}$$

If $0 < \alpha < \frac{n-p}{p}$, we see that $u \in W^{1,p}(U)$, even though it is unbounded on each open subset of U.

Theorem (Properties of weak derivatives)

Assume $u, v \in W^{k,p}(U)$, and the multi-index α satisfies $|\alpha| \le k$. Then

1. $D^\alpha u \in W^{k-|\alpha|,p}(U)$ *and* $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta} u$ *for all multi-indices* α, β *such that* $|\alpha| + |\beta| \le k$.
2. *For each* $\lambda, \mu \in \mathbb{R}$, $\lambda u + \mu v \in W^{k,p}(U)$ *and* $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$.
3. *If* $V \subset U$ *is open, then* $u \in W^{k,p}(V)$.
4. *If* $\zeta \in C_c^\infty(U)$, *then* $\zeta u \in W^{k,p}(U)$ *and*

$$D^\alpha(\zeta u) = \sum_{\beta \le \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} D^\beta \zeta D^{\alpha-\beta} u \qquad \text{(Leibniz' formula).}$$

Proof of 1).

Let $\phi \in C_c^\infty(U)$, then $D^\beta \phi \in C_c^\infty(U)$.

$$\begin{aligned}
\int_U D^\alpha u D^\beta \phi dx &= (-1)^{|\alpha|} \int_U u D^{\alpha+\beta} \phi dx \\
&= (-1)^{|\alpha|}(-1)^{|\alpha+\beta|} \int_U D^{\alpha+\beta} u \phi dx \\
&= (-1)^{|\beta|} \int_U D^{\alpha+\beta} u \phi dx.
\end{aligned}$$

Hence, $D^\beta(D^\alpha u) = D^{\alpha+\beta} u$ in the weak sense. $\square$

The other parts are left as an exercise.

Theorem (Banach space structure)

For each $k \geq 1$ and $1 \leq p \leq \infty$, the Sobolev space $W^{k,p}(U)$ is a Banach space.

Proof.

1) Prove that $\|u\|_{W^{k,p}(U)}$ is a norm:
The definition implies $\|\lambda u\|_{W^{k,p}(U)} = |\lambda|\|u\|_{W^{k,p}(U)}$ and $\|u\|_{W^{k,p}(U)} = 0$ if and only if $u = 0$ a.e. The triangle inequality follows from Minkowski's inequality. For $u, v \in W^{k,p}(U)$ with $1 \le p < \infty$

$$\begin{aligned}
\|u+v\|_{W^{k,p}(U)} &= \Big(\sum_{|\alpha|\le k} \|D^\alpha u + D^\alpha v\|^p_{L^p(U)}\Big)^{1/p} \\
&\le \Big(\sum_{|\alpha|\le k} (\|D^\alpha u\|_{L^p(U)} + \|D^\alpha v\|_{L^p(U)})^p\Big)^{1/p} \\
&\le \Big(\sum_{|\alpha|\le k} \|D^\alpha u\|^p_{L^p(U)}\Big)^{1/p} + \Big(\sum_{|\alpha|\le k} \|D^\alpha v\|_{L^p(U)})^p\Big)^{1/p} \\
&= \|u\|_{W^{k,p}(U)} + \|v\|_{W^{k,p}(U)}.
\end{aligned}$$

2) Prove that $W^{k,p}(U)$ is complete:
Assume that $\{u_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $W^{k,p}(U)$.
Then for each $|\alpha| \leq k$, $\{D^\alpha u_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $L^p(U)$. The $L^p(U)$ space is complete, so for each $|\alpha| \leq k$ there exist a function $u_\alpha \in L^p(U)$ such that

$$D^\alpha u_m \to u_\alpha \qquad \text{in } L^p(U).$$

Define $u = u_{(0,\ldots,0)}$, i.e. the case without derivatives.

$$u_m \to u \qquad \text{in } L^p(U).$$

3) Prove that $u \in W^{k,p}(U)$ and $D^\alpha u = u_\alpha$ in the weak sense for all $|\alpha| \leq k$:
Let $\phi \in C_c^\infty(U)$.

$$\lim_{m\to\infty} \Big|\int_U (u - u_m) D^\alpha \phi dx\Big| \leq \lim_{m\to\infty} \|u - u_m\|_{L^p(U)} \|D^\alpha \phi\|_{L^q(U)} = 0,$$
$$\lim_{m\to\infty} \Big|\int_U (u_\alpha - D^\alpha u_m) \phi dx\Big| \leq \lim_{m\to\infty} \|u_\alpha - D^\alpha u_m\|_{L^p(U)} \|\phi\|_{L^q(U)} = 0,$$

$$\begin{aligned}\int_U u D^\alpha \phi dx &= \lim_{m\to\infty} \int_U u_m D^\alpha \phi dx \\ &= \lim_{m\to\infty} (-1)^{|\alpha|} \int_U D^\alpha u_m \phi dx = (-1)^{|\alpha|} \int_U u_\alpha \phi dx.\end{aligned}$$

4) The completeness follows because $D^\alpha u_m \to D^\alpha u$ in $L^p(U)$ for all $|\alpha| \leq k$, i.e. $u_m \to u$ in $W^{k,p}(U)$. $\square$

Approximation

- Sobolev functions can be fairly ill-behaved and difficult to use directly in calculations.

Approximation

- Sobolev functions can be fairly ill-behaved and difficult to use directly in calculations.
- Smooth functions are better for calculations.

Approximation

- Sobolev functions can be fairly ill-behaved and difficult to use directly in calculations.
- Smooth functions are better for calculations.
- Sobolev functions can be be approximated by smooth functions.

Approximation

- Sobolev functions can be fairly ill-behaved and difficult to use directly in calculations.
- Smooth functions are better for calculations.
- Sobolev functions can be be approximated by smooth functions.
- Local approximation in U.

Approximation

- Sobolev functions can be fairly ill-behaved and difficult to use directly in calculations.
- Smooth functions are better for calculations.
- Sobolev functions can be be approximated by smooth functions.
- Local approximation in U.
- Global approximation in bounded U.

Approximation

- Sobolev functions can be fairly ill-behaved and difficult to use directly in calculations.
- Smooth functions are better for calculations.
- Sobolev functions can be be approximated by smooth functions.
- Local approximation in U.
- Global approximation in bounded U.
- Global approximation in bounded $\bar{U}$ with C^1 boundary.

Mollifiers

Let $U_\epsilon = \{x \in U | \operatorname{dist}(x, \partial U) > \epsilon\}$.

Definition

For $u \in L^1_{\text{loc}}(U)$, we define its mollification to be

$$u^\epsilon(x) = (\eta_\epsilon * u)(x) = \int_U \eta_\epsilon(x-y)u(y)dy = \int_{B(0,\epsilon)} \eta_\epsilon(y)u(x-y)dy \quad \forall x \in U_\epsilon.$$

Theorem (Properties of Mollifiers)

1. *$u^\epsilon \in C^\infty(U_\epsilon)$ for each $\epsilon > 0$,*
2. *$u^\epsilon \to u$ a.e. as $\epsilon \to 0$,*
3. *If $u \in C(U)$, then $u^\epsilon \to u$ uniformly on compact sets of U as $\epsilon \to 0$.*
4. *If $u \in L^p_{loc}(U)$ for $1 \leq p < \infty$, then $u^\epsilon \to u$ in $L^p_{loc}(U)$, as $\epsilon \to 0$.*

Proof

1) Fix $x \in U_\epsilon$, $i \in \{1, \ldots, n\}$, and $h > 0$ so small that $x + he_i \in U_\epsilon$. Then

$$\begin{aligned}\frac{u^\epsilon(x+he_i) - u^\epsilon(x)}{h} &= \frac{1}{\epsilon^n}\int_U \frac{1}{h}\left(\eta\left(\frac{x+he_i-y}{\epsilon}\right) - \eta\left(\frac{x-y}{\epsilon}\right)\right)u(y)dy \\ &= \frac{1}{\epsilon^n}\int_V \frac{1}{h}\left(\eta\left(\frac{x+he_i-y}{\epsilon}\right) - \eta\left(\frac{x-y}{\epsilon}\right)\right)u(y)dy,\end{aligned}$$

for some open set $V \Subset U$, for instance $V = B^0(x+he_i, \epsilon) \cup B^0(x, \epsilon)$. As

$$\frac{1}{h}\left(\eta\left(\frac{x+he_i-y}{\epsilon}\right) - \eta\left(\frac{x-y}{\epsilon}\right)\right) \to \frac{1}{\epsilon}\frac{\partial \eta}{\partial x_i}\left(\frac{x-y}{\epsilon}\right) = \epsilon^n \frac{\partial \eta_\epsilon}{\partial x_i}(x-y)$$

as $h \to 0$ uniformly on V, $\frac{\partial u^\epsilon}{\partial x_i}(x)$ exists and equals $\int_U \frac{\partial \eta_\epsilon}{\partial x_i}(x-y)u(y)dy$.
A similar argument shows that $D^\alpha u^\epsilon(x)$ exists, and equals

$$\int_U D^\alpha \eta_\epsilon(x-y)u(y)dy \qquad x \in U_\epsilon.$$

2) Lebesque's differentiation theorem gives

$$\lim_{r\to 0} \fint_{B(x,r)} |u(y) - u(x)| dy = 0 \text{ a.e. } x \in U.$$

Fix such a point x

$$\begin{aligned}
|u^\epsilon(x) - u(x)| &= \Big| \int_{B(x,\epsilon)} \eta_\epsilon(x-y)(u(y)-u(x))dy \Big| \\
&\le \frac{1}{\epsilon^n} \int_{B(x,\epsilon)} \eta\Big(\frac{x-y}{\epsilon}\Big) |u(y)-u(x)| dy \\
&\le C \fint_{B(x,\epsilon)} |u(y)-u(x)| dy \to 0 \text{ as } \epsilon \to 0.
\end{aligned}$$

3) Let $u \in C(U)$. For any $V \Subset U$ we can choose W such that $V \Subset W \Subset U$ and note that u is uniformly continuous on W. This means that the limit

$$\lim_{r\to 0} \fint_{B(x,r)} |u(y) - u(x)| dy = 0,$$

holds uniformly on V. Hence, $u^\epsilon \to u$ uniformly on V.

4) Let $u \in L^p_{\mathrm{loc}}(U)$ for $1 \le p < \infty$ and $V \Subset U$. Choose W s.t. $V \Subset W \Subset U$. Now we want to prove that for small enough $\epsilon > 0$ we have

$$\|u^\epsilon\|_{L^p(V)} \le \|u\|_{L^p(W)}. \tag{2}$$

For $x \in V$, we have

$$\begin{aligned}
|u^\epsilon(x)| = \Big|\int_{B(x,\epsilon)} \eta_\epsilon(x-y)u(u)dy\Big| &\le \int_{B(x,\epsilon)} \eta_\epsilon^{1-1/p}(x-y)\eta_\epsilon^{1/p}(x-y)|u(u)|dy \\
&\le \underbrace{\Big(\int_{B(x,\epsilon)} \eta_\epsilon(x-y)dy\Big)}_{1}^{1-1/p}\Big(\int_{B(x,\epsilon)} \eta_\epsilon(x-y)|u(y)|^p dy\Big)^{1/p}
\end{aligned}$$

$$\int_V |u^\epsilon(x)|^p dx \le \int_V \Big(\int_{B(x,\epsilon)} \eta_\epsilon(x-y)|u(y)|^p dy\Big)dx \le \int_W |u(y)|^p \underbrace{\Big(\int_{B(y,\epsilon)} \eta_\epsilon(x-y)dx\Big)}_{1} dy$$

Fix $V \Subset W \Subset U$, $\delta > 0$ and choose $v \in C(W)$ s. t. $\|u - v\|_{L^p(W)} < \delta$. Then

$$\begin{aligned} \|u^\epsilon - u\|_{L^p(V)} &\leq \|u^\epsilon - v^\epsilon\|_{L^p(V)} + \|v^\epsilon - v\|_{L^p(V)} + \|v - u\|_{L^p(V)} \\ &\leq 2\|u - v\|_{L^p(W)} + \|v^\epsilon - v\|_{L^p(V)} \\ &\leq 2\delta + \|v^\epsilon - v\|_{L^p(V)}. \end{aligned}$$

Since $v^\epsilon \to v$ uniformly on V, we have $\limsup_{\epsilon \to 0} \|u^\epsilon - u\|_{L^p(V)} \leq 2\delta$. □

Approximation by smooth functions

Theorem (1. Local approximation by smooth functions)

Assume $u \in W^{k,p}(U)$, $1 \leq p < \infty$, *and let*

$$u^\epsilon = \eta_\epsilon * u \quad \textit{in } U_\epsilon.$$

Then $u^\epsilon \to u$ *in* $W^{k,p}_{loc}(U)$, *as* $\epsilon \to 0$.

Proof

We now want to prove that if $|\alpha| \leq k$, then

$$D^\alpha u^\epsilon = \eta_\epsilon * D^\alpha u \in U_\epsilon, \tag{3}$$

i.e. the αth partial derivative of u^ϵ is the ϵ-mollification of the weak αth partial derivative of u.

For $x \in U_\epsilon$, we have

$$\begin{aligned} D^\alpha u^\epsilon(x) &= D^\alpha \int_U \eta_\epsilon(x-y)u(y)dy \\ &= \int_U D_x^\alpha \eta_\epsilon(x-y)u(y)dy = (-1)^{|\alpha|} \int_U D_y^\alpha \eta_\epsilon(x-y)u(y)dy. \end{aligned}$$

For a fixed $x \in U_\epsilon$ the function $\phi(y) = \eta_\epsilon(x-y) \in C_c^\infty(U)$. The definition of a weak derivative gives

$$\int_U D_y^\alpha \eta_\epsilon(x-y)u(y)dy = (-1)^{|\alpha|}\int_U \eta_\epsilon(x-y)D^\alpha u(y)dy.$$

Thus

$$D^\alpha u^\epsilon(x) = (-1)^{|\alpha|+|\alpha|}\int_U \eta_\epsilon(x-y)D^\alpha u(y)dy = (\eta_\epsilon * D^\alpha u)(x).$$

Hence, we have (3).
Now choose an open set $V \Subset U$ then from the properties of mollifiers we get $D^\alpha u^\epsilon = \eta_\epsilon * D^\alpha u \to D^\alpha u$ in $L^p(V)$ as $\epsilon \to 0$, for each $|\alpha| \leq k$. Hence,

$$\|u^\epsilon - u\|_{W^{k,p}(V)}^p = \sum_{|\alpha|\leq k} \|D^\alpha u^\epsilon - D^\alpha u\|_{L^p(V)}^p \to 0 \text{ as } \epsilon \to 0. \tag{4}$$

$\square$

Theorem (2. Global approximation by smooth functions)

Assume that U is open and bounded. Let $u \in W^{k,p}(U)$, $1 \leq p < \infty$. Then there exist functions $u_m \in C^\infty(U) \cap W^{k,p}(U)$ such that

$$u_m \to u \text{ in } W^{k,p}(U).$$

Remark

The functions u_m might not be smooth on $\bar{U}$.

Proof

Let

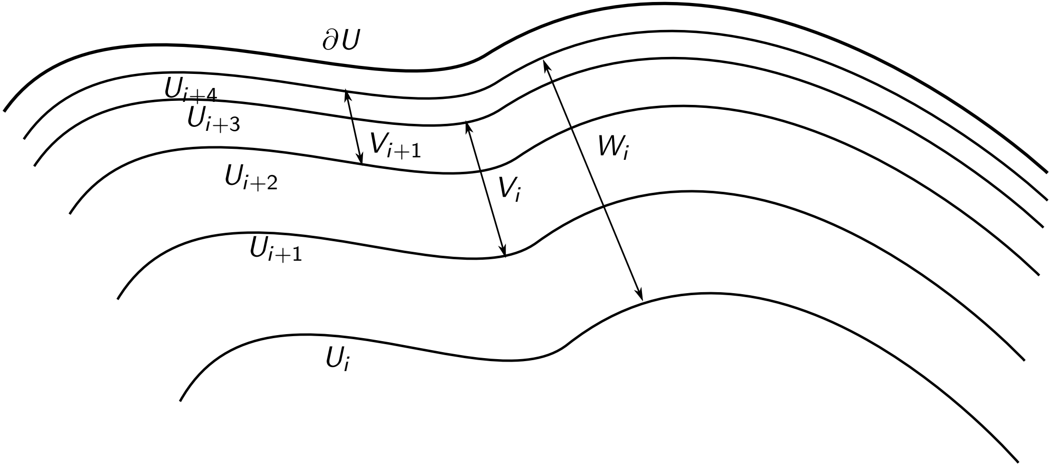
$$U_i = \{x \in U | \operatorname{dist}(x, \partial U) > 1/i\} \quad (i = 1, 2, \dots)$$
$$V_i = U_{i+3} - \bar{U}_{i+1}, \qquad V_0 = U_3.$$

We get $U = \bigcup_{i=0}^{\infty} V_i$.
Let $\{\zeta_i\}_{i=0}^{\infty}$ be a C^{∞} partition of unity subordinate to $\{V_i\}_{i=0}^{\infty}$, i.e.

$$\begin{cases} 0 \le \zeta_i \le 1, & \zeta_i \in C_c^{\infty}(V_i) \\ \sum_{i=0}^{\infty} \zeta_i = 1 & \text{on } U. \end{cases}$$

From $u \in W^{k,p}(U)$ and the product properties of the weak derivative

$$\zeta_i u \in W^{k,p}(U) \text{ and } \operatorname{spt}(\zeta_i u) \subset V_i.$$


∂U
U_{i+4}
U_{i+3}
V_{i+1}
W_i
U_{i+2}
V_i
U_{i+1}
U_i

Fix $\epsilon > 0$. Choose $\epsilon_i > 0$ small enough so $u^i = \eta_{\epsilon_i} * (\zeta_i u)$ satisfies

$$\begin{cases} \|u^i - \zeta_i u\|_{W^{k,p}(U)} \leq \epsilon 2^{-i-1} & (i = 0, 1, \dots) \\ \operatorname{spt} u^i \subset W_i & (i = 1, \dots), \end{cases}$$

where $W_i = U_{i+4} - \bar{U}_i \supset V_i$, $(i = 1, \dots)$.
Let $v = \sum_{i=0}^{\infty} u^i$. For any $V \Subset U$ there are only finitely many non-zero terms in the sum. Therefore $v \in C^\infty(U)$. Also $u = \sum_{i=0}^{\infty} \zeta_i u$ have finitely many non-zero terms when restricted to V.

$$\|v - u\|_{W^{k,p}(V)} \leq \sum_{i=0}^{\infty} \|u^i - \zeta_i u\|_{W^{k,p}(U)} \leq \epsilon \sum_{i=0}^{\infty} 2^{-i-1} = \epsilon.$$

Taking the supremum over all sets $V \Subset U$ we get $\|v - u\|_{W^{k,p}(U)} \leq \epsilon$. $\square$

Theorem (3. Global approximation by functions in $C^\infty(\bar{U})$)

Assume U is bounded and ∂U is C^1. Let $u \in W^{k,p}(U)$, $1 \leq p < \infty$. Then there exist functions $u_m \in C^\infty(\bar{U})$ such that

$$u_m \to u \text{ in } W^{k,p}(U).$$

Proof

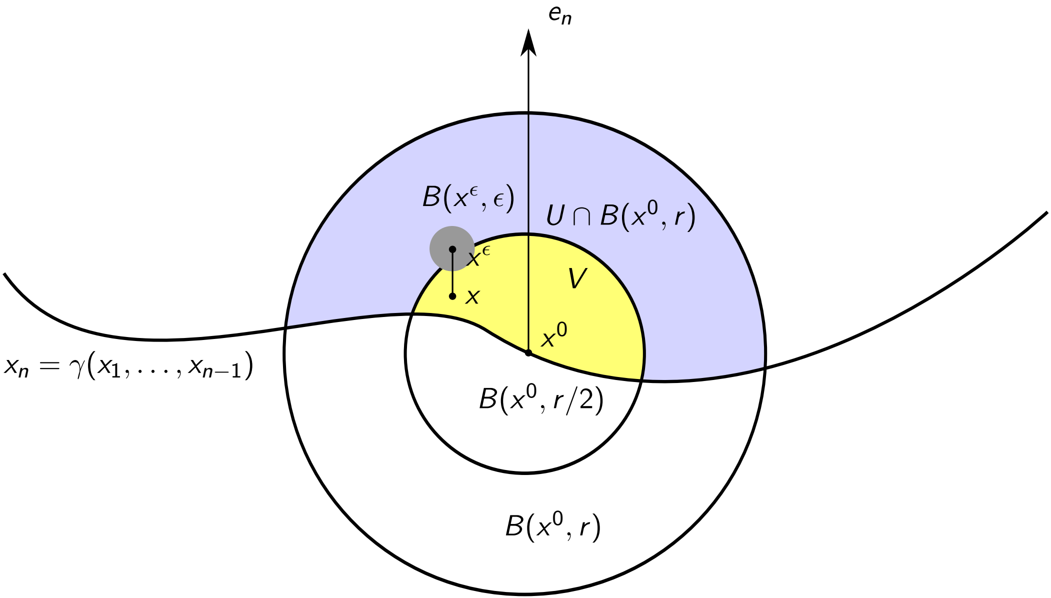
Fix any point $x^0 \in \partial U$. Then there exist a radius $r > 0$ and a C^1 function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that (after relabelling the coordinate axes) we have

$$U \cap B^0(x^0, r) = \{x \in B^0(x^0, r) | x_n > \gamma(x_1, \dots, x_{n-1})\}$$

Set $V = U \cap B^0(x^0, r/2)$, we get $\bar{V} = \{x \in B(x^0, r/2) | x_n \geq \gamma(x_1, \dots, x_{n-1})\}$. Define

$$x^\epsilon = x + \lambda\epsilon e_n \quad (x \in \bar{V}, \epsilon > 0).$$

Observe that for some fixed, sufficiently large $\lambda > 0$ the ball $B^0(x^\epsilon, \epsilon) \subset U \cap B^0(x^0, r)$ for all $x \in \bar{V}$ and all small $\epsilon > 0$.


e_n
$B(x^\epsilon, \epsilon)$
$U \cap B(x^0, r)$
x^ϵ
V
x
x^0
$x_n = \gamma(x_1, \ldots, x_{n-1})$
$B(x^0, r/2)$
$B(x^0, r)$

Now define the translated function $u_\epsilon(x) = u(x^\epsilon)$ for $x \in \bar{V}$.
Let $v^\epsilon = \eta_\epsilon * u_\epsilon \in C^\infty(\bar{V})$. (By the translation, we have room for the mollification.)
We now want to prove $v^\epsilon \to u$ in $W^{k,p}(V)$:
For any $|\alpha| \leq k$ we have

$$\|D^\alpha v^\epsilon - D^\alpha u\|_{L^p(V)} \leq \|D^\alpha v^\epsilon - D^\alpha u_\epsilon\|_{L^p(V)} + \|D^\alpha u_\epsilon - D^\alpha u\|_{L^p(V)}.$$

The second term goes to zero with ϵ since translation is continuous in the L^p norms.
The first terms also vanishes in the limit due to Theorem 1.

Select $\epsilon > 0$. Since ∂U is compact, we can find finitely many points $x_i^0 \in \partial U$ and radii $r_i > 0$, corresponding sets $V_i = U \cap B^0(x_i^0, r_i/2)$, and functions $v_i \in C^\infty(\bar{V}_i)$ $(i = 1, \ldots, N)$ such that $\partial U \subset \bigcup_{i=1}^N B^0(x_i^0, r_i/2)$ and

$$\|v_i - u\|_{W^{k,p}(V_i)} \leq \epsilon.$$

Also choose $V_0 \Subset U$ such that $U \subset \bigcup_{i=0}^N V_i$ and use Theorem 1 to get $v_0 \in C^\infty(V_0)$ satisfying

$$\|v_0 - u\|_{W^{k,p}(V_0)} \leq \epsilon.$$

Let $\{\zeta_i\}_{i=0}^N$ be a C^∞ partition of unity on $\bar{U}$, subordinate to V_0 and $\{B^0(x_i^0, r_i/2)\}_{i=1}^N$. Let $v = \sum_{i=0}^N \zeta_i v_i \in C^\infty(\bar{U})$. For $|\alpha| \le k$ we get (using $U \cap \operatorname{spt} \zeta_i \subset V_i$)

$$\begin{aligned} \|D^\alpha v - D^\alpha u\|_{L^p(U)} &\le \sum_{i=0}^N \|D^\alpha(\zeta_i v_i) - D^\alpha(\zeta_i u)\|_{L^p(V_i)} \\ &\le C \sum_{i=0}^N \|v_i - u\|_{W^{k,p}(V_i)} \le CN\epsilon. \end{aligned}$$

$\square$

Extensions

Extending a function $u \in W^{1,p}(U)$ to $\mathbb{R}^n$ by setting out to zero on $\mathbb{R}^n - U$ does not work due to weak derivatives.

Theorem (Extension Theorem)

Assume that U is bounded and ∂U is C^1. Select a bounded open set V such that $U \Subset V$. Then there exist a linear operator

$$E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$$

such that for each $u \in W^{1,p}(U)$

- *$Eu = u$ a.e. in U*
- $\operatorname{spt} Eu \subset V$
- $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}$.

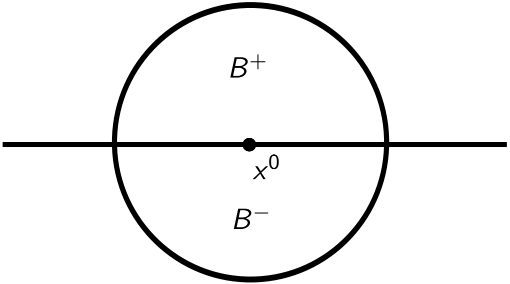
(C depends on p, U and V but not u.)
Eu is called an extension of u to $\mathbb{R}^n$.

Fix $x^0 \in \partial U$ and suppose first

$$\partial U \text{ is flat near } x^0, \text{ lying in the plane } \{x_n = 0\}. \tag{5}$$

Then we may assume there exists an open ball $B = B^0(x^0, r)$, such that

$$\begin{cases} B^+ = B \cap \{x_n \geq 0\} \subset \bar{U} \\ B^- = B \cap \{x_n \leq 0\} \subset \mathbb{R}^n - U \end{cases}$$



Temporarily assume that $u \in C^1(\bar{U})$ and define

$$\bar{u}(x) = \begin{cases} u(x) & \text{if } x \in B^+ \\ -3u(x_1, \dots, x_{n-1}, -x_n) + 4u(x_1, \dots, x_{n-1}, -\frac{x_n}{2}) & \text{if } x \in B^-. \end{cases}$$

This is called a higher-order reflection of u from B^+ to B^-.
We now want to show $\bar{u} \in C^1(B)$:
Define $u^- = \bar{u}|_{B^-}$ and $u^+ = \bar{u}|_{B^+}$.

$$\frac{\partial u^-}{\partial x_n}(x) = 3\frac{\partial u}{\partial x_n}(x_1, \dots, x_{n-1}, -x_n) - 2\frac{\partial u}{\partial x_n}(x_1, \dots, x_{n-1}, -\frac{x_n}{2})$$

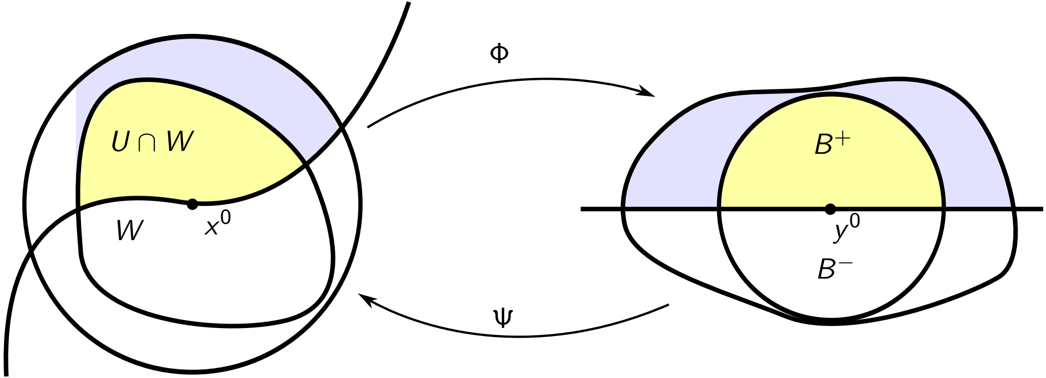
This gives $u^-_{x_n} = u^+_{x_n}$ on $\{x_n = 0\}$. Since $u^- = u^+$ we also get $u^-_{x_i} = u^+_{x_i}$ on $\{x_n = 0\}$.
Together, we get $D^\alpha u^- = D^\alpha u^+$ on $\{x_n = 0\}$ for $|\alpha| \leq 1$.
We can conclude $\bar{u} \in C^1(B)$.

Direct calculations also gives

$$\|\bar{u}\|_{W^{1,p}(B)} \leq C\|u\|_{W^{1,p}(B^+)}$$

for some constant C which does not depend on u.

Now, we can consider the case when ∂U is not flat near x^0. We can find a C^1 mapping Φ with inverse Ψ such that Φ straightens out ∂U near x^0. We write $y = \Phi(x)$, $x = \Psi(y)$, $u'(y) = u(\Psi(y))$. Then $u'(y)$ can be handled as the previous case to extend u' from B^+ to a function $\bar{u}' \in C^1(B^0(y^0, r))$ with an estimate $\|\bar{u}'\|_{W^{1,p}(B^0(y^0,r))} \leq C\|u'\|_{W^{1,p}(B^+)}$. Let $W = \Psi(B^0(y_0, r))$.



Converting back to x variables, we obtain an extension $\bar{u}$ of u to W, with

$$\|\bar{u}\|_{W^{1,p}(W)} \leq C\|u\|_{W^{1,p}(U)}.$$

Since ∂U is compact, there exist finitely many points $x_i^0 \in \partial U$, open sets W_i, and extensions $\bar{u}_i$ of u to W_i ($i = 1, \ldots, N$), as before, such that $\partial U \subset \bigcup_{i=1}^N W_i$. Take $W_0 \Subset U$ so that $U \subset \bigcup_{i=0}^N W_i$, and let $\{\zeta_i\}_{i=0}^N$ be an associated partition of unity. Write $\bar{u} = \sum_{i=0}^N \zeta_i \bar{u}_i$, where $\bar{u}_0 = u$. Then we get the estimate

$$\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}$$

for some constant C not depending on u. Furthermore we can arrange $\operatorname{spt} \bar{u} \subset V \Supset U$.

Write $Eu = \bar{u}$ and observe that the mapping $u \mapsto Eu$ is linear.
Instead of assuming $u \in C^1(\bar{U})$, we now assume $u \in W^{1,p}(U)$, $1 \leq p < \infty$ and choose $u_m \in C^\infty(\bar{U})$ such that $u_m \to u$ in $W^{1,p}(U)$. We get

$$\|Eu_m - Eu_l\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u_m - u_l\|_{W^{1,p}(U)}.$$

Thus $\{Eu_m\}_{m=1}^\infty$ is a Cauchy sequence so it converges to $\bar{u} = Eu$. This extension does not depend on the approximating sequence and satisfies the conclusions in the theorem. $\square$

Remark

The theorem can without much change be extended to $W^{2,p}(U)$. However, for $W^{k,p}(U)$ with $k > 2$ one needs a more complicated reflection technique.

Traces

Theorem (Trace Theorem)

Assume U is bounded and ∂U is C^1. Then there exists a bounded linear operator

$$T : W^{1,p}(U) \to L^p(\partial U)$$

such that $Tu = u|_{\partial U}$ if $u \in W^{1,p}(U) \cap C(\bar{U})$ and

$$\|Tu\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)} \quad \forall u \in W^{1,p}(U)$$

where C only depends on p and U.
We call Tu the trace of u on ∂U.

Traces

Theorem (Trace Theorem)

Assume U is bounded and ∂U is C^1, $1 \leq p < \infty$. Then there exists a bounded linear operator

$$T : W^{1,p}(U) \to L^p(\partial U)$$

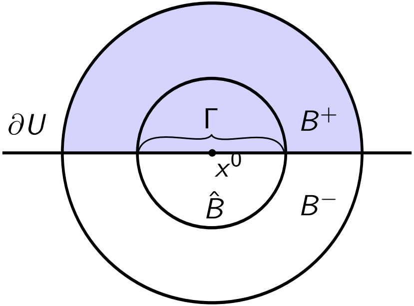
such that $Tu = u|_{\partial U}$ if $u \in W^{1,p}(U) \cap C(\bar{U})$ and

$$\|Tu\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)} \quad \forall u \in W^{1,p}(U)$$

where C only depends on p and U.
We call Tu the trace of u on ∂U.

Proof

Assume first that $u \in C^1(\bar{U})$, $x^0 \in \partial U$ and $\partial U = \{x_n = 0\}$ in an open ball $B = B^0(x_0, r)$ such that $U \cap B = \{x_n > 0\} \cap B$. Let $\hat{B} = B^0(x_0, r/2)$, $\Gamma = \hat{B} \cap \partial U$. Select $\zeta \in C_c^\infty(B)$, with $\zeta \geq 0$ in B and $\zeta = 1$ on $\hat{B}$.



Set $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1} = \{x_n = 0\}$. Then by Young's inequality we get

$$\begin{aligned}
\int_\Gamma |u|^p dx' &\le \int_{\{x_n=0\}} \zeta |u|^p dx' = -\int_{B^+} (\zeta |u|^p)_{x_n} dx \\
&= -\int_{B^+} |u|^p \zeta_{x_n} + p|u|^{p-1}(\operatorname{sgn} u) u_{x_n} \zeta \, dx \\
&\le \int_{B^+} |u|^p \zeta_{x_n} + |u_{x_n}|^p + \frac{p}{q} |u|^{q(p-1)} \zeta^q dx \\
&\le C \int_{B^+} |u|^p + |Du|^p dx,
\end{aligned}$$

where $q^{-1} = 1 - p^{-1}$.

If $x^0 \in \partial U$, but ∂U is not flat near x^0, we straighten out the bounday near x^0 to obtain the previous setting. Changing the variables we still get a bound

$$\int_\Gamma |u|^p dx' \leq C \int_U |u|^p + |Du|^p dx,$$

where Γ is some open neighbourhood of x^0 in ∂U.
Since ∂U is compact, we can choose finitely many points $x_i^0 \in \partial U$ and corresponding open subsets $\Gamma_i \subset \partial U$, $(i = 1, \dots, N)$ such that $\partial U = \bigcup_{i=1}^n \Gamma_i$ and

$$\|u\|_{L^p(\Gamma_i)} \leq C\|u\|_{W^{1,p}(U)}.$$

With the notation $Tu = u|_{\partial U}$, we have

$$\|Tu\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)},$$

for some constant C not depending on u.

If we assume $u \in W^{1,p}(U)$ instead of $u \in C^1(\bar{U})$, then there exist functions $u_m \in C^\infty(\bar{U})$ such that $u_m \to u$ in $W^{1,p}(U)$. Due to the estimate

$$\|Tu_m - Tu_l\|_{L^p(\partial U)} \leq C\|u_m - u_l\|_{W^{1,p}(U)},$$

the sequence $\{Tu_m\}_{m=1}^\infty$ is a Cauchy sequence in $L^p(\partial U)$. We can therefore define

$$Tu = \lim_{m\to\infty} Tu_m,$$

where the limit is taken in $L^p(\partial U)$. This limit does not depend on the choice of approximating functions.

Finally, if $u \in W^{1,p}(U) \cap C(\bar{U})$, then the functions $u_m \in C^\infty(\bar{U})$ converge uniformly to u on $\bar{U}$. Hence, $Tu = u|_{\partial U}$. □

Theorem (Trace-zero functions on $W^{1,p}$)

Assume U is bounded and ∂U is C^1, $u \in W^{1,p}$, $1 \leq p < \infty$. Then

$$u \in W_0^{1,p}(U) \text{ if and only if } Tu = 0 \text{ on } \partial U.$$

Proof

Suppose first that $u \in W_0^{1,p}(U)$. Then by definition we have functions $u_m \in C_c^\infty(U)$ such that

$$u_m \to u \text{ in } W^{1,p}(U).$$

As $Tu_m = 0$ on ∂U and $T : W^{1,p}(U) \to L^p(\partial U)$ is a bounded linear operator, we get $Tu = 0$ on ∂U.

Now, we want to prove the converse.
Assume $Tu = 0$ on ∂U. Using partitions of unity and straightening out the boundary, we may as well assume

$$\begin{cases} u \in W^{1,p}(\mathbb{R}^n_+), u \text{ has compact support in } \bar{\mathbb{R}}^n_+, \\ Tu = 0 \text{ on } \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}. \end{cases}$$

Then, there exist functions $u_m \in C^1(\bar{\mathbb{R}}^n_+)$ such that

$$u_m \to u \text{ in } W^{1,p}(\mathbb{R}^n_+)$$

and, since $Tu = 0$ on $\mathbb{R}^{n-1}$

$$Tu_m = u_m|_{\mathbb{R}^{n-1}} \to Tu = 0 \text{ in } L^p(\mathbb{R}^{n-1}).$$

Now, if $x' \in \mathbb{R}^{n-1}$, $x_n \geq 0$, we have

$$|u_m(x', x_n)| \leq |u_m(x', 0)| + \int_0^{x_n} |u_{m,x_n}(x', t)| dt$$
$$\leq |u_m(x', 0)| + x_n^{1-1/p} \Big(\int_0^{x_n} |Du_m(x', t)|^p dt \Big)^{1/p}.$$

For fixed x_n this gives

$$\|u_m(x', x_n)\|_{L^p(\mathbb{R}^{n-1})} \leq \|u_m(x', 0)\|_{L^p(\mathbb{R}^{n-1})} + \Big(x_n^{p-1} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} |Du_m(x', t)|^p dx' dt \Big)^{1/p}.$$

Letting $m \to \infty$ we get

$$\|u(x', x_n)\|_{L^p(\mathbb{R}^{n-1})}^p \leq x_n^{p-1} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} |Du(x', t)|^p dx' dt \tag{6}$$

for a. e. $x_n > 0$.

Let $\zeta \in C^\infty(\mathbb{R}_+)$ satisfy

$$\zeta = 1 \text{ on } [0,1], \zeta = 0 \text{ on } \mathbb{R}_+ - [0,2], 0 \le \zeta \le 1,$$

and define

$$\begin{cases} \zeta_m(x) = \zeta(mx_n) \quad (x \in \mathbb{R}^n_+) \\ w_m(x) = u(x)(1 - \zeta_m(x)). \end{cases} \quad \Rightarrow \begin{cases} w_{m,x_n} = u_{x_n}(1 - \zeta_m) - mu\zeta'(mx_n) \\ D_{x'} w_m = D_{x'} u(1 - \zeta_m). \end{cases}$$

We get $|Dw_m - Du| \le |\zeta_m||Du| + m|u||\zeta'(mx_n)|$ and

$$\|Dw_m - Du\|_{L^p(\mathbb{R}^n_+)} \le \underbrace{\|\zeta_m Du\|_{L^p(\mathbb{R}^n_+)}}_{\to 0} + C\Big(m^p \int_0^{2/m} \int_{\mathbb{R}^{n-1}} |u|^p dx' dx_n\Big)^{1/p}.$$

since $\zeta_m \neq 0$ only if $0 \le x_n \le 2/m$.

Using (6) we get

$$
\begin{aligned}
\|Dw_m - Du\|^p_{L^p(\mathbb{R}^n_+)} &\le Cm^p \int_0^{2/m} \|u(x',x_n)\|^p_{L^p(\mathbb{R}^{n-1})} dx_n \\
&\le Cm^p \int_0^{2/m} x_n^{p-1} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} |Du(x',t)|^p dx' dt dx_n \\
&\le Cm^p \Big(\int_0^{2/m} x_n^{p-1} dx_n \Big) \Big(\int_0^{2/m} \int_{\mathbb{R}^{n-1}} |Du(x',t)|^p dx' dt \Big) \\
&\le C \Big(\int_0^{2/m} \int_{\mathbb{R}^{n-1}} |Du(x',t)|^p dx' dt \Big) \to 0 \text{ as } m \to \infty.
\end{aligned}
$$

Since also $w_m \to u$ in $L^p(\mathbb{R}^n_+)$ we get $w_m \to u$ in $W^{1,p}(\mathbb{R}^n_+)$.
Note that $w_m = 0$ if $0 < x_n < 1/m$ so we can mollify w_m to produce $u_m \in C_c^\infty(\mathbb{R}^n_+)$ such that $u_m \to u$ in $W^{1,p}(\mathbb{R}^n_+)$. We therefore conclude $u \in W_0^{1,p}(\mathbb{R}^n_+)$. □

Sobolev inequalities

Sometimes one would like to know if a function belongs to $W^{k,p}(U)$, does it then automatically belong to some other space.
We will focus our attention on the cases $1 \leq p < n$.

- We would like to establish an inequality

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)},$$

for certain constants $1 \leq p < n$, $1 \leq q < \infty$ and $C > 0$ and all $u \in C_c^\infty(\mathbb{R}^n)$.

- We would like to establish an inequality

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)},$$

for certain constants $1 \leq p < n$, $1 \leq q < \infty$ and $C > 0$ and all $u \in C_c^\infty(\mathbb{R}^n)$.
- This can only work for a specific q depending on p and n.

Assume $1 \leq p < n$ and

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)} \quad \forall u \in C_c^\infty(\mathbb{R}^n).$$

Let $u \in C_c^\infty(\mathbb{R}^n)$, not identically zero. Define for $\lambda > 0$ a rescaling $u_\lambda(x) = u(\lambda x)$. The inequality gives $\|u_\lambda\|_{L^q(\mathbb{R}^n)} \leq C\|Du_\lambda\|_{L^p(\mathbb{R}^n)}$.

$$\int_{\mathbb{R}^n} |u_\lambda|^q dx = \int_{\mathbb{R}^n} |u(\lambda x)|^q dx = \frac{1}{\lambda^n} \int_{\mathbb{R}^n} |u(y)|^q dy,$$
$$\int_{\mathbb{R}^n} |Du_\lambda|^p dx = \lambda^p \int_{\mathbb{R}^n} |Du(\lambda x)|^p dx = \frac{\lambda^p}{\lambda^n} \int_{\mathbb{R}^n} |Du(y)|^p dy.$$

Hence,

$$\frac{1}{\lambda^{n/q}}\|u\|_{L^q(\mathbb{R}^n)} \leq C\frac{\lambda}{\lambda^{n/p}}\|Du\|_{L^p(\mathbb{R}^n)} \quad \Rightarrow \quad \|u\|_{L^q(\mathbb{R}^n)} \leq C\lambda^{1-n/p+n/q}\|Du\|_{L^p(\mathbb{R}^n)}$$

Unless $1 - n/p + n/q = 0$ we can get a contradiction (u identically zero) if we let $\lambda \to 0$ or $\lambda \to \infty$. Hence, we must choose $q = \frac{np}{n-p}$.

Definition

If $1 \leq p < n$, the Sobolev conjugate of p is

$$p^* = \frac{np}{n-p}.$$

Observe that $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ and $p^* > p$.

Theorem (Gagliardo-Nirenberg-Sobolev inequality)

Assume $1 \leq p < n$. There exists a constant C, depending only on p and n, such that

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)} \quad \forall u \in C_c^\infty(\mathbb{R}^n).$$

Remark

Compact support is needed because constant functions does not satisfy the inequality.
The constant C does not depend on the size of the support of u.

Proof

First assume $p = 1$.

Since u has compact support, for each $i = 1, \ldots, n$ and $x \in \mathbb{R}^n$ we have

$$u(x) = \int_{-\infty}^{x_i} u_{x_i}(x_1, \ldots, y_i, \ldots, x_n) dy_i,$$

which gives

$$|u(x)| \leq \int_{-\infty}^{\infty} |Du(x_1, \ldots, y_i, \ldots, x_n)| dy_i.$$

Hence,

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du(x_1, \ldots, y_i, \ldots, x_n)| dy_i \right)^{\frac{1}{n-1}}.$$

Integrate with respect to x_1

$$\int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}} dx_1 \leq \int_{-\infty}^{\infty} \prod_{i=1}^{n} \Big(\int_{-\infty}^{\infty} |Du| dy_i \Big)^{\frac{1}{n-1}} dx_1$$
$$\leq \Big(\int_{-\infty}^{\infty} |Du| dy_1 \Big)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \Big(\int_{-\infty}^{\infty} |Du| dy_i \Big)^{\frac{1}{n-1}} dx_1$$
$$\leq \Big(\int_{-\infty}^{\infty} |Du| dy_1 \Big)^{\frac{1}{n-1}} \Big(\prod_{i=2}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dx_1 dy_i \Big)^{\frac{1}{n-1}},$$

where the last inequality comes from the general Hölder inequality with $p = n - 1$:

$$\int_U \prod_{i=2}^{n} |u_i| dx_1 \leq \prod_{i=2}^{n} \|u_i\|_{L^{n-1}(U)}$$

Integrate with respect to x_2

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|u|^{\frac{n}{n-1}}dx_1dx_2 \le \Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|dx_1dy_2\Big)^{\frac{1}{n-1}}\int_{-\infty}^{\infty}\prod_{i=1,i\neq 2}^{n} I_i^{\frac{1}{n-1}}dx_2,$$

$$I_1 = \int_{-\infty}^{\infty}|Du|dy_1, \qquad I_i = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|dx_1dy_i \quad (i = 3, \dots, n).$$

Applying the general Hölder inequality again gives

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|u|^{\frac{n}{n-1}}dx_1dx_2 \le \Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|dx_1dy_2\Big)^{\frac{1}{n-1}}\Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|dy_1dx_2\Big)^{\frac{1}{n-1}}$$
$$\times\prod_{i=3}^{n}\Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|dx_1dx_2dy_i\Big)^{\frac{1}{n-1}}.$$

Continue integrating with respect to $x_3, \ldots, x_n$ eventually gives

$$\begin{aligned} \int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx &\leq \prod_{i=1}^{n} \Big(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |Du| dx_1 \ldots dy_i \ldots dx_n \Big)^{\frac{1}{n-1}} \\ &= \Big(\int_{\mathbb{R}^n} |Du| dx \Big)^{\frac{n}{n-1}} \end{aligned} \tag{7}$$

This is the desired inequality for $p = 1$.

Now consider the case $1 < p < n$. Apply the estimate (7) to $v = |u|^\gamma$ for some $\gamma > 1$.

$$\Big(\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}} dx\Big)^{\frac{n-1}{n}} \leq \int_{\mathbb{R}^n} |D|u|^\gamma| dx = \gamma \int_{\mathbb{R}^n} |u|^{\gamma-1} |Du| dx$$
$$\leq \gamma \Big(\int_{\mathbb{R}^n} |u|^{(\gamma-1)\frac{p}{p-1}} dx\Big)^{\frac{p-1}{p}} \Big(\int_{\mathbb{R}^n} |Du|^p dx\Big)^{\frac{1}{p}}.$$

Choose γ so that $\frac{\gamma n}{n-1} = (\gamma - 1)\frac{p}{p-1} \Rightarrow \gamma = \frac{p(n-1)}{n-p} > 1 \Rightarrow \frac{\gamma n}{n-1} = \frac{np}{n-p} = p^*$.

$$\Big(\int_{\mathbb{R}^n} |u|^{p^*} dx\Big)^{\frac{1}{p*}} \leq \gamma \Big(\int_{\mathbb{R}^n} |Du|^p dx\Big)^{\frac{1}{p}}.$$

This gives the desired inequality with $C = \gamma = \frac{p(n-1)}{n-p}$. $\square$

Theorem (Estimates for $W^{1,p}$, $1 \leq p < n$.)

Let $U \subset \mathbb{R}^n$ be bounded, open, such that ∂U is C^1. Assume $1 \leq p < n$ and $u \in W^{1,p}(U)$. Then $u \in L^{p^*}$, with the estimate

$$\|u\|_{L^{p^*}(U)} \leq C\|u\|_{W^{1,p}(U)},$$

where C only depends on p, n and U.

Proof

The conditions in the extension theorem are satisfied, so we get $\bar{u} = Eu \in W^{1,p}(\mathbb{R}^n)$ such that

$$\begin{cases} \bar{u} = u \text{ in } U, \\ \operatorname{spt} \bar{u} \text{ is compact}, \\ \|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}. \end{cases}$$

Because $\bar{u}$ has compact support, we can use the local approximation theorem, and get a sequence for functions $u_m \in C_c^\infty(\mathbb{R}^n)$ such that

$$u_m \to \bar{u} \text{ in } W^{1,p}(\mathbb{R}^n).$$

Now, from the Gagliardo-Nirenberg-Sobolev inequality we get

$$\|u_m - u_l\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du_m - Du_l\|_{L^p(\mathbb{R}^n)} \quad \forall l, m \geq 1.$$

Hence, $\{u_m\}_{m=1}^\infty$ is a Cauchy sequence also in $L^{p^*}(\mathbb{R}^n)$ and therefore

$$u_m \to \bar{u} \text{ in } L^{p^*}(\mathbb{R}^n).$$

We also have

$$\|u_m\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du_m\|_{L^p(\mathbb{R}^n)} \quad \forall m \geq 1.$$

Taking the limit $m \to \infty$ gives

$$\|\bar{u}\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|D\bar{u}\|_{L^p(\mathbb{R}^n)},$$

$$\|u\|_{L^{p^*}(U)} = \|\bar{u}\|_{L^{p^*}(U)} \leq \|\bar{u}\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|D\bar{u}\|_{L^p(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}.$$

□

Theorem (Estimates for $W_0^{1,p}$, $1 \leq p < n$.)

Let $U \subset \mathbb{R}^n$ be bounded, open, such that ∂U is C^1. Assume $1 \leq p < n$ and $u \in W_0^{1,p}(U)$. Then we have the estimate

$$\|u\|_{L^q(U)} \leq C\|Du\|_{L^p(U)}$$

for each $1 \leq q \leq p^*$, C only depends on p, q, n and U.
In particular, for all $1 \leq p \leq \infty$,

$$\|u\|_{L^p(U)} \leq C\|Du\|_{L^p(U)}.$$

Remark

This is often called a Poincaré inequality.

Proof.

Since $u \in W_0^{1,p}(U)$, there are functions $u_m \in C_c^\infty(U)$ such that $u_m \to u$ in $W^{1,p}(U)$. Extend u_m to be 0 on $\mathbb{R}^n - U$ and apply the Gagliardo-Nirenberg-Sobolev inequality to get

$$\|u_m\|_{L^{p^*}(U)} \leq C\|Du_m\|_{L^p(U)}.$$

The argument with the Cauchy sequeces gives $u_m \to u$ in $L^{p^*}(U)$.
Hence, in the limit we get

$$\|u\|_{L^{p^*}(U)} \leq C\|Du\|_{L^p(U)}.$$

The domain U has finite measure which gives $\|u\|_{L^q} \leq C\|u\|_{L^{p^*}(U)}$ if $1 \leq q \leq p^*$. $\square$

Remark

On $W_0^{1,p}(U)$, the norms $\|Du\|_{L^p(U)}$ and $\|u\|_{W^{1,p}(U)}$ are equivalent.

Remark

From the limit $p^* = \frac{np}{n-p} \to \infty$ as $p \to n$, one would guess $u \in W^{1,n}(U)$ implies $u \in L^\infty(U)$, but this is not true for $n > 1$. A counterexample is given by $u = \log\log\left(1 + \frac{1}{|x|}\right) \in W^{1,n}(B^0(0,1))$, but does not belong to $L^\infty(B^0(0,1))$. (One of the exercises to prove this.)

Theorem (General Sobolev inequalities)

Let $U \subset \mathbb{R}^n$ be open, bounded such that ∂U is C^1. Assume $u \in W^{k,p}(U)$ with

$$k < \frac{n}{p},$$

then $u \in L^q(U)$, where

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n},$$

and we have the estimate

$$\|u\|_{L^q(U)} \leq C\|u\|_{W^{k,p}(U)},$$

where C only depends on k, p, n and U.

Proof.

From the assumptions we get $D^\alpha u \in W^{1,p}(U)$ for all $|\alpha| \leq k-1$. We therefore have

$$\|D^\alpha u\|_{L^{p^*}(U)} \leq C\|u\|_{W^{k,p}(U)} \quad \text{if } |\alpha| \leq k-1,$$

which means $u \in W^{k-1,p^*}(U)$. Similarly, we find $u \in W^{k-2,p^{**}}(U)$, where

$$\frac{1}{p^{**}} = \frac{1}{p^*} - \frac{1}{n} = \frac{1}{p} - \frac{2}{n}.$$

After k steps like this we find $u \in W^{0,q}(U) = L^q(U)$ for $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$. The estimate follows from the chain of estimates in the k steps. $\square$

Theorem (Morrey's inequality)

Assume $U \subset \mathbb{R}^n$ is bounded open with C^1 boundary and $n < p \leq \infty$, and $u \in W^{1,p}(U)$. Then u has a version $u^* \in C^{0,\gamma}(\bar{U})$, for $\gamma = 1 - \frac{n}{p}$, and a constant C depending only on p, n and U such that

$$\|u^*\|_{C^{0,\gamma}(\bar{U})} \leq C\|u\|_{W^{1,p}(U)}$$

The Hölder space $C^{0,\gamma}(\bar{U})$ has the norm

$$\|u\|_{C^{0,\gamma}(\bar{U})} = \|u\|_{C(\bar{U})} + \sup_{\substack{x,y \in U \\ x \neq y}} \left\{ \frac{|u(x) - u(y)|}{|x - y|^{\gamma}} \right\}$$

Theorem (Consequence of Morrey's inequality)

Let $U \subset \mathbb{R}^n$ be a bounded, open with C^1 boundary. Assume $u \in W^{k,p}(U)$ with $n < kp \leq \infty$, then

$$\|u\|_{L^\infty(U)} \leq C\|u\|_{W^{k,p}(U)},$$

where the constant C only depends on k, p, n and U.

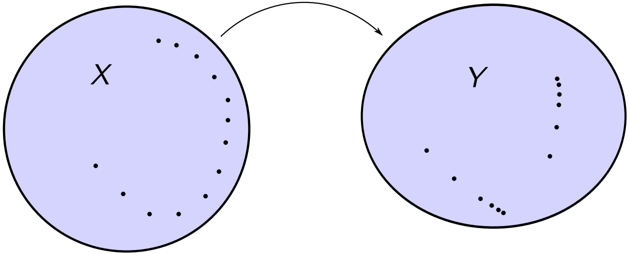
The L^∞ norm is a part of the Hölder norm.

Compactness

Definition

Let X and Y be Banach spaces, $X \subset Y$. We say that X is compactly embedded in Y ($X \Subset Y$) if

1. $\|x\|_Y \leq C\|x\|_X$ for all $x \in X$
2. Each bounded sequence in X is precompact in Y, i.e. has a subsequence that converges in Y.



Theorem (Rellich-Kondrachov Compactness Theorem)

Assume $U \subset \mathbb{R}^n$ is bounded open with C^1 boundary and $1 \leq p < n$. Then

$$W^{1,p}(U) \Subset L^q(U)$$

for each $1 \leq q < p^* = \frac{np}{n-p}$.

Proof

Fix $1 \leq q < p^*$. The Sobolev inequality gives

$$W^{1,p}(U) \subset L^q(U), \quad \|u\|_{L^q(U)} \leq C\|u\|_{W^{1,p}(U)}.$$

Assume $\{u_m\}_{m=1}^{\infty}$ is a bounded sequence in $W^{1,p}(U)$.
We would like to find a subsequence $\{u_{m_j}\}_{j=1}^{\infty}$ which converges in $L^q(U)$.
By the extension theorem, we can assume $u_m \in W^{1,p}(\mathbb{R}^n)$ and u_m is compactly supported in V_ϵ for all $1 \leq n < \infty$ and some bounded open set $V \subset \mathbb{R}^n$.
Here $V_\epsilon = \{x \in V | \operatorname{dist}(x, \partial V) > \epsilon\}$.
The mollified functions $u_m^\epsilon = \eta_\epsilon * u_m \in C_c^\infty(V)$.

We want to prove $u_m^\epsilon \to u_m$ in $L^q(V)$ uniformly in m.
First note that if u_m is smooth, then

$$
\begin{aligned}
u_m^\epsilon(x) - u_m(x) &= \int_{B(0,1)} \eta(y)(u_m(x-\epsilon y) - u_m(x))dy \\
&= \int_{B(0,1)} \eta(y) \int_0^1 \frac{d}{dt}(u_m(x-\epsilon ty))dtdy \\
&= -\epsilon \int_{B(0,1)} \eta(y) \int_0^1 Du_m(x-\epsilon ty)\cdot y \; dtdy. \\
\int_V |u_m^\epsilon(x) - u_m(x)|dx &\leq \epsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_V |Du_m(x-\epsilon ty)|dxdtdy \\
&\leq \epsilon \int_V |Du_m(z)|dz.
\end{aligned}
$$

By approximation by smooth functions this estimate holds also if $u_m \in W^{1,p}(V)$.

Together with the fact that V is bounded and the assumtion that $\{u_m\}_{m=1}^{\infty}$ is bounded in $W^{1,p}(V)$ this gives

$$\|u_m^\epsilon - u_m\|_{L^1(V)} \leq \epsilon \|Du_m\|_{L^1(V)} \leq \epsilon C \|Du_m\|_{L^p(V)} \leq \epsilon C \underbrace{\sup_m \|u_m\|_{W^{1,p}(V)}}_{<\infty}$$

Hence, $u_m^\epsilon \to u_m$ in $L^1(V)$ uniformly in m.
Since $1 \leq q < p^*$, we can use the interpolation inequality for L^p-norms to get

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \leq \|u_m^\epsilon - u_m\|_{L^1(V)}^{\theta} \|u_m^\epsilon - u_m\|_{L^{p^*}(V)}^{1-\theta},$$

where $\frac{1}{q} = \theta + \frac{(1-\theta)}{p^*}$, $0 < \theta < 1$. We can estimate the $L^{p^*}(V)$ norm with the Gagliardo-Nirenberg-Sobolev inequality and bound it by a constant because $\{u_m\}_{m=1}^{\infty}$ is bounded in $W^{1,p}(V)$.

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \leq C \|u_m^\epsilon - u_m\|_{L^1(V)}^{\theta} \leq \epsilon^{\theta} C.$$

Hence, $u_m^\epsilon \to u_m$ in $L^q(V)$ uniformly in m.

We now want to show that for each fixed $\epsilon > 0$, the sequence $\{u_m^\epsilon\}_{m=1}^\infty$ is uniformly bounded and equicontinuous.

If $x \in \mathbb{R}^n$, then

$$|u_m^\epsilon(x)| \leq \int_{B(x,\epsilon)} \eta_\epsilon(x-y)|u_m(y)|dy \leq \|\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)}\|u_m\|_{L^1(V)} \leq \frac{C}{\epsilon^n} < \infty,$$

where C does not depend on m. Similarly

$$|Du_m^\epsilon(x)| \leq \int_{B(x,\epsilon)} |D\eta_\epsilon(x-y)||u_m(y)|dy \leq \|D\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)}\|u_m\|_{L^1(V)} \leq \frac{C}{\epsilon^{n+1}} < \infty.$$

Hence, the sequence $\{u_m\}_{m=1}^\infty$ is uniformly bounded and equicontinuous.

Now, fix $\delta > 0$. We will show there exists a subsequence $\{u_{m_j}\}_{j=1}^{\infty}$ such that

$$\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^p(V)} \leq \delta.$$

To see this, we use the uniform convergence in $L^q(V)$ to select $\epsilon > 0$ such that

$$\|u_m^\epsilon - u_m\|_{L^p(V)} \leq \delta/2 \qquad \forall m.$$

As the functions $\{u_m^\epsilon\}_{m=1}^{\infty}$ are all supported in the bounded set V, uniformly bounded and equicontinuous, we can use the Arzela-Ascoli theorem to obtain a subsequence $\{u_{m_j}^\epsilon\}_{j=1}^{\infty}$ which converges uniformly on V. In particular

$$\limsup_{j,k\to\infty} \|u_{m_j}^\epsilon - u_{m_k}^\epsilon\|_{L^q(V)} = 0.$$

Together we get

$$\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^p(V)} \leq \delta.$$

Now, we can use a diagonal argument: For each $l = 1, \ldots,$ we can choose $\delta_l = 1/l$, and a subsequence $\{u^{\epsilon}_{m_{l,j}}\}_{j=1}^{\infty}$

$$\limsup_{j,k\to\infty} \|u_{m_{l,j}} - u_{m_{l,k}}\|_{L^p(V)} \leq \delta_l = 1/l.$$

Hence, the diagonal sequence $\{u_{m_l} = u_{m_{l,l}}\}_{l=1}^{\infty}$ converges in $L^p(V)$. $\square$

Theorem

Assume $U \subset \mathbb{R}^n$ *is bounded open with* C^1 *boundary. Then* $W^{1,p}(U) \Subset L^p(U)$ *for all* $1 \leq p \leq \infty$.

Proof

If $1 \leq p < n$, then $p^* > p$, this follows from the Rellich-Kondrachov compactness theorem.

In general we have $\|u\|_{L^p(U)} \leq \|u\|_{W^{1,p}(U)}$, so we just need to check that bounded sequences in $W^{1,p}(U)$ are precompact in $L^p(U)$.

For the case $n \leq p < \infty$, we assume that a sequence $\{u_m\}_{m=1}^{\infty}$ is bounded in $W^{1,p}(U)$, but then it is also bounded in $W^{1,\tilde{p}}(U)$, where

$$\tilde{p} = \left(\frac{1}{n} + \frac{1}{2p}\right)^{-1} < n.$$

The Rellich-Kondrachov compactness theorem gives a subsequence that converges in $L^p(U)$, because $p < \tilde{p}^* = 2p$.

$$W^{1,p}(U) \subset W^{1,\tilde{p}}(U) \Subset L^p(U)$$

The case $p = \infty$ follows from Arzela-Ascoli.

For the case $p > n$ we alternatively use

$$W^{1,p}(U) \subset C^{0,\gamma}(\bar{U}) \Subset C(\bar{U}) \subset L^p(U)$$

where the first embedding is Morrey's inequality, the compact embedding follows from Arzela-Ascoli. $\square$

Theorem

Assume $U \subset \mathbb{R}^n$ *is bounded open (no condition on the boundary). Then* $W_0^{1,p}(U) \Subset L^p(U)$ *for all* $1 \leq p \leq \infty$.

Proof.

Only the extension theorem needed the C^1 boundary condition. For $u \in W_0^{1,p}(U)$ we can extend with 0. $\square$

Notation

The average of u over U will be denoted $(u)_U = \fint_U u dy$. Also $(u)_{x,r} = \fint_{B(x,r)} u dy$

Theorem (Poincaré inequality)

Let U be a bounded, connected, open subset of $\mathbb{R}^n$, with C^1 boundary. Assume $1 \leq p \leq \infty$. Then there exists a constant C, depending only on n, p and U, such that

$$\|u - (u)_U\|_{L^p(U)} \leq C\|Du\|_{L^p(U)}$$

for each function $u \in W^{1,p}(U)$.

Proof

We argue by contradiction. If the estimate would be false, there would exist for each integer $k = 1, \ldots,$ a function $u_k \in W^{1,p}(U)$ satisfying

$$\|u_k - (u_k)_U\|_{L^p(U)} > k\|Du_k\|_{L^p(U)}.$$

Let

$$v_k = \frac{u_k - (u_k)_U}{\|u_k - (u_k)_U\|_{L^p(U)}}$$

Then $(v_k)_U = 0$ and $\|v_k\|_{L^p(U)} = 1$ and $\|Dv_k\|_{L^p(U)} < 1/k$.
In particular the functions $\{v_k\}_{k=1}^{\infty}$ are bounded in $W^{1,p}(U)$.
$W^{1,p}(U) \Subset L^p(U)$ gives that there exist a subsequence $\{v_{k_j}\}_{j=1}^{\infty}$ and a function $v \in L^p(U)$ such that

$$v_{k_j} \to v \text{ in } L^p(U).$$

We get $(v)_U = 0$ and $\|v\|_{L^p(U)} = 1$.

On the other hand $\|Dv_k\|_{L^p(U)} < 1/k$ implies

$$\int_U v\phi_{x_i}dx = \lim_{j\to\infty}\int_U v_{k_j}\phi_{x_i}dx = -\lim_{j\to\infty}\int_U v_{k_j,x_i}\phi dx = 0 \quad \forall\phi \in C_c^\infty(U).$$

Consequently, $v \in W^{1,p}(U)$ with $Dv = 0$ a.e. Thus v is constant, since U is connected. Together with $(v)_U = 0$ we get $v = 0$ a.e. which means $\|v\|_{L^p(U)} = 0$, but we had $\|v\|_{L^p(U)} = 1$, a contradition. $\square$

Theorem (Poincaré inequality on a ball)

Assume $1 \leq p \leq \infty$. Then there exists a constant C, depending only on n and p, such that

$$\|u - (u)_{x,r}\|_{L^p(B^0(x,r))} \leq Cr\|Du\|_{L^p(B^0(x,r))}$$

for each ball $B(x,r) \subset \mathbb{R}^n$ and each function $u \in W^{1,p}(B^0(x,r))$.

Proof.

The case $U = B(0,1)$ follows from the previous theorem. If $u \in W^{1,p}(B^0(x,r))$, we define

$$v(y) = u(x + ry) \quad y \in B(0,1).$$

Then $v \in W^{1,p}(B^0(0,1))$, and

$$\|v - (v)_{0,1}\|_{L^p(B^0(0,1))} \leq C\|Dv\|_{L^p(B^0(0,1))}.$$

Changing variables gives the desired estimate. $\square$

Definition

A locally integrable function u has *bounded mean oscillation* $u \in BMO(\mathbb{R}^n)$ if

$$\sup_{B(x,r)} \left\{ ⨍_{B(x,r)} |u - (u)_{x,r}| dy \right\} < \infty$$

Theorem

Assume $u \in W^{1,n}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. *Then* $u \in BMO(\mathbb{R}^n)$.

Proof

The Poincaré inequality with $p = 1$ on an arbitrary ball $B(x, r)$ gives

$$\begin{aligned} ⨍_{B(x,r)} |u - (u)_{x,r}| dy &\leq Cr ⨍_{B(x,r)} |Du| dy \\ &\leq Cr \left(⨍_{B(x,r)} |Du|^n dy \right)^{1/n} \leq C \left(\int_{\mathbb{R}^n} |Du|^n dy \right)^{1/n} \end{aligned}$$

Fractional Sobolev spaces

Theorem (H^k in terms of Fourier transform.)

Let k be a nonnegative integer.

1. *A complex valued function $u \in L^2(\mathbb{R}^n)$ belongs to $H^k(\mathbb{R}^n)$ if and only if*

$$(1 + |y|^k)\hat{u} \in L^2(\mathbb{R}^n).$$

2. *In addition, there exists a positive constant C such that*

$$\frac{1}{C}\|u\|_{H^k(\mathbb{R}^n)} \leq \|(1 + |y|^k)\hat{u}\|_{L^2(\mathbb{R}^n)} \leq C\|u\|_{H^k(\mathbb{R}^n)}$$

for each $u \in H^k(\mathbb{R}^n)$.

Proof

Assume first that $u \in H^k(\mathbb{R}^n)$. Then for each multiindex $|\alpha| \leq k$, we have $D^\alpha u \in L^2(\mathbb{R}^n)$. If $v \in C^k$ with compact support, we get

$$\widehat{D^\alpha v}(y) = (iy)^\alpha \hat{v}(y).$$

Approximating with smooth functions gives $\widehat{D^\alpha u}(y) = (iy)^\alpha \hat{u}(y) \in L^2(\mathbb{R}^n)$ for $|\alpha| \leq k$ due to Plancherel's Theorem. In particular

$$\||y_i|^k |\hat{u}|\|^2_{L^2(\mathbb{R}^n)} = \|(\partial_{x_i})^k u\|^2_{L^2(\mathbb{R}^n)}.$$

Which gives

$$\||y|^k |\hat{u}|\|^2_{L^2(\mathbb{R}^n)} \leq C\|D^k u\|^2_{L^2(\mathbb{R}^n)}.$$

Hence,

$$\|(1 + |y|^k)\hat{u}\|_{L^2(\mathbb{R}^n)} \leq \|u\|_{L^2(\mathbb{R}^n)} + \||y|^k |\hat{u}|\|_{L^2(\mathbb{R}^n)} \leq C\|u\|_{H^k(\mathbb{R}^n)} < \infty$$

Conversely, assume $(1+|y|^k)\hat{u} \in L^2(\mathbb{R}^n)$ and $|\alpha| \le k$. Then

$$\|(iy)^\alpha \hat{u}\|^2_{L^2(\mathbb{R}^n)} \le \int_{\mathbb{R}^n} |y|^{2|\alpha|} |\hat{u}|^2 dy \le C\|(1+|y|^k)\hat{u}\|^2_{L^2(\mathbb{R}^n)}$$

Let u_α be the inverse Fourier transform of $(iy)^\alpha \hat{u}$. Then for each $\phi \in C_c^\infty(\mathbb{R}^n)$ we get

$$\int_{\mathbb{R}^n} (D^\alpha \phi)\bar{u} dx = \int_{\mathbb{R}^n} (\widehat{D^\alpha \phi})\bar{\hat{u}} dy = \int_{\mathbb{R}^n} (iy)^\alpha \hat{\phi} \bar{\hat{u}} dy = (-1)^{|\alpha|} \int_{\mathbb{R}^n} \phi \bar{u_\alpha} dx.$$

Hence, $u_\alpha = D^\alpha u$ in the weak sense, and $D^\alpha u \in L^2(\mathbb{R}^n)$. This means $u \in H^k(\mathbb{R}^n)$. $\square$

We can use this idea to define *fractional Sobolev spaces*.

Definition

Assume $0 < s \in \mathbb{R}$ and $u \in L^2(\mathbb{R}^n)$. Then $u \in H^s(\mathbb{R}^n)$ if $(1 + |y|^s)\hat{u} \in L^2(\mathbb{R}^n)$. For non-integer s, we set

$$\|u\|_{H^s(\mathbb{R}^n)} = \|(1 + |y|^s)\hat{u}\|_{L^2(\mathbb{R}^n)}.$$

Theorem

If $u \in H^s(\mathbb{R}^n)$ *with* $s > n/2$, *then* $u \in L^\infty(\mathbb{R}^n)$ *and*

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C\|u\|_{H^s(\mathbb{R}^n)},$$

where C *only depends on* s *and* n.

Proof

Assume $u \in H^s(\mathbb{R}^n)$ with $s > n/2$.

$$\|\hat{u}\|_{L^1(\mathbb{R}^n)} \leq \|(1+|y|^s)^{-1}\|_{L^2(\mathbb{R}^n)}\|(1+|y|^s)\hat{u}\|_{L^2(\mathbb{R}^n)}$$
$$= \int_{\mathbb{R}^n}(1+|y|^s)^{-2}dy\|u\|_{H^s(\mathbb{R}^n)} \leq C\|u\|_{H^s(\mathbb{R}^n)} < \infty.$$

Hence, $\hat{u} \in L^1(\mathbb{R}^n)$ and for a. e. $x \in \mathbb{R}^n$, we can use the inverse Fourier transform

$$|u(x)| = \frac{1}{(2\pi)^{n/2}}\left|\int_{\mathbb{R}^n} e^{ix\cdot y}\hat{u}(y)dy\right| \leq \frac{1}{(2\pi)^{n/2}}\|\hat{u}\|_{L^1(\mathbb{R}^n)} \leq C\|u\|_{H^s(\mathbb{R}^n)}.$$

Hence,

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C\|u\|_{H^s(\mathbb{R}^n)},$$

□

Definition

The dual space of $H_0^1(U)$ is $H^{-1}(U)$.
$\langle\ ,\ \rangle$ denotes the pairing between $H^{-1}(U)$ and $H_0^1(U)$.
For $f \in H^{-1}(U)$ we define the norm

$$\|f\|_{H^{-1}(U)} = \sup\Big\{\langle f, u\rangle | u \in H_0^1(U), \|u\|_{H_0^1(U)} \leq 1\Big\}$$

Theorem (Characterization of H^{-1})

1. *Assume $f \in H^{-1}(U)$. Then there exist functions $f^0, f^1, \dots, f^n \in L^2(U)$ such that*

$$\langle f, v\rangle = \int_U f^0 v + \sum_{i=1}^{n} f^i v_{x_i}\, dx \qquad (v \in H_0^1(U)). \tag{8}$$

2. *Furthermore,*

$$\|f\|_{H^{-1}(U)} = \inf\Big\{\Big(\int_U \sum_{i=0}^{n} |f^i|^2\Big)^{1/2} \Big| f \text{ satisfies (8) for } f^0, \dots, f^n \in L^2(U)\Big\}.$$

Proof.

From the book. $\square$

Extra material.

Theorem (Morray's inequality for $C^1(\mathbb{R}^n)$)

Assume $n < p \leq \infty$. Then there exists a constant C, depending only on p and n, such that

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in C^1(\mathbb{R}^n)$, where $\gamma = 1 - n/p$.

Proof

Choose a ball $B(x,r) \in \mathbb{R}^n$.

We want to prove

$$\fint_{B(x,r)} |u(y) - u(x)| dy \leq C \int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}} dy, \tag{9}$$

where C only depends on n.

Fix any point $w \in \partial B(0,1)$. Then if $0 < s < r$,

$$\begin{aligned} |u(x+sw) - u(x)| &= \left| \int_0^s \frac{d}{dt} u(x+tw) dt \right| \\ &= \left| \int_0^s Du(x+tw) \cdot w dt \right| \leq \int_0^s |Du(x+tw)| dt. \end{aligned}$$

Integrate over $\partial B(0,1)$

$$\int_{\partial B(0,1)} |u(x+sw)-u(x)|dS \leq \int_0^s \int_{\partial B(0,1)} |Du(x+tw)|dSdt$$
$$= \int_0^s \int_{\partial B(0,1)} |Du(x+tw)|\frac{t^{n-1}}{t^{n-1}}dSdt$$

Let $y = x + tw \Rightarrow t = |x-y|$. From polar coordinates we get

$$\int_{\partial B(0,1)} |u(x+sw)-u(x)|dS \leq \int_{B(x,s)} \frac{|Du(y)|}{|x-y|^{n-1}}dy \leq \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}}dy.$$

Multiply by s^{n-1} and integrate s from 0 to r gives

$$\int_{B(x,r)} |u(y)-u(x)|dy \leq \frac{r^n}{n} \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}}dy.$$

Hence, we have (9).

Fix $x \in \mathbb{R}^n$ and apply (9).

$$
\begin{aligned}
|u(x)| &\leq \fint_{B(x,1)} |u(x) - u(y)| dy + \fint_{B(x,1)} |u(y)| dy \\
&\leq C \int_{B(x,1)} \frac{|Du(y)|}{|x-y|^{n-1}} dy + C\|u\|_{L^p(B(x,1))} \\
&\leq C \Big(\int_{\mathbb{R}^n} |Du|^p dy \Big)^{1/p} \Big(\int_{B(x,1)} \frac{dy}{|x-y|^{(n-1)p/(p-1)}} \Big)^{(p-1)/p} + C\|u\|_{L^p(B(x,1))} \\
&\leq C\|u\|_{W^{1,p}(\mathbb{R}^n)},
\end{aligned}
$$

because $p > n$ implies $(n-1)p/(p-1) < n$ and the integral over the ball is finite. Hence,

$$\|u\|_{C(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

Choose two arbitrary points $x, y \in \mathbb{R}^n$ and let $r = |x - y|$, $W = B(x,r) \cap B(y,r)$. Then

$$|u(x) - u(y)| \leq \fint_W |u(x) - u(z)| dz + \fint -_W |u(y) - u(z)| dz.$$

Apply (9)

$$\begin{aligned} \fint_W |u(x) - u(z)| dz &\leq C \fint_{B(x,r)} |u(x) - u(z)| dz \\ &\leq C \Big(\int_{B(x,r)} |Du|^p \Big)^{1/p} \Big(\int_{B(x,r)} \frac{dz}{|x - z|^{(n-1)p/(p-1)}} \Big)^{(p-1)/p} \\ &\leq C \big(r^{n-(n-1)p/(p-1)} \big)^{(p-1)/p} \|Du\|_{L^p(\mathbb{R}^n)} \\ &= C r^{1-n/p} \|Du\|_{L^p(\mathbb{R}^n)} \end{aligned}$$

The same estimate holds if we change x to y.

We get

$$|u(x) - u(y)| \leq Cr^{1-n/p} \|Du\|_{L^p(\mathbb{R}^n)} = C|x-y|^{1-n/p} \|Du\|_{L^p(\mathbb{R}^n)}.$$

We can conclude

$$\begin{aligned} \|u\|_{C^{0,1-n/p}(\mathbb{R}^n)} &= \|u\|_{C(\mathbb{R}^n)} + \sup_{\substack{x,y \in \mathbb{R}^n \\ x \neq y}} \left\{ \frac{|u(x) - u(y)|}{|x-y|^{1-n/p}} \right\} \\ &\leq C\|u\|_{W^{1,p}(\mathbb{R}^n)} + C\|Du\|_{L^p(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}. \end{aligned}$$

$\square$