Geometry and Calculus of Variations
Lecture notes *

March 7, 2012

Abstract

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1 Lecture

1.1 Basic notations

We begin with introducing some basic notations which will be used throughout the notes.

- We work in $\mathbb{R}^n$ $(n = 2, 3)$

- Points in $\mathbb{R}^n$ will be denoted by **column vectors**

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = (x_1, x_2, \dots, x_n)^T$$

- The **inner product** is

$$x \cdot y = \sum_{i=1}^{n} x_i y_i$$

- The **magnitude** of $x \in \mathbb{R}^n$ is

$$|x| = \sqrt{x \cdot x} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

- The **distance** between x and y, $x, y \in \mathbb{R}^n$ is

$$|x - y| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

*This is a preliminary version of the notes. It will be updated weekly.

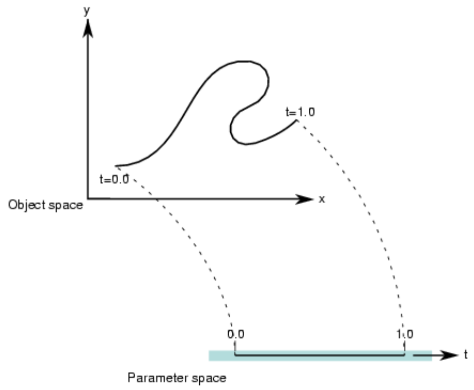

Figure 1: In this figure $I = (0, 1)$.

1.2 Plane Curves (Curves in $\mathbb{R}^2$)

Definition:

- A **curve** $\vec{x}$ in $\mathbb{R}^2$ is a smooth mapping $t \mapsto \vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ where $t \in I, \vec{x}(t) \in \mathbb{R}^2$, I is an interval of $\mathbb{R}$. Here smooth means that $\vec{x}$ is differentiable as often as we like on I.
- t is called the **parameter**. It is convenient to think of t as time.
- The **trace** of $\vec{x}$ is the image set $\vec{x}(t)$. The curves are arrowed in direction of increasing t.

Examples:

- For semicircle $\vec{x}(t) = (\cos t, \sin t)^T, t \in (0, \pi) = I$.
- The straight line $\vec{x}(t) = \vec{a} + t\vec{u}$ for some fixed vectors $\vec{a}$ and $\vec{u}, t \in (-\infty, \infty)$.

1.3 Tangent vectors

Definition:

- A **tangent vector to** $\mathbb{R}^2$ is a pair $(p, \vec{v})$, where
  - $p \in \mathbb{R}^2$ is **the point of application**
  - $\vec{v} \in \mathbb{R}^2$ is the vector acting at p
- We say that two **tangent vectors** $(p, \vec{v}), (q, \vec{w})$ **are equal** if and only if $p = q$ and $\vec{v} = \vec{w}$.
- We do the usual vector Algebra with tangent vectors at given point.
- For fixed p the **tangent space to** $\mathbb{R}^2$ **at** p is the set of all tangent vectors at p and is denoted by

$$T_p\mathbb{R}^2$$

- A **vectorfield** $\vec{V}$ is a function which assigns a tangent vector $\vec{V}(p) = (p, \vec{v}(p))$ to each $p \in \mathbb{R}^2$.
- The **velocity** of a curve $\vec{x}(t)$ at time t is the tangent vector

$$\vec{x}'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}$$

  at the point $\vec{x}(t)$.
- The **speed** is defined as

$$v = |\vec{x}'(t)|.$$

  v is the rate with respect to t at which the point $\vec{x}(t)$ describes the curve $\vec{x}$.

Interpretation: By definition

$$\vec{x}'(t) = \lim_{\Delta t \to 0} \frac{\vec{x}(t + \Delta t) - \vec{x}(t)}{\Delta t}.$$

The secants passing through $\vec{x}(t)$ and $\vec{x}(t + \Delta t)$ tend to the tangent at $\vec{x}(t)$ as $\Delta t \to 0$.

Examples: It is clear that for semicircle $\vec{x}'(t) = (-\sin t, \cos t)^T$ and for straight line $\vec{x}'(t) = \vec{u}$.

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2.1 Regularity

Definition:

- The curve $\vec{x} : I \mapsto \mathbb{R}^2$ is
  - **regular at** t_0 if $v(t_0) \neq 0$
  - **regular** if $v \neq 0$ on I.

- Regularity is used to avoid nasty behaviour: even though $\vec{x}$ is smooth, the trace of $\vec{x}$ mayw have corners or worse.

Example 1:

$$\vec{x}(t) = \begin{bmatrix} t^3 \\ |t^3| \end{bmatrix}$$

This curve is not regular at $t = 0$. It is easy to see that $x_1(t) = t^3$ and $x_2 = |t^3|$ are twice continuously differentiable. Note that after excluding the parameter t we get that $x_2 = |x_1|$.

Example 2:

$$\vec{x}(t) = \begin{bmatrix} t^3 \\ t^2 \end{bmatrix}$$

This curve is not regular at $t = 0$.

2.2 Implicitly defined curves

Definition:

- Let $F : \mathbb{R}^2 \mapsto \mathbb{R}$. A point $p \in \mathbb{R}^2$ is **a regular point** of F if $\nabla F \neq 0$, i.e.

$$\left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}\right) \neq 0$$

- Let $F : \mathbb{R}^2 \mapsto \mathbb{R}$ and let $C \subset \mathbb{R}^2$ be the zero set of F, i.e.

$$C = \{p \in \mathbb{R}^2, F(p) = 0\}.$$

Let C be nonempty If F is regular at each $p \in C$ then C is a **regular implicitly defined curve.**

As the example of

$$F(x, y) = \begin{cases} 0 & \text{if } x^2 + y^2 \leq 1 \\ x^2 + y^2 - 1 & \text{if } x^2 + y^2 > 1 \end{cases}$$

shows the regularity assumption on the F is necessary. Note that the zero set of F is the unit disk-Not a bit curve-like!

Theorem: A regular implicitly defined curve is <u>locally</u> the trace of a regular (parametrized) curve.

Proof: The key idea is to use the implicit function theorem: if $F(x, y) = 0$ defines an implicit curve C and $F_y \neq 0$ on C then then y can be solved in x, i.e. there exists a function f such that C is the graph of $y = f(x)$. If $F_x \neq 0$ then x is a function of y, i.e $x = g(y)$.

Now given $(x_0, y_0) \in C$ and assume that $F_y(x, y) \neq 0$ in some rectangular neighbourhood of $R = (x_0 - \alpha, x_0 + \alpha) \times (y_0 - \beta, y_0 + \beta), \alpha, \beta > 0$ centered at (x_0, y_0). Then according the implicit function theorem $C \cap R$ is the graph of some function $y = f(x)$ and clearly as parametrisation we can take

$$\vec{x}(t) = \begin{bmatrix} t \\ f(t) \end{bmatrix}$$

with $t = x$. When $F_x \neq 0$ then the parametrisation is

$$\vec{x}(t) = \begin{bmatrix} g(t) \\ t \end{bmatrix}$$

and $y = t$ is the parameter.

Geometric interpretation: Consider surface $S : z = F(x, y)$ and introduce $\phi(x, y, z) = z - F(x, y)$. Clearly the C curve is the intersection of $\phi = 0$ and $z = 0$. The normal of the surface S is $(-F_x, -F_y, 1)$. At the points where $\nabla F = 0$ allows possibility of zero set line disc since at those points the normal is parallel to z-axis.

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3.1 Orientation

In what follows C denotes the curve.

Theorem:

- **An orientation of C:** of a regular implicitly defined curve with no self-intersection is a choice of direction in which C is to be traversed.
- The curve C parametrized by t (i.e.$\vec{x}(t)$ is given) is **naturally oriented** in the x-direction of increasing t (see Lecture 1).

3.2 Reparametrisation

First lets consider the following example.

- For semicircle we have
$$\vec{x}(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$
$t \in (0, \pi)$. Introduce a new parameter u :
$$t = \frac{u}{2} \qquad \text{i.e.} \qquad u = 2t.$$
Denote $h(u) = u/2$, then the composed map
$$(\vec{x} \circ h)(u) \equiv \vec{X}(u) = \begin{bmatrix} \cos \frac{u}{2} \\ \sin \frac{u}{2} \end{bmatrix},$$
is another parametrization of the semicircle with parameter $u \in (0, 2\pi)$.
Hence $\vec{X}(u), u \in (0, 2\pi)$ has exactly the same trace as $\vec{x}(t), t(0, \pi)$ but x with different parameter u.
- We conclude that different parametrizations can define the same curve!

Now we give the precise definition:

Definition:

- Let $\vec{x}(t) : I \mapsto \mathbb{R}^2$ be a curve. If $h : J \mapsto I$ is a smooth bijection, with smooth inverse, of an interval J onto I then $\vec{X} = \vec{x} \circ h$ is a reparametrization of $\vec{x}$ by h.

Remarks:

- Recall that bijection is an one-to-one mapping hence the inverse is well-defined.
- if $h \neq 0$ and $\vec{x}$ is regular (i.e $|\vec{x}'(t) \neq 0$) then so is $\vec{X}$. Indeed it follows from Chain Rule
$$\vec{X}'(u) = \vec{x}'(h(u))h'(u), \qquad |\vec{X}'(u) = |\vec{x}'(h(u))||h'(u)| \neq 0.$$
- If $h' > 0$ then $\vec{X}$ has the same orientation with $\vec{x}$. Otherwise if $h' < 0$ then $\vec{X}$ has opposite orientation with $\vec{x}$.
h is said to be **orientation preserving** if $h' > 0$ and **reversing** otherwise.

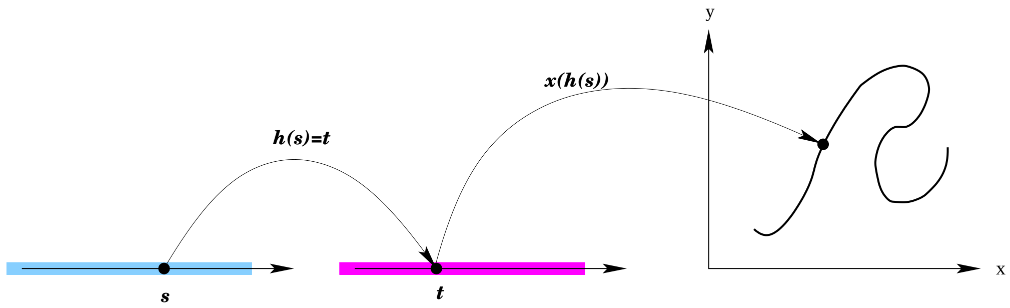

Figure 2: Reparametrisation through $h(s)$.

Examples: 1) For the reparamatrization of the semicircle $t = h(u) = u/2$ considered at the beginning of the subsection we have that h is orientation preserving since $h' = 1/2$.

2) If one defines $h(u) = 2\pi - u, u \in (\pi, 2\pi)$ and $t = h(u), t \in (0, \pi)$ then $\vec{X}(u) = \vec{x}(h(u))$ is a reparametrization of the upper semicircle which is orientation reversing since $h(u) = -1$.

3.3 Arc-Length

Definition:

- Let $\vec{x}$ be an oriented curve (i.e. we know in which direction it is traversed). The **arclength** from $A = \vec{x}(a)$ to $B = \vec{x}(b)$ is
$$s(b) = \int_a^b v(t)dt = \int_a^b |\vec{x}'(t)|dt = \int_a^b \sqrt{[x_1'(t)]^2 + [x_2'(t)]^2}dt$$
- The curve C is arc-length parametrized if the parameter s is the arc-length from some point A belongs to the trace of C and hence $s = 0$ at A.
- Symbol s is usually reserved for arc-length.

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Theorem:
Parameter s arc length parametrizes C if and only if

i) $v = 1$ (unit speed)

ii) $s = 0$ at some point $A \in$ trace of C.

Proof: $\Rightarrow$ Assume that s is the arc-length then let's show that $i)$ and $ii)$ are satisfied. $ii)$ is satisfied by the definition of the arclength, hence we need to show oly $i)$. Assuming that the point A in the definition of the arc length corresponds to $s = 0$ we have

$$s = \int_0^s |\vec{x}'(u)|du$$

differentiation with respect s gives

$$1 = |\vec{x}(s)|.$$

$\Leftarrow$ To prove the converse statement we need to show that *i)&ii)* imply that curve is arc-length parametrized. Thus let's assume that we are given a parametrization t such that $|\vec{x}'(t)| = 1$ and arc-length is 0 at $t = 0$. Then we have

$$t = \int_0^t du = \int_0^t |\vec{x}'(u)|du = \text{arc-length}.$$

Theorem: A regular curve can always be arc length parametrized.

Proof: Given a regular curve $\vec{x}(t)$ then

$$s(t) = \int_a^t |\vec{x}'(u)|du \qquad \Rightarrow \qquad \frac{ds}{dt} = |\vec{x}'(t)| \neq 0.$$

Hence s is strictly increasing function of t, and by inverse function theorem $t = h(s)$ and h is arc-length reparametrizes $\vec{x}$.

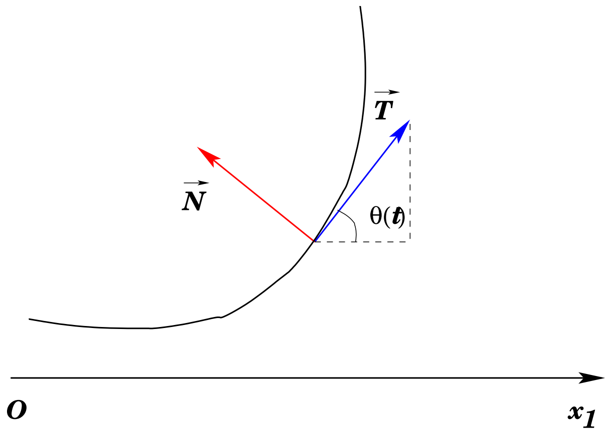
- **Remark:** Since $\frac{ds}{dt} > 0 \ \Rightarrow \vec{X}$ and $\vec{x}$ have the same orientation.
- **Example:** Let arc length reparametrize the following curve

$$\vec{x}(t) = \begin{bmatrix} 3t \\ 2t^{3/2} \end{bmatrix} \qquad \Rightarrow \qquad \vec{x}'(t) = \begin{bmatrix} 3 \\ 3t^{1/2} \end{bmatrix}, \qquad t > 0$$

Then $v(t) = 3\sqrt{1+t}$ and for the arc-length we have $s(t) = 3\int_0^t \sqrt{1+t}dt = 2[(1+t)^{3/2} - 1]$. Now from this formula express t as a function of s:

$$s = 2[(1+t)^{3/2} - 1], \qquad \Rightarrow \qquad \frac{s}{2} + 1 = (1+t)^{3/2}, \qquad \Rightarrow \qquad \left(\frac{s}{2} + 1\right)^{\frac{2}{3}} = 1 + t, \qquad t = \left(\frac{s}{2} + 1\right)^{\frac{2}{3}} - 1$$

hence $h(s) = (\frac{s}{2} + 1)^{2/3} - 1$.



4.1 Frenet Frame

Definition:

- Let $\vec{x}$ be a regular curve. **The unit tangent vector is**
$$\vec{T}(t) = \frac{\vec{x}'(t)}{v(t)}.$$
Note that if $t = s$-arclength then $\vec{T} = \vec{x}'$.
- **The unit normal vector** is $\vec{N} = J\vec{T}$ where J is the 90^o anticlockwise rotation
$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
- **Frenet frame at** $\vec{x}(t)$ is $(\vec{T}(t), \vec{N}(t))$: is an orthogonal 2×2 orthogonal matrix i.e. its columns are unit vectors which are mutually orthogonal.
- In the example above with
$$\vec{x}(t) = \begin{bmatrix} 3t \\ 2t^{3/2} \end{bmatrix} \qquad t > 0$$
it is easy to see that
$$\vec{T}(t) = \frac{1}{\sqrt{1+t}} \begin{bmatrix} 1 \\ t^{1/2} \end{bmatrix}, \qquad \vec{N}(t) = \frac{1}{\sqrt{1+t}} \begin{bmatrix} -t^{1/2} \\ 1 \end{bmatrix}$$

4.2 Rate of change of $\vec{T}$

- First we notice that if $\vec{a}(t)$ is a unit vector then $2\vec{a}(t) \cdot \vec{a}'(t) = 0$. Indeed if
$$\vec{a}(t) = \begin{bmatrix} a_1(t) \\ a_2(t) \end{bmatrix} \Rightarrow a_1^2(t) + a_2^2(t) = 1, \text{ differentiating } \Rightarrow \ 0 = 2(a_1a_1' + a_2a_2') = 2\vec{a} \cdot \vec{a}'$$
- since $\vec{T}$ is the unit tangent vector then the computation above implies that $\vec{T}' \perp \vec{T}$
- introduce $\theta(t)$ **inclination** of $\vec{T}$ to Ox_1 direction, then

$$\vec{T}(t) = \begin{bmatrix} \cos\theta(t) \\ \sin\theta(t) \end{bmatrix}, \qquad \vec{N}(t) = \begin{bmatrix} -\sin\theta(t) \\ \cos\theta(t) \end{bmatrix}$$

- Next differentiating this we get

$$\vec{T}'(t) = \begin{bmatrix} -\sin\theta(t) \\ \cos\theta(t) \end{bmatrix} \theta'(t) = \vec{N}(t)\theta'(t)$$

- Thus $\theta'(t)$ measures the rate of change of direction but depends on parametrization. To deal with this we first take the arc length parametrization.

Definition: Let $\vec{x} : J \mapsto \mathbb{R}^2$ be a regular curve parametrized by arc-length. **The curvature** k is the coefficient k in the equation

$$\vec{T}' = k\vec{N} \tag{1}$$

- **Remark 1:** Since $\vec{N}$ is a unit vector we have after taking the inner product of $\vec{N}$ and $\vec{T}'$ and using equation (1) we get $k = \vec{T}' \cdot \vec{N}$. k can be interpreted as the rate with respect to distance at which $\vec{T}$ turns towards $\vec{T}'$.

- **Remark 2:** Next we want to show that

$$\vec{N}' = -k\vec{T} \tag{2}$$

Indeed by definition $\vec{N} = J\vec{T}$ which after differentiation and using equation (1) yields

$$\vec{N}' = J\vec{T}' = Jk\vec{N} = JkJ\vec{T} = kJ^2\vec{T} = -k\vec{T}$$

where the last line follows from

$$J^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- Combining equations (1) and (2) we get the **structure equations** of the arc-length parametrized curve

Definition:
The equations (1) and (2) are equivalent to

$$(\vec{T}', \vec{N}') = \underbrace{(\vec{T}, \vec{N})}_{Frenet frame} \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}, \qquad \vec{T}(s) = \vec{x}'(s) \tag{3}$$

and (3) are called the **structure equations** of arc-length parametrized curve $\vec{x}(s)$.

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Remarks:

- $k(s)$ measures rate of turn of $\vec{T}(s)$. For instance if $\vec{x}(s) = s\vec{u} + \vec{a}$, i.e. a straight line, where $\vec{a}$ is fixed vector and $\vec{u}$ is an unit vector, then $\vec{x}'(s) = \vec{u} = \vec{T}(s)$. Thus $\vec{T}'(s) = 0 \Rightarrow k(s) = 0$. Hence $|k(s)| = |\vec{T}'(s)|$ gives a measure of how rapidly the curve pulls away from the tangent line.

- For circle of radius r we have $\vec{x}(t) = r(\cos t, \sin t)^T$. Thus $\vec{x}'(t) = r(-\sin t, \cos t)^T$ and the speed is $v(t) = r$. By arc-length formula we get that

$$s = \int_0^t v(u)du = rt, \qquad \Rightarrow \qquad t = \frac{s}{r}$$

(we measure arc-length from $(1,0)$). Hence the arc-length parametrisation is $\vec{x}_1(s) = r(\cos\frac{s}{r}, \sin\frac{s}{r})^T$. Now we differentiate $\vec{x}_1$ by s to get the unit tangent $\vec{T}_1(s) = (-\sin\frac{s}{r}, \cos\frac{s}{r})$. Finally

$$\vec{T}_1'(s) = -\frac{1}{r}\begin{bmatrix} \cos\frac{s}{r} \\ \sin\frac{s}{r} \end{bmatrix} = \frac{1}{r}\underbrace{\begin{bmatrix} -\cos\frac{s}{r} \\ -\sin\frac{s}{r} \end{bmatrix}}_{\vec{N}_1(s)}$$

Thus $k_1(s) = \frac{1}{r} > 0$. Notice that for $\vec{x}_2(s) = r\begin{bmatrix} -\cos\frac{s}{r} \\ \sin\frac{s}{r} \end{bmatrix}$ with reversed orientation we have $k_2(s) = -\frac{1}{r}$.
See Figure 3.

- Since $k(s) = \vec{T}'(s) \cdot \vec{N}(s) \Rightarrow k(s) > 0$ (resp. $k(s) < 0$) if $\vec{T}(s)$ turns towards (resp. away from) $\vec{N}(s)$.

5.1 Arbitrary speed curves

Let $\vec{x}(t)$ be a regular curve, then there exists an arc-length parametrization $\vec{x}_1(s)$ such that

$$\vec{x}(t) = \vec{x}_1(s(t)), \qquad s(t) = \int_a^t |\vec{x}'(u)|du.$$

Differentiating

$$\frac{d\vec{x}(t)}{dt} = \underbrace{\frac{d\vec{x}_1(s(t))}{ds}}_{\vec{T}_1(s(t))} \underbrace{\frac{ds}{dt}}_{v(t)-speed}$$

or

$$\vec{T}_1(s(t)) = \frac{\vec{x}'(t)}{v(t)} \equiv \vec{T}(t)$$

Thus the tangents are identical

$$\vec{T}_1(s(t)) = \vec{T}(t).$$

Next let's compare the normals

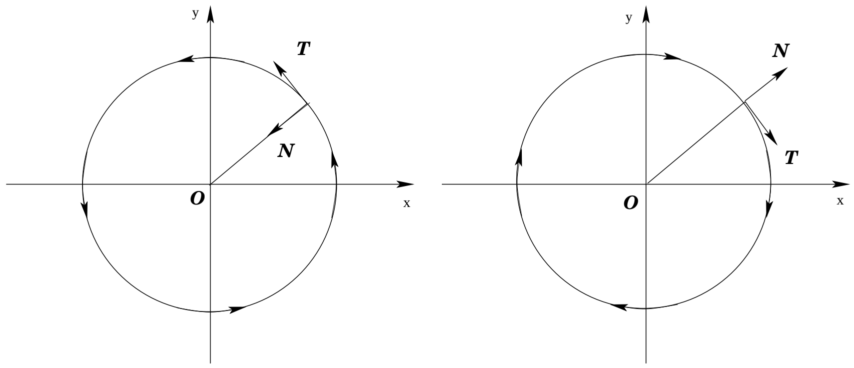

Figure 3: In the example $k > 0$ for $\vec{x}_1(s)$ and $k < 0$ for $\vec{x}_2(s)$.

Since the unit normal is the 90 degree rotation of tangent we conclude that the normal are identical

$$\vec{N}_1(s(t)) = \vec{N}(t).$$

Finally let's examine the curvature

We have

$$\frac{d\vec{T}(t)}{dt} = \frac{d\vec{T}_1}{ds} \underbrace{\frac{ds}{dt}}_{v(t)-speed}$$

Recalling that $\frac{d\vec{T}_1(s)}{ds} = k_1(s)\vec{N}_1(s)$ and using the equality for normals and tangents we get

$$\begin{aligned} \vec{T}'(t) &= k_1(s(t))\vec{N}_1(s(t))v(t) = \\ &= k_1(s(t))\vec{N}(t)v(t) = \\ &= k(t)\vec{N}(t)v(t) \end{aligned}$$

where

$$k(t) = k_1(s(t))$$

This is the definition of the curvature for arbitrary speed curve.

By a similar computation we have

$$\vec{N}'(t) = -k(t)\vec{T}(t)v(t).$$

Thus the structure equation in the matrix form are

$$(\vec{T}'(t), \vec{N}'(t)) = \underbrace{(\vec{T}(t), \vec{N}(t))}_{Frenet frame} \begin{pmatrix} 0 & -k(t)v(t) \\ k(t)v(t) & 0 \end{pmatrix}, \qquad \vec{x}'(t) = v(t)\vec{T}(t)$$

Conclusions:

- From the definition of Frenet frame $(\vec{T}, \vec{N})$ it follows that Frenet frame does not depend on the parametrization.
- From the definition of $k(t)$ it follows that curvature does not depend on the parametrization.

The next lemma provides a useful formula for curvature for arbitrary parametrisations:

Lemma 1 *For a regular curve $\vec{x}(t)$ we have that*

$$k(t) = \frac{1}{v^3(t)} \vec{x}''(t) \cdot J\vec{x}'(t) = \frac{\det(\vec{x}', \vec{x}'')}{v^3}.$$

Proof: Since $\vec{x}'(t) = \vec{T}(t)v(t)$ we have from product rule

$$\vec{x}''(t) = \vec{T}'(t)v(t) + \vec{T}(t)v'(t) = k\vec{N}(t)v^2(t) + \vec{T}(t)v'(t)$$

Taking the inner product with $\vec{N}$ we get

$$k = \frac{1}{v^2}\vec{x}'' \cdot \vec{N} = \frac{1}{v^2}\vec{x}'' \cdot J\frac{\vec{x}'}{v} = \frac{1}{v^3}\vec{x}'' \cdot J\vec{x}'.$$

Example:

- Let
$$\vec{x}(t) = \begin{bmatrix} t \\ f(t) \end{bmatrix} \quad \Rightarrow \quad \vec{x}'(t) = \begin{bmatrix} 1 \\ f'(t) \end{bmatrix}, \quad \vec{x}''(t) = \begin{bmatrix} 0 \\ f''(t) \end{bmatrix}$$
Then the speed is $v(t) = \sqrt{1 + (f'(t))^2}$. Hence
$$k(t) = \frac{1}{(1 + (f'(t))^2)^{3/2}} \begin{bmatrix} 0 \\ f''(t) \end{bmatrix} \cdot J \begin{bmatrix} 1 \\ f'(t) \end{bmatrix} = \frac{f''(t)}{(1 + (f'(t))^2)^{3/2}}.$$

Theorem 2 $\vec{T}, \vec{N}, k$ *are all unchanged (resp. reversed) under orientation preserving (resp. reversing) reparametrization.*

We split the proof into 2 parts.

Step 1. Let's assume that $\vec{x}_2$ is the orientation preserving reparametrization of $\vec{x}_1$, i.e. $\vec{x}_2 = \vec{x}_1 \circ h$. Then if $\vec{x}_3$ is the arc-length parametrization then $\vec{x}_3(s) = \vec{x}_2(H(s))$ then it is also an arc-length parametrization for $\vec{x}_1$ since

$$\vec{x}_3(s) = \vec{x}_2(H(s)) = \vec{x}_1(h(H(s))).$$

We know that

- $\vec{x}_1$ and $\vec{x}_3$ have the same $\vec{T}, \vec{N}, k$
- $\vec{x}_2$ and $\vec{x}_3$ have the same $\vec{T}, \vec{N}, k$

hence so do $\vec{x}_1$ and $\vec{x}_2$.

Step 2. For reversing case it is enough to consider the arc-length parametrization in view of step 1. Let $\vec{x} : I \mapsto \mathbb{R}^2$. Introduce $\vec{x}_1(s) = \vec{x}(-s) \Rightarrow \vec{x}_1'(s) = -\vec{x}'(-s) \Rightarrow \vec{T}_1 = -\vec{T} \Rightarrow \vec{N}_1 = -\vec{N}$. Moreover $\vec{x}_1''(s) = \vec{x}''(-s)$. Thus

$$k_1(s) = \vec{x}_1'' \cdot J\vec{x}_1' = \vec{x}'' \cdot (J(-\vec{x}')) = -k(-s).$$

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6.1 Rotation index

- **Definition.** For speed one curve $\vec{x}$ **an argument function** is a continuous function θ such that
$$\vec{T}(s) = \begin{bmatrix} \cos\theta(s) \\ \sin\theta(s) \end{bmatrix}$$
- θ is unique up to an overall addition of a multiple of 2π.
- **Proposition:**
$$\frac{d\theta}{ds} = k.$$
We have $\vec{T}' = \theta'\vec{N}$ but by definition $\vec{T}' = k\vec{N}$. (see also subsection 4.2)
- **Note:** For arbitrary speed curve $\theta' = vk$.
- **Definition:** Curve $\vec{x} : \mathbb{R} \mapsto \mathbb{R}^2$ is closed if $\vec{x}$ is periodic
$$\vec{x}(s+L) = \vec{x}(s), \forall s, L > 0$$
- The smallest such L is **the length of** $\vec{x}$.
- **The rotation index** $\gamma(\vec{x})$ of the closed curve $\vec{x}$ is the integer
$$\gamma(\vec{x}) = \frac{1}{2\pi}(\theta(L) - \theta(0))$$
- **Interpretation:** Regard $\vec{T}(s)$ as position vector $O\vec{P}'$ of point P' on unit circle. Then γ measures total number of turns P' encircles 0 anticlockwise as $\vec{x}$ is described once, i.e. the total change of direction anticlockwise measured in whole rotations.

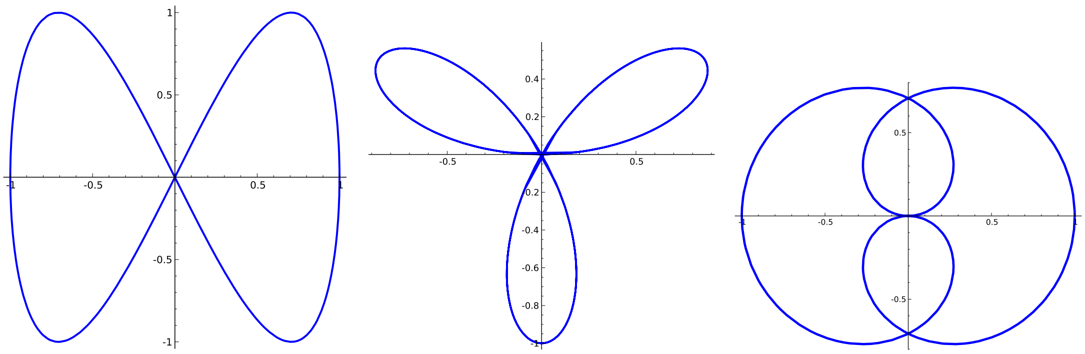
Theorem 3 *For closed curve* $\vec{x}$
$$\gamma(\vec{x}) = \frac{1}{2\pi}\int_0^L k(s)ds.$$

We have
$$\frac{1}{2\pi}\int_0^L k(s)ds = \frac{1}{2\pi}\int_0^L \theta'(s)ds = \frac{1}{2\pi}(\theta(L) - \theta(0)) = \gamma(\vec{x}).$$

Example: Consider circle of radius r, then
$$\vec{x}(s) = r\begin{bmatrix} \cos(s/r) \\ \sin(s/r) \end{bmatrix}$$
then
$$\vec{T}(s) = \begin{bmatrix} -\sin(s/r) \\ \cos(s/r) \end{bmatrix} = \begin{bmatrix} \cos(s/r + \pi/2) \\ \sin(s/r + \pi/2) \end{bmatrix}$$
so we can take $\theta(s) = (s/r + \pi/2)$.

Now let us consider the following simple example

- $k = \frac{1}{r}$
- $L = 2\pi r$



(a) $\gamma = 0$ (b) $r(\theta) = \sin(3\theta), \gamma = 2$ (c) $r(\theta) = \sin(\frac{1}{2}\theta), \theta \in (0, 4\pi), \gamma = 4$

Figure 4: Curves with various rotation indeces

- $\gamma = 1$

Example: Figure 4 illustrates a curve for which the tangent $\vec{T}$ makes two full rounds, the blue and red parts, hence the rotation index of the curve is 2.

For this particular Example we take

6.2 Families of curves

- **Defn:** A family of curves is a smooth map

$$\vec{X} : (\lambda, t) \mapsto \vec{X}(\lambda, t) \ni \mathbb{R}^2$$

where (λ, t) belongs to some rectangle D in the (λ, t)-plane.

- For each fixed λ define the curve $\vec{x}_\lambda(t)$ by $\vec{x}_\lambda : t \mapsto \vec{X}(\lambda, t)$
- Regard $\vec{X}$ as the family of all the curves $\vec{x}_\lambda$.
- **Example:** Take

$$\vec{X}(\lambda, t) = \begin{bmatrix} \lambda \\ \lambda^2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2\lambda \end{bmatrix}.$$

This has form $\vec{\mathscr{X}}(\lambda) + t\vec{\mathscr{X}}'(\lambda)$ where $\vec{\mathscr{X}}(\lambda)$ is the parabola $(\lambda, \lambda^2)^T$ so $\vec{X}$ is the family of tangent lines to $\vec{\mathscr{X}}$

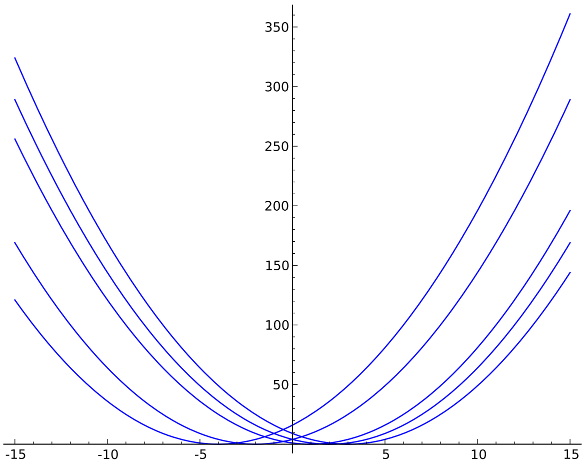
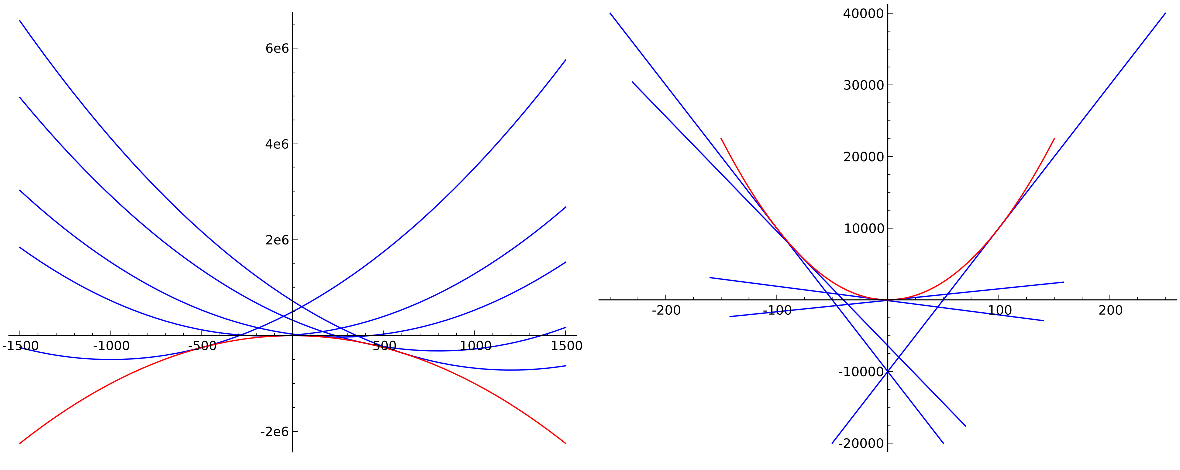


Figure 5: This is the family of parabolas $\vec{X}(\lambda, t) = (t, (t-\lambda)^2)^T$.



(a) A gull

(b) A tiger

Figure 6: Families of curves

7 Lecture 7

7.1 Envelope

- **Definition:** A curve $\vec{u}$ is an **envelope** for the family $\vec{X}$ if at each point of $\vec{u}$ it is tangent to a member of the family $\vec{X}$ but is not a member of the family.

7.2 Enveloping condition

Remark:
Suppose $\vec{u}$ touches $\vec{x}_\lambda(t)$ at some point $P(\lambda) = \vec{x}(t_\lambda)$ on the curve where $t_\lambda = T(\lambda)$ (for some function T) i.e. at the point $P(\lambda) = \vec{X}(\lambda, T(\lambda))$. Then $P(\lambda)$ is a point of trace $\vec{u}$ but this is true for each λ so $\vec{u}$ has parametrization $\lambda \mapsto \vec{X}(\lambda, T(\lambda))$.

Next we give necessary condition for enveloping.

Theorem:
Jacobian Matrix: If $\vec{u} : \lambda \mapsto \vec{X}(\lambda, T(\lambda))$ is an envelope for $\vec{X}$ then

$$\det \begin{bmatrix} \frac{\partial \vec{X}}{\partial \lambda} \frac{\partial \vec{X}}{\partial t} \end{bmatrix} = \det \begin{bmatrix} \frac{\partial X^1}{\partial \lambda} & \frac{\partial X^1}{\partial t} \\ \frac{\partial X^2}{\partial \lambda} & \frac{\partial X^2}{\partial t} \end{bmatrix} = 0 \qquad \text{at} \qquad (\lambda, T(\lambda)).$$

- **Proof:** It suffices to prove that $\frac{\partial \vec{X}}{\partial \lambda}$ and $\frac{\partial \vec{X}}{\partial t}$ are parallel (i.e. are collinear vectors). At the points where $\vec{u}$ is tangent to a member of the family curve the tangent vectors are colinear.

  We have
  $$\frac{d\vec{u}(\lambda)}{d\lambda} = \frac{\partial \vec{X}(\lambda, T(\lambda))}{\partial \lambda} + \frac{\partial \vec{X}(\lambda, T(\lambda))}{\partial t} T'(\lambda).$$
  Next $\vec{x}_\lambda$, at the contact point $t = T(\lambda)$, has tangent
  $$\left.\frac{d\vec{x}_\lambda(t)}{dt}\right|_{t=T(\lambda))} = \left.\frac{\partial \vec{X}(\lambda, t)}{\partial t}\right|_{t=T(\lambda))} = \frac{\partial \vec{X}(\lambda, T(\lambda))}{\partial t}.$$
  For $\vec{u}$ to be tangent to $\vec{x}_\lambda$ at the contact point $P(\lambda)$ we require the velocities (which are collinear to corresponding unit tangents)
  $$\frac{d\vec{u}(\lambda)}{d\lambda} = C_1 \frac{d\vec{x}_\lambda(T(\lambda))}{dt}$$
  for some constant C_1, i.e. the velocities must be linearly dependent!

  Summarizing
  $$C_1 \frac{\partial \vec{X}(\lambda, T(\lambda))}{\partial t} = \boxed{C_1 \frac{d\vec{x}_\lambda(T(\lambda))}{dt} = \frac{d\vec{u}(\lambda)}{d\lambda}} = \frac{\partial \vec{X}(\lambda, T(\lambda))}{\partial \lambda} + \frac{\partial \vec{X}(\lambda, T(\lambda))}{\partial t} T'(\lambda)$$
  or equivalently
  $$(C_1 - T'(\lambda)) \frac{\partial \vec{X}(\lambda, T(\lambda))}{\partial t} = \frac{\partial \vec{X}(\lambda, T(\lambda))}{\partial \lambda}.$$

- **Note** this is a necessary condition for enveloping but we use it to try to locate (recover) envelopes.

Example: Take

$$\vec{X}(\lambda, t) = \begin{bmatrix} \lambda \\ \lambda^2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2\lambda \end{bmatrix}.$$

We have

$$\frac{\partial \vec{X}}{\partial \lambda} = \begin{bmatrix} 1 \\ 2\lambda \end{bmatrix} + \begin{bmatrix} 0 \\ 2t \end{bmatrix}, \qquad \frac{\partial \vec{X}}{\partial t} = \begin{bmatrix} 1 \\ 2\lambda \end{bmatrix}$$

Thus by Theorem we have that

$$\det \begin{bmatrix} 1 & 1 \\ 2\lambda & 2\lambda + 2t \end{bmatrix} = 0 \Longleftrightarrow t = 0 \qquad \text{(This is } T(\lambda)!\text{)}.$$

Substituting back into $\vec{X}$ gives envelope as $\vec{u}(\lambda) = \vec{X}(\lambda, T(\lambda)) = (\lambda, \lambda^2)^T$ which is the parabola.

Example: Let r be a given constant and let

$$\vec{X}(\lambda, t) = \begin{bmatrix} \lambda + r\cos t \\ r\sin t \end{bmatrix}.$$

Clearly the parametrisation of each member of this family, for fixed λ, is $x_1(t) = \lambda + r\cos t, x_2(t) = r\sin t$ or equivalently

$$(x_1 - \lambda)^2 + x_2^2 = r^2\cos^2 t + r^2\sin^2 t = r^2.$$

Thus for fixed λ $\vec{x}_\lambda(t)$ is a circle centers at $(\lambda, 0)$ of radius r. The det $= 0$ condition implies that $r\cos t = 0 \Rightarrow t = \pm\frac{\pi}{2}$. Substituting into $\vec{X}$ we get that the envelope $\vec{u}(\lambda) = \begin{bmatrix} \lambda \\ \pm r \end{bmatrix}$, i.e. the envelope consists of a pair of parallel horizontal lines.
See Figure 5.

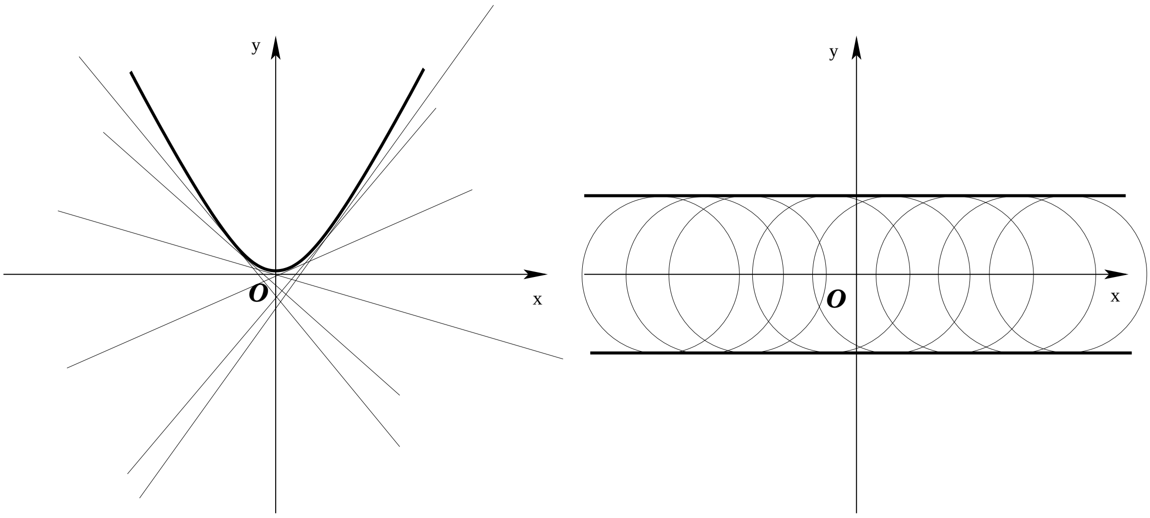


Figure 7: Examples of envelopes

7.3 Envelope of implicitly defined curves

- Similarly extend an implicitly defined curve $f_\lambda(\vec{x}) = 0$ to a family of implicitly defined curves given by
$$F(\vec{x}, \lambda) = 0$$
- For each fixed value of λ thus defines an implicitly defined curve $f_\lambda(\vec{x}) = 0$ where
$$f(\vec{x}, \lambda) = F(\vec{x}, \lambda).$$
- **Enveloping condition:** Analysis similar to above shows that a necessary condition for enveloping is
$$F(x, y, \lambda) = 0, \qquad \frac{\partial F(x, y, \lambda)}{\partial \lambda} = 0,$$
where we set $\vec{x} = (x, y)$.
- **Example:** Let $F(x, y, \lambda) = (x - \lambda)^2 + y^2 - r^2$ (see the previous example) where r is a given positive constant. As we have seen each member of this implicit family of curves is a circle. Now let's find the envelope. We have $\dfrac{\partial F}{\partial \lambda} = -2(x - \lambda) = 0$ hence $x = \lambda$ substituting this into $F(x, y, \lambda) = 0$ we get that $y^2 - r^2 = 0$ or $y = \pm r$- a pair of parallel lines.

  See Figure 5.

8 Lecture 8

8.1 Calculus of Variations

- Given a functional $J[y]$ (that is a function of $y(x), x \in [a, b]$).
- y satisfies boundary conditions $y(a) = u, y(b) = v$, the numbers u, v are given.

- **Question:** Find y that extremises $J[y]$(i.e. gives max or min value to J.)

Defn:(Extremals) Let $J[y]$ be a functional and y satisfies the boundary conditions as indicated above. Then y is an extremal of J if

$$\frac{d}{d\varepsilon}J[y+\varepsilon h]\Big|_{\varepsilon=0} = 0 \tag{1}$$

for all smooth functions h such that $h(a) = h(b) = 0$.

Lemma: If y extremises J then y must be an extremal of J (that is (1) is satisfied).

- **Proof:** Assume that y gives the maximal value of J. Then $J[y+\varepsilon h] leq J[y]$ for all functions h as indicated above. Thus

$$J[y+\varepsilon h] - J[y] \leq 0 \tag{$\star$}$$

Take $\varepsilon > 0$ and divide the above inequality by ε and send ε to zero

$$\frac{d}{d\varepsilon}J[y+\varepsilon h] = \lim_{\varepsilon\downarrow 0}\frac{J[y+\varepsilon h]-J[y]}{\varepsilon} \leq 0 \tag{$\star\star$}$$

Next take $\varepsilon <$ and divide ($\star$) by ε. Since $\varepsilon < 0$ after division we have reversed inequality. Sending ε to 0 yields

$$\frac{d}{d\varepsilon}J[y+\varepsilon h] = \lim_{\varepsilon\uparrow 0}\frac{J[y+\varepsilon h]-J[y]}{\varepsilon} \geq 0 \tag{$\star\star\star$}$$

Combining ($\star\star$) with ($\star\star\star$) we finish the proof. If y gives minimal value then all inequalities ($\star$), ($\star\star$) and ($\star\star\star$) must be reversed.

- **Notes:** In calculus if f is a function then the analogous condition is $f'(x)$ to have extreme local value at x.

  If (1) is satisfied then y is said to be a **stationary point of** J.

  (1) is necessary condition but not always sufficient.

  The following table links the concepts of Extremum and Extremal of a functional to the customary concepts of max/min and critical points for the functions

Function $f(x)$	Functional $J[y]$
A point of absolute max or min at $x \in [a,b]$	y is an Extremum of $J[y]$
x is a critical point $f'(x) = 0$	y is an extremal $\frac{d}{d\varepsilon}J[y+\varepsilon h]\Big\vert_{\varepsilon=0} = 0$

- **Example:**

Let $J[y] = \int_a^b (1+(y'(x))^2)dx$ and boundary conditions are $y(a) = u, y(b) = v$. Then for h such that $h(a) = h(b) = 0$ we have

$$\begin{aligned} J[y+\varepsilon h] &= \int_a^b \left[1+(y'(x)+\varepsilon h'(x))^2\right] dx \\ &= \int_a^b \left[1+(y'(x))^2+2\varepsilon y'(x)h'(x)+\varepsilon^2(h'(x))^2\right] dx \\ &= \int_a^b \left[1+(y'(x))^2\right] dx + 2\varepsilon \int_a^b y'(x)h'(x)dx + \varepsilon^2 \int_a^b (h'(x))^2 dx \end{aligned}$$

Thus

$$0 = \frac{d}{d\varepsilon} J[y+\varepsilon h]\Big|_{\varepsilon=0} = 2\int_a^b y'(x)h'(x)dx$$

Use integration by parts to get that

$$\int_a^b y'(x)h'(x)dx = y'(x)h(x)\Big|_a^b - \int_a^b y''(x)h(x)dx = -\int_a^b y''(x)h(x)dx = 0$$

since $h(a) = h(b) = 0$ and thus we conclude that

$$\int_a^b y''(x)h(x)dx = 0.$$

Now choose $h(x) = y''(x)(x-a)(b-x)$ and substitute into the last equality to obtain

$$\int_a^b (y''(x))^2(x-a)(b-x)dx = 0$$

the integrand is nonnegative function thus $(y''(x))^2(x-a)(b-x) = 0 \Leftarrow y''(x) = 0$. Solving the ODE $y''(x) = 0$ with boundary conditions $y(a) = u, y(b) = v$ we have that y is a linear function.

8.2 Euler-Lagrange Equations

- In the previous example the problem of finding extremals was reduced to solving a linear second order ODE coupled with boundary data. Thus we want to obtain the corresponding ODE for functionals of general type.
- **Theorem:** Let $J[y] = \int_a^b F(x,y(x),y'(x))dx$ where $F(x,y,z)$ is a smooth function of (x,y,z) (here z is the dummy variable for y'). Assume that $y(a) = u, y(b) = v$.

  Then extremals of J are the solutions of

  $$\frac{\partial F(x,y(x),y'(x))}{\partial y} - \frac{d}{dx}\left(\frac{\partial F(x,y(x),y'(x))}{\partial y'}\right) = 0 \qquad y(a) = u, y(b) = v.$$

- This theorem says that to find extremals it is enough to solve the corresponding boundary value problem for the ODE-the Euler -Lagrange equation.
- **Sketch of proof:** By equation (1) we have

$$\frac{d}{d\varepsilon} J[y+\varepsilon h] = \frac{d}{d\varepsilon}\int_a^b F(x,y(x)+\varepsilon h(x),y'(x)+\varepsilon h'(x))dx$$

Since F is smooth we can interchange integration with differentiation to get

$$\begin{aligned}\frac{d}{d\varepsilon}J[y+\varepsilon h] &= \int_a^b \frac{d}{d\varepsilon}F(x, y(x)+\varepsilon h(x), y'(x)+\varepsilon h'(x))dx \\ &= \int_a^b \left(\frac{\partial F(\dots)}{\partial y}h(x) + \frac{\partial F(\dots)}{\partial y'}h'(x)\right)dx\end{aligned}$$

here we used the chain rule and we used notation $\dots = (x, y(x)+\varepsilon h(x), y'(x)+\varepsilon h'(x))$ for brevity. Integration by parts and the condition $h(a) = h(b) = 0$ imply

$$\begin{aligned}\int_a^b \frac{\partial F(\dots)}{\partial y'}h'(x)dx &= \frac{\partial F(\dots)}{\partial y'}h(x)\Big|_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial F(\dots)}{\partial y'}\right)h(x)dx \\ &= -\int_a^b \frac{d}{dx}\left(\frac{\partial F(\dots)}{\partial y'}\right)h(x)dx\end{aligned}$$

Using this computation and sending ε to 0 we conclude

$$\int_a^b \left[\frac{\partial F(x, y(x), y'(x))}{\partial y} - \frac{d}{dx}\left(\frac{\partial F(x, y(x), y'(x))}{\partial y'}\right)\right]h(x)dx = 0$$

for arbitrary smooth function h such that $h(a) = h(b) = 0$. If F is thrice continuously differentiable function of x, y and z, and y is thrice continuously differentiable function of x then we can take

$$h(x) = \left[\frac{\partial F(x, y(x), y'(x))}{\partial y} - \frac{d}{dx}\left(\frac{\partial F(x, y(x), y'(x))}{\partial y'}\right)\right](x-a)(b-a).$$

Notice that under these conditions h is continuously differentiable function of x.

The general result (i.e. the when F and y are not thrice differentiable) follows from the theorem of du Bois Reymond.

9 Lecture 9

9.1 Special cases of E-L equations

F **is independent of** y: In this case $\frac{\partial F}{\partial y} = 0$ hence from E-L equation we get

$$-\frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \qquad \Rightarrow \qquad \boxed{\frac{\partial F}{\partial y'} = k,\ k \text{ is a constant}}$$

thus $\dfrac{\partial F}{\partial y'}$ is constant for extremals y.

F **is independent of** x**:** In this case $\frac{\partial F}{\partial x} = 0$. Let us compute

$$\begin{aligned}\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) &= \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial y'}\frac{dy'}{dx} - \frac{dy'}{dx}\frac{\partial F}{\partial y'} - y'\frac{d}{dx}\frac{\partial F}{\partial y'} \\ &= \frac{\partial F}{\partial y}\frac{dy}{dx} - y'\frac{d}{dx}\frac{\partial F}{\partial y'} \\ &= y'\underbrace{\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right)}_{E-L\ equation} = 0\end{aligned}$$

$$\boxed{F - y'\frac{\partial F}{\partial y'} = k, \quad k \text{ is a constant}}$$

9.2 Examples

- $J[y] = \int_a^b (1 + [y'(x)]^2)dx, F(x, y, y') = 1 + [y'(x)]^2$ and it's independent of x and y thus

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 - \frac{d}{dx}(2y') = 0$$

  hence $y'' = 0$ implying that y is a linear function.

- Arc-length. $J[y] = \int_a^b \sqrt{1 + [y'(x)]^2}dx$, $F(x, y, y') = \sqrt{1 + [y'(x)]^2}$ independent of x and y thus from E-L equation we get

$$0 = \frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 - \frac{d}{dx}\left[\frac{1}{2}\frac{2y'}{\sqrt{1 + [y'(x)]^2}}\right]$$

  thus

$$\frac{y'}{\sqrt{1 + [y'(x)]^2}} = k$$

  hence y' is constant and thereby y is a linear function.

- Minimal area of surface of revolution. $J[y] = \int_a^b 2\pi y\sqrt{1 + [y'(x)]^2}dx$.

  F is independent of x hence (as indicated above) $F - y'\frac{\partial F}{\partial y'} = k$ which results

$$y = k\sqrt{1 + [y'(x)]^2}$$

  To work out an expression for y' we square both sides to get

$$y^2 = k^2(1 + [y']^2) \qquad \Rightarrow \qquad \frac{y^2}{k^2} - 1 = [y']^2 \qquad \Rightarrow \qquad y' = \sqrt{\frac{y^2}{k^2} - 1}$$

  We now solve the ODE

$$\frac{dy}{dx} = \sqrt{\frac{y^2}{k^2} - 1} \qquad \Rightarrow \qquad \frac{dy}{\sqrt{\frac{y^2}{k^2} - 1}} = dx \qquad \Rightarrow \qquad \int \frac{dy}{\sqrt{\frac{y^2}{k^2} - 1}} = \int dx$$

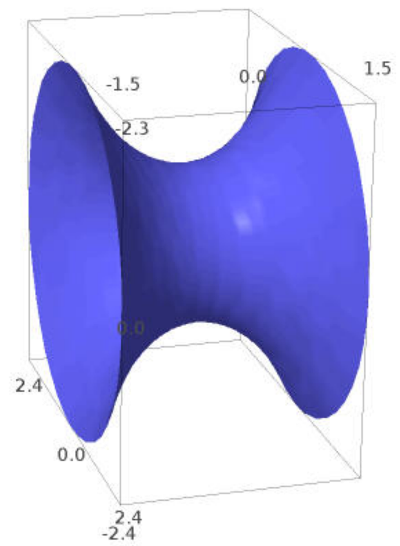


Figure 8: Catenoid

which gives the inverse of cosh

$$k \cosh^{-1} \frac{y}{k} = x - x_0$$

and x_0 is an arbitrary constant (the integration constant). Hence

$$y = k \cosh\left(\frac{x - x_0}{k}\right)$$

Therefore we obtain a 2-parameter (which are x_0 and k) family $\mathcal{F}$ of extremals.

9.3 Several dependent variables

Definition: $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ is said to be an extremal of $J[\vec{x}]$ if

$$\frac{d}{d\varepsilon} J[\vec{x} + \varepsilon \vec{h}]\Big|_{\varepsilon=0} = 0 \tag{4}$$

for all $\vec{h}$ such that on the boundary $\vec{h}(a) = \vec{h}(b) = 0$
If (4) is satisfied then we say that $\vec{x}$ makes J stationary.

10 Lecture 10

Theorem: If $\vec{x}$ extremises J then $\vec{x}$ is an extremal. The proof is exactly the same as for the scalar case.

E-L equations Assume that $F(t,\vec{x},\vec{x}')$ is a smooth function of all its arguments and $J[\vec{x}] = \int_a^b F(t,\vec{x},\vec{x}')dt$ where $\vec{x} : [a,b] \mapsto \mathbb{R}^n$, $\vec{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ with boundary conditions $\vec{x}(a) = \vec{u}, \vec{x}(b) = \vec{v}$ (i.e. $x_1(a) = u_1, \ldots, x_n(a) = u_n, \ldots$)
Then the extremals of J are the solutions of the n-coupled (in general) second order ODE's

$$\boxed{\frac{\partial F}{\partial x_i} - \frac{d}{dt}\frac{\partial F}{\partial x_i'} = 0, \qquad i = 1,2,\ldots,n.}$$

- **Proof:** Interchanging differentiation with integration we obtain

$$\begin{aligned}
\frac{d}{d\varepsilon}J[\vec{x}+\varepsilon\vec{h}] &= \frac{d}{d\varepsilon}\int_a^b F(t,\vec{x}(t)+\varepsilon\vec{h}(t),\vec{x}'(t)+\varepsilon\vec{h}'(t))dt \\
&= \int_a^b \frac{d}{d\varepsilon}F(t,\vec{x}(t)+\varepsilon\vec{h}(t),\vec{x}'(t)+\varepsilon\vec{h}'(t))dt \\
&= \int_a^b \sum_{i=1}^n \left(\frac{\partial F}{\partial x_i}h_i + \frac{\partial F}{\partial x_i'}h_i'\right)dt \\
&= \underbrace{\frac{\partial F}{\partial x_i'}h_i\Big|_a^b}_{\text{is 0 since } \vec{h}(a)=\vec{h}(b)=0} - \int_a^b \sum_{i=1}^n h_i\left(\frac{\partial F}{\partial x_i} - \frac{d}{dt}\frac{\partial F}{\partial x_i'}\right)dt
\end{aligned}$$

- **Example:** For Arc-length we have $J[\vec{x}] = \int_a^b \sqrt{(x_1'(t))^2 + (x_2'(t))^2}dt$. Then the E-L equations are

$$\frac{\partial F}{\partial x_i} - \frac{d}{dt}\frac{\partial F}{\partial x_i'} = 0 \qquad i = 1,2. \Rightarrow \begin{cases} -\dfrac{x_1'(t)}{\sqrt{(x_1'(t))^2 + (x_2'(t))^2}} = k_1 \\ -\dfrac{x_2'(t)}{\sqrt{(x_1'(t))^2 + (x_2'(t))^2}} = k_2 \end{cases}$$

If we assume that $x_1' = 0$ then the first constant $k_1 = 0$ and hence x_1 is constant substituting this into the second equation gives that x_2 is linear.
Now assume that $x_1' \neq 0$ then

$$\frac{x_2'}{x_1'} = \frac{k_2}{k_1} = constant$$

hence the curve has constant slope $\Rightarrow$ it is line.

10.1 Special cases

- F **is independent of** $\vec{x}$: In this case $\frac{\partial F}{\partial x_i} = 0$ hence from E-L equation we get

$$-\frac{d}{dt}\frac{\partial F}{\partial x_i'} = 0 \qquad \Rightarrow \qquad \boxed{\frac{\partial F}{\partial x_i'} = k_i, i = 1,2,\ldots,n, \ k_i \text{ is a constant}}$$

thus $\dfrac{\partial F}{\partial x_i'}$ is constant for extremals $\vec{x}$.

- F **is independent of** t**:** In this case $\frac{\partial F}{\partial t} = 0$. Let us compute

$$
\begin{aligned}
\frac{d}{dt}\left(F - \sum_{i=1}^{n} x_i' \frac{\partial F}{\partial x_i'}\right) &= \sum_{i=1}^{n}\left(\frac{\partial F}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial F}{\partial x_i'}\frac{dx_i'}{dt} - \frac{dx_i'}{dt}\frac{\partial F}{\partial x_i'} - x_i'\frac{d}{dt}\frac{\partial F}{\partial x_i'}\right) \\
&= \sum_{i=1}^{n}\left(\frac{\partial F}{\partial x_i}\frac{dx_i}{dt} - x_i'\frac{d}{dt}\frac{\partial F}{\partial x_i'}\right) \\
&= \sum_{i=1}^{n} x_i' \underbrace{\left(\frac{\partial F}{\partial x_i} - \frac{d}{dt}\frac{\partial F}{\partial x_i'}\right)}_{E-L\ equation} = 0
\end{aligned}
$$

$$
\boxed{F - \sum_{i=1}^{n} x_i' \frac{\partial F}{\partial x_i'} = k, \quad k \text{ is a constant}}
$$

10.2 Application to Mechanics

- Particle of mass m moves in a force field with potential energy $V(\vec{x})$. Then the force acting on the particle is $-\nabla_x V(\vec{x})$. From Newton's second law we have

$$m\vec{x}''(t) = -\nabla_x V(\vec{x}).$$

- **Hamolton's principle:** $\vec{x}(t)$ is the extremal of the action functional

$$J[\vec{x}] = \int_a^b \left[\underbrace{\frac{1}{2}m(x'(t))^2}_{kinetic\ energy} - \underbrace{V(\vec{x}(t))}_{potential\ energy}\right] dt$$

- **Proof:** From E-L equations for $F = \frac{1}{2}m(x'(t))^2 - V(\vec{x}(t))$ we have that

$$\frac{\partial V}{\partial x_i} - \frac{d}{dt}(mx_i') = 0$$

which gives Newton's second law.

- **Conservation of energy** F is independent of t thus (see above) we have

$$\frac{1}{2}m(x'(t))^2 - V(\vec{x}(t)) - \sum_{i=1}^{n} x_i' m x_i' = -(\frac{1}{2}m(x'(t))^2 + V(\vec{x}(t))) = constant$$

Thus the energy is independent of t.

- **Polar coordinates** Express $\vec{x}$ in terms of the polar coordinates $r(t)$ and $\theta(t)$.

$$
\begin{aligned}
\vec{x}(t) &= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} r(t)\cos\theta(t) \\ r(t)\sin\theta(t) \end{bmatrix} \\
\vec{x}'(t) &= r'(t)\begin{bmatrix} \cos\theta(t) \\ \sin\theta(t) \end{bmatrix} + r(t)\theta'(t)\begin{bmatrix} -\sin\theta(t) \\ \cos\theta(t) \end{bmatrix} \\
|\vec{x}'(t)|^2 &= (r')^2 + r^2(\theta')^2
\end{aligned}
$$

Then the action functional takes form

$$J[r,\theta] = \int_a^b \frac{1}{2}[(r')^2 + r^2(\theta')^2] - V(r,\theta)dt$$

and E-L equations are

$$\begin{aligned} \frac{\partial F}{\partial r} - \frac{d}{dt}\frac{\partial F}{\partial r'} = 0 \\ \frac{\partial F}{\partial \theta} - \frac{d}{dt}\frac{\partial F}{\partial \theta'} = 0 \end{aligned} \quad (5)$$

10.3 Central Field and conservation of angular momentum

A central field has V depending only on distance r measured from some center (e.g. gravity force directed only to earth's center) thus $V = V(r)$ and F is independent of θ implying (by second E-L equation) that $\frac{\partial F}{\partial \theta'} = constant$ or equivalently

$$mr^2\theta' = constant$$

In mechanics this is interpreted as Conservation of Angular Momentum.

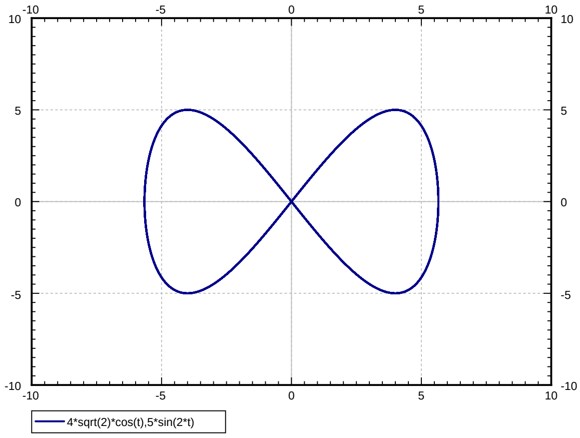


Figure 9: Lemniscate