Geometry and Calculus of Variations
Lecture notes *

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Abstract

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1 Lecture 11

1.1 Space Curves

- **Definition:** A space curve is a smooth map $\vec{x} : I \mapsto \mathbb{R}^3$, where I is a given interval in $\mathbb{R}$. We allow $I = \mathbb{R}$.
- **Definition:** The following is defined analogously to planar curves: **velocity, speed, regular curve, orientation, reparametrization, arc-length, unit tangent, frame.**
- **Example:** Let $\vec{x}(t) = (t, t^2, t^3)^T$ (cubic curve). It is easy to see that $\vec{x}'(t) = (1, 2t, 3t^2)^T, v(t) = \sqrt{1 + 4t^2 + 9t^4}$ and for the arclength we have $s(t) = \int_0^t \sqrt{1 + 4u^2 + 9u^4}du$
- **Example:** $\vec{x}(t) = (\cos t \cos 4t, \cos t \sin 4t, \sin t)$. Notice that $|\vec{x}(t)| = 1$ i.e. lies on the unit sphere. $\vec{x}(t)$ encircles the x_3 axis while climbing from equator to north pole. By direct computation

$$\vec{x}' = (-\sin t \cos 4t, -\sin t \sin 4t, \cos t)^T + \cos t(-4 \sin t, 4 \cos t, 0)^T.$$

$v(t) = \sqrt{1 + 16 \cos^2 t}$

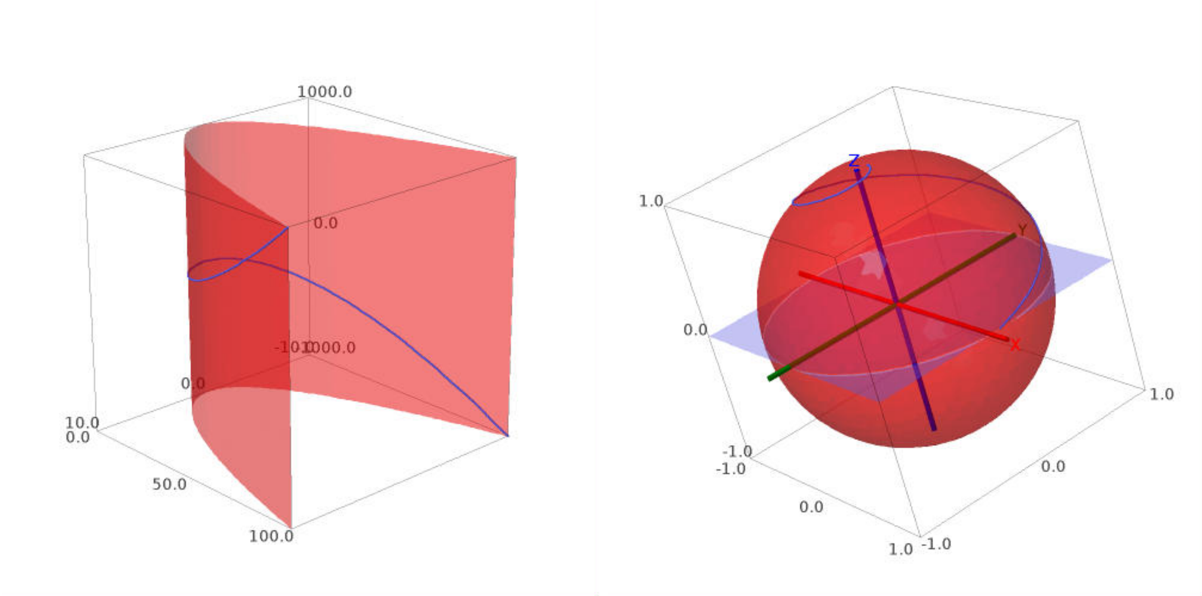
1.2 Curvature

- Unit tangent $\vec{T} = \vec{x}'/v$ is defined as before.
- **Defn:** Let $\vec{x}$ be a regular space curve. **Curvature** k is defined by

$$k(t) = \frac{1}{v}|\vec{T}'|.$$

Notice that for plane curves $k = \pm\frac{1}{v}|\vec{T}'|$ whereas for space curves $k \geq 0$.

*This is a preliminary version of the notes. It will be updated weekly.



(a) The cubic curve (b) The spheric curve

Figure 1: Examples of space curves

1.3 Biregular curves

- We want to define a Frenet frame along $\vec{x}$, i.e. a group of three mutually orthogonal unit vectors in $\mathbb{R}^3$. We can do so if $\vec{x}$ is **biregular**.
- **Definition:** Regular curve, i.e. the unit tangent $\vec{T}$ exists, is **biregular** if $\vec{T}' \neq 0$.
- Next we define the **Unit normal**: unlike plane curves there is a whole plane, called **normal plane** containing vectors perpendicular to $\vec{T}$. Among the normal vectors we give special names to two particular ones: normal and binormal.

  For biregular $\vec{x}$ the **unit normal** is the vector in the direction $\vec{T}'$

  $$\vec{N} = \frac{\vec{T}'}{|\vec{T}'|}.$$

  Clearly $\vec{N}$ exists, for $\vec{T}' \neq 0$. From the definition it follows that $\vec{N}$ and $\vec{T}$ are orthogonal (to see this differentiate $|\vec{T}(t)|^2 = 1$ by t).

- At each point $\vec{x}(t)$ we now have 2 orthogonal unit vectors $\vec{T}$ and $\vec{N}$. To complete the Frenet frame we introduce the **binormal** $\vec{B}$.

  $$\vec{B} = \vec{T} \times \vec{N}$$

  it is the vector product of $\vec{T}$ and $\vec{N}$.

- **The Frenet Frame** is the orthogonal set (3×3 orthogonal matrix) $(\vec{T}, \vec{N}, \vec{B})$ of tangent vectors to $\mathbb{R}^3$ at $\vec{x}(t)$. Recall that tangent vectors to $\mathbb{R}^3$ at $\vec{x}(t)$ need not point along curve, but unit tangent vector $\vec{T}$ does.

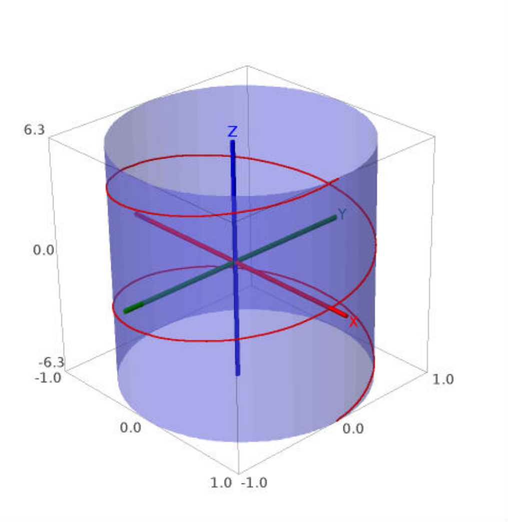


Figure 2: Helix with $k = a = 1$.

1.4 Formulae for $\vec{T}$ and $\vec{N}$

Let $\vec{x}(t), t \in I$ be a parametrisation of the curve $\vec{x}$ then differentiating $\vec{x}$ and using definitions of $\vec{T}, \vec{N}$ and k we conclude

$$\begin{aligned}\vec{x}' &= v\vec{T}\\ \vec{x}'' &= v'\vec{T} + v\vec{T}' = v'\vec{T} + v^2 k\vec{N}.\end{aligned}$$

In particular if $s = t$, i.e. the curve is arc length parametrised, then

$$\begin{aligned}\vec{x}' &= \vec{T}\\ \vec{x}'' &= \vec{T}' = k\vec{N},\end{aligned}$$

So $\vec{N}$ is unit vector orthogonal $\vec{T}$ in plane formed by $\vec{x}''$ and $\vec{x}' = v\vec{T}$.

Examples: 1) Helix: $\vec{x}(t) = (a\cos t, a\sin t, kt)$. Clearly $\vec{x}''(t) = (-a\cos t, -a\sin t, 0)$ points along $\vec{N}$. 2) Cubic curve: $\vec{x}(t) = (t, t^2, t^3)^T$ we have at $t = 0$, $\vec{x}'(0) = (1, 0, 0), \vec{x}''(0) = (0, 1, 0)$.

2 Lecture 12

2.1 Structural equations

We have defined the Frenet frame for a biregular curve $\vec{x}$ as orthogonal 3×3 matrix $(\vec{T}, \vec{N}, \vec{B})$ where

$$\begin{aligned}\vec{T} &= \frac{\vec{x}'}{v}\\ \vec{N} &= \frac{\vec{T}'}{|\vec{T}'|}\\ \vec{B} &= \vec{T}\times\vec{N}\end{aligned}$$

We want to derive the structural equations that is to relate $(\vec{T}', \vec{N}', \vec{B}')$ to orthogonal 3×3 matrix $(\vec{T}, \vec{N}, \vec{B})$.

- **Step 1: Equation for** $\vec{T}'$ From the definition of curvature and the unit normal we have

  $$\vec{T}' = |\vec{T}'|\vec{N} \qquad k = \frac{|\vec{T}'|}{v}$$

  combining these two equations we get that

  $$\vec{T}' = kv\vec{N}$$

- **Step 2: Equation for** $\vec{B}'$. It is quite easy to see that for $\vec{a}(t)$ and $\vec{b}(t)$

  $$\frac{d}{dt}(\vec{a}\times\vec{b}) = \vec{a}'\times\vec{b} + \vec{a}\times\vec{b}'$$

  recalling the definition of $\vec{B}$ and utilizing this formula we get

  $$\vec{B}' = \frac{d}{dt}(\vec{T}\times\vec{N}) = \vec{T}'\times\vec{N} + \vec{T}\times\vec{N}' = \vec{T}\times\vec{N}'$$

  since $\vec{T}'$ and $\vec{N}$ are parallel. Thus $\vec{T}\times\vec{N}'$ is parallel to $\vec{N}$ which yields

  $$\vec{B}' = -\tau v\vec{N}$$

  for some scalar function τ.

- **Step 3: Equation for** $\vec{N}'$ Notice that $\vec{N} = \vec{B}\times\vec{T}$. Thus

  $$\vec{N}' = \vec{B}'\times\vec{T} + \vec{B}\times\vec{T}' = -\tau v\vec{N}\times\vec{T} + \vec{B}\times(kv\vec{N}) = -kv\vec{T} + \tau v\vec{B}$$

- Summarizing we get the structural equations

  $$(\vec{T}', \vec{N}', \vec{B}') = (\vec{T}, \vec{N}, \vec{B})v\begin{bmatrix} 0 & -k & 0\\ k & 0 & -\tau\\ 0 & \tau & 0\end{bmatrix} \tag{1}$$

- **Defn:** $\tau : I \mapsto \mathbb{R}$ is the **torsion**.

- **Defn:** The plane determined by the unit tangent vector $\vec{T}$ and the unit normal vector $\vec{N}$ is called the osculating plane.

Theorem: Curvature and torsion are unchanged by reparametrization.

Proof:

Step 1: Velocity and speed Let us take $t = g(u), u$ is the new parameter and $g' \neq 0$ then $\vec{x}_1(u) = \vec{x}(g(u))$ and

$$\vec{x}_1' = \frac{d\vec{x}}{dt} g' \qquad \frac{v}{v_1} = \frac{1}{|g'|}$$

Step 2: Tangent By definition

$$\vec{T}_1 = \frac{\vec{x}_1'}{v_1} \qquad \vec{T} = \frac{1}{v}\frac{d\vec{x}}{dt}$$

thus

$$\vec{T}_1 = \frac{\vec{x}_1'}{v_1} = \frac{\frac{d\vec{x}}{dt} g'}{v_1} = \frac{\vec{T} v g'}{v_1} = \frac{\vec{T} v g'}{v|g'|} = \varepsilon \vec{T}$$

where $\varepsilon = \frac{g'}{|g'|} = \pm 1$: $+1$ if g is orientation preserving or -1 orientation reversing.

Step 3: Curvature The the u-derivative of the tangents equality and utilize the structural equations

$$v_1 k_1 \vec{N}_1 = \frac{d\vec{T}_1}{du} = \varepsilon \frac{d\vec{T}}{dt} g' = \varepsilon v k g' \vec{N}$$

Taknig the magnitude of both sides and noting that both $\vec{N}_1$ and $\vec{N}$ are unit vectors we conclude

$$v_1 k_1 = |\varepsilon| v k |g'| \Rightarrow k_1 = \frac{v}{v_1} k |g'| = k$$

Moreover

$$\vec{N}_1 = \frac{\varepsilon v k g'}{v_1 k_1} \vec{N} = \varepsilon \frac{v}{v_1} g' \vec{N} = \varepsilon \frac{g'}{|g'|} = \varepsilon^2 \vec{N} = \vec{N}.$$

Step 4: Torsion Finally we want to show that torsion is unchanged. By definition $\vec{B}_1 = \vec{T}_1 \times \vec{N}_1 = \varepsilon \vec{T} \times \vec{N} = \varepsilon \vec{B}$.
Differentiating with respect u then $\vec{B}_1' = \varepsilon \frac{d\vec{B}}{dt} g'$ and using the structural equations

$$\vec{B}_1' = -v_1 \tau_1 \vec{N}_1 = \varepsilon(-v\tau\vec{N}) g' = \varepsilon B' g'$$

taking the magnitude

$$\tau_1 = \varepsilon \frac{v}{v_1} g' \tau = \varepsilon \frac{g'}{|g'|} \tau = \varepsilon^2 \tau = \tau$$

and the proof of the Theorem is now complete.

2.2 Interpretation of curvature and Torsion

The trajectory of a particle moving in space produces a curve. The **osculating plane** is formed by unit vectors $\vec{T}$ and $\vec{N}$. For general space curves the particle tries to escape the osculating plane, for otherwise it is a plane curve.

When k is constantly zero, $\vec{T}$ never changes and the "curve" is a straight line. As the name "curvature" suggests, k measures the rate at which any nonstraight curve tends to depart from its tangent.

When τ is constantly zero, the osculating plane never changes, and we have plane curve, with constant binormal $\vec{B}$. Thus the torsion measures the rate at which a twisted curve tends to depart from its osculating plane.

Without loss of generality we may take the arc length parametrisation of $\vec{x}$ and assume that $\vec{T}, \vec{N}, \vec{B}$ at $\vec{x}(s_0)$ coincide with the directions of x, y and z-axis. Let $\vec{X}$ be the projection of the curve onto the osculating plane at $\vec{x}(s_0)$. Then

$$\vec{X}(s) = \vec{x}(s) - \vec{x}(s_0) - \vec{B}[(x(s) - x(s_0)) \cdot \vec{B}].$$

Then

$$\vec{X}' = \vec{x}' - \vec{B}(\vec{x}' \cdot \vec{B}),$$
$$\vec{X}'' = \vec{x}'' - \vec{B}(\vec{x}'' \cdot \vec{B})$$

In particular at $s = s_0$, using the fact that $\vec{B}$ is orthogonal to $\vec{T}$ and $\vec{N}$, we get $\vec{X}' = \vec{x}' = \vec{T}, \vec{X}'' = \vec{x}'' = k\vec{N}$. Thus the curvature of $\vec{X}$ at $s = s_0$ is $k = \det[\vec{X}', \vec{X}'']$.

Thus the projection of the curve onto the osculating plane at $\vec{x}(t_0)$ has curvature k.

Now consider the projection onto the **normal plane** passing through $\vec{x}(t_0)$, i.e. formed by $\vec{N}(t_0)$ and $\vec{B}(t_0)$. There the projected curve has Frenet frame $(\mathbf{t}, \mathbf{n})$, with $\mathbf{t} = \vec{N}(t_0), \mathbf{n} = \vec{B}(t_0)$ at $t = t_0$ satisfies

$$\mathbf{t}' = \tau\mathbf{n}, \qquad \mathbf{n}' = -\tau\mathbf{t}.$$

thus τ measures the rotation in **normal plane** about $\vec{T}$-axis as we pove along the curve.

2.3 Spherical image and Darboux vector

Let us define the vector

$$\omega = v\tau\vec{T} + vk\vec{B},$$

then $|\omega| = v\sqrt{k^2 + \tau^2}$ and

$$\begin{aligned} \vec{T}' &= \omega \times \vec{T} \\ \vec{N}' &= \omega \times \vec{N} \\ \vec{B}' &= \omega \times \vec{B} \end{aligned}$$

which can be readily verified. ω is called **Darboux vector**. It is the angular velocity of the particle when the point is moving along the curve.

2.4 Formulae for k and r

Proposition 1 *Let $\vec{x}$ be a biregular curve then*

$$k = \frac{1}{v^3}|\vec{x}' \times \vec{x}''| \qquad \tau = \frac{(\vec{x}' \times \vec{x}'') \cdot \vec{x}'''}{v^6 k^2}$$

Proof: Let's differentiate $\vec{x}$ three times and use structural equations

$$\begin{aligned} \vec{x}' &= v\vec{T} \\ \vec{x}'' &= v'\vec{T} + v\vec{T}' = v'\vec{T} + v^2k\vec{N}, \\ \vec{x}''' &= v''\vec{T} + v'\vec{T}' + 2vv'k\vec{N} + v^2k'\vec{N} + v^2k\vec{N}', \\ &= v''\vec{T} + v'vk\vec{N} + 2vv'k\vec{N} + v^2k'\vec{N} + v^2k(-kv\vec{T} + \tau v\vec{B}). \end{aligned}$$

Hence $\vec{x}' \times \vec{x}'' = v\vec{T} \times (v'\vec{T} + v^2k\vec{N}) = v^3k\vec{T} \times \vec{N} = v^3k\vec{B}$. Now taking the magnitude of this vector we conclude $k = \frac{1}{v^3}|\vec{x}' \times \vec{x}''|$.

Finally using the fact that $\vec{B}$ is orthogonal to $\vec{N}$ and $\vec{T}$ and taking the scalar product of $\vec{x}' \times \vec{x}''$ with $\vec{x}'''$ we conclude

$$[\vec{x}' \times \vec{x}''] \cdot \vec{x}''' = v^3kv^3k\tau = v^6k^2\tau$$

and the proof follows.

3 Lecture 13

3.1 Local behaviour

Theorem 2 *If we choose the Frenet frame (the trihedron) such that $\vec{T}(0) = \vec{e}_1, \vec{N}(0) = \vec{e}_2, \vec{B}(0) = \vec{e}_3$ and take the arc-length parametrization then in this coordinates*

$$\boxed{\vec{x}(s) \approx s\vec{e}_1 + \frac{k}{2}s^2\vec{e}_2 + \frac{kr}{6}s^3\vec{e}_3}$$

here $\vec{e}_1, \vec{e}_2, \vec{e}_3$ is the standard orthogonal basis in $\mathbb{R}^3$.

To see this we use Taylor's expansion

$$\vec{x}(s) = \vec{x}(0) + \vec{x}'(0)s + \frac{\vec{x}''(0)}{2!}s^2 + \frac{\vec{x}'''(0)}{3!}s^3 + \dots$$

and use the structural equations (remember $v = 1$!)

$$\begin{aligned} \vec{x}' &= \vec{T} \\ \vec{x}'' &= \vec{T}' = k\vec{N} \\ \vec{x}''' &= k'\vec{N} + k\vec{N}' \\ &= k'\vec{N} + k(-k\vec{T} + r\vec{B}). \end{aligned}$$

Thus assuming that $\vec{x}(0) = 0$ we get

$$\begin{aligned} \vec{x}(s) &= \vec{T}s + \frac{k\vec{N}}{2!}s^2 + \frac{k'\vec{N} + k(-k\vec{T} + r\vec{B})}{3!}s^3 + \dots \\ &= \vec{T}(s - \frac{k^2}{3!}s^3) + \vec{N}(\frac{k}{2!}s^2 + \frac{k'}{3!}s^3) + \frac{rk}{3!}s^3 + \dots \end{aligned}$$

Assuming that s is very small we can ignore the higher powers of s within the coefficients of $\vec{T}, \vec{N}$ and $\vec{B}$ thus

$$\vec{x}(s) = s\vec{e}_1 + \frac{k}{2}s^2\vec{e}_2 + \frac{kr}{6}s^3\vec{e}_3 + \dots = \begin{bmatrix} ss \\ \frac{k}{2}s^2 \\ \frac{k\tau}{6}s^3 \end{bmatrix}$$

And the result follows. Thus locally any biregular curve is cubic!

3.2 Implicitly defined curves

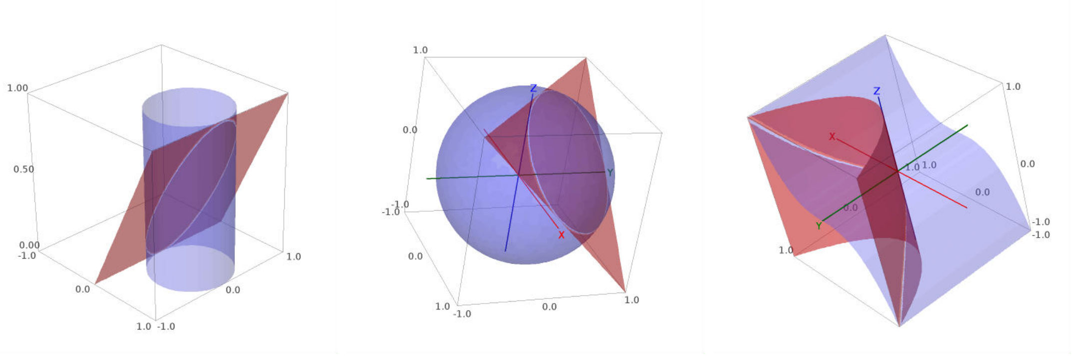
Let f_1, f_2 be functions of three variables, $f_1 : (x, y, z) \mapsto \mathbb{R}$, $f_2 : (x, y, z) \mapsto \mathbb{R}$. Consider $\mathcal{C}$ the intersection of the zero level sets of f_1 and f_2, then it is an implicitly defined space curve, provided that it is not empty.

The tangent plane to the graph of f_1 has normal ∇f_1 thus the tangent plane, Π_1, at (x_0, y_0, z_0) is $\nabla f_1(x_0, y_0, z_0)\cdot(x-x_0, y-y_0, z-z_0) = 0$. Similarly Π_2, given by $\nabla f_2(x_0, y_0, z_0)\cdot(x-x_0, y-y_0, z-z_0) = 0$ is the tangent plane to the graph of f_2 at (x_0, y_0, z_0). If $(x_0, y_0, z_0) \in \mathcal{C}$ then the both Π_1 and Π_2 are tangent tp $\mathcal{C}$ at (x_0, y_0, z_0). Clearly the tangent vector is collinear to the vector products of the normals of Π_1 and Π_2, thus

$$\vec{T} = \frac{\nabla f_1(x_0, y_0, z_0) \times \nabla f_2(x_0, y_0, z_0)}{|\nabla f_1(x_0, y_0, z_0) \times \nabla f_2(x_0, y_0, z_0)|}.$$

Defn: An implicitly defined space curve is said to be regular if $\nabla f_1 \times \nabla f_2 \neq 0$.

Clearly the tangent vector is defined for regular implicitly defined space curves.



(a) (b) (c)

Figure 3: Examples of implicit space curves

3.3 Examples

a) Let $f_1 = x_1^2 + x_2^2 - x_1$ and $f_2 = x_3 - x_1$

b) Let $f_1 = x_1^2 + x_2^2 + x_3^2 - 1$ and $f_2 = x_1 + x_2 + x_3 - 1$. Thus $f_1 = 0$ is the unit sphere and $f_2 = 0$ is a plane so intersection is a circle in $\mathbb{R}^3$.

c) Another example is the intersection of $S_1 : x_2 = x_1^2$ and $S_2 : x_3 = x_1^3$.

3.4 Torsion Examples

Example 1: Biregular plane curve has $\tau = 0$. This is a simple exercise.

The converse statement is also true. A biregular curve with $\tau = 0$ is planar! (Hint: Use structural equations).

Example 2: A helix has constant curvature and torsion. It is staightforward to verify that for helix $\vec{x}(t) = (a\cos t, a\sin t, bt)^T$

$$k = \frac{a}{a^2 + b^2}, \qquad \tau = \frac{b}{a^2 + b^2}.$$

Conversely let us show that if k and τ are constants then $\vec{x}$ is a helix. If $\tau = 0$ then $\vec{x}$ is a plane curve with constant curvature thus it is a circle, for $k = \frac{d\theta}{ds}$, θ is an argument function (Lecture 6) and hence $\theta = ks + C$ where C is an arbitarary constant. Hence $\vec{x}' = \vec{T} = (\cos(kt + C), \sin(kt + C))^T$ implying that $\vec{x} = \frac{1}{k}(\sin(kt + C) + A, -\cos(kt + C)) + B)^T$ which is a circle of radius $R = 1/k$ centered at (A, B).

Thus we assume that $\tau \neq 0$. Let's take the arc length parametrisation of the curve. Then Darboux vector $\omega = \tau\vec{T} + k\vec{B}$ is constant, for $\omega' = \tau\vec{T}' + k\vec{B}' = \tau(k\vec{N}) + k(-\tau\vec{N}) = 0$. Without loss of generality we may assume that ω is collinear to e_3 so $\omega = e_3\sqrt{k^2 + \tau^2}$.

Next $\omega \cdot \vec{T} = (\tau\vec{T} + k\vec{B}) \cdot \vec{T} = \tau$, thus the unit tangent vector makes constant angle α with the e_3 axis and $\cos\alpha = \frac{\tau}{\sqrt{k^2+\tau^2}}$.

Furthermore

$$\frac{dx_3}{ds} = \frac{d(\vec{x} \cdot \omega)}{ds} = \vec{x}' \cdot \omega = \vec{T} \cdot \omega = \cos\alpha$$

thus $x_3(s) = s\cos\alpha + C$, where C is an arbitrary constant.

Next let us consider the projection of the curve onto the plane orthogonal to $\omega = e_3$, i.e. $\vec{X} = \vec{x} - (\vec{x} \cdot \omega)\omega$. For this plane curve we have

$$\vec{X}' = \vec{x}' - (\vec{x}' \cdot \omega)\omega, \qquad \vec{X}'' = \vec{x}''$$

In particular the speed of $\vec{X}$ is $V = \sqrt{(\vec{x}')^2 - 2(\vec{x}' \cdot \omega)^2 + (\vec{x}' \cdot \omega)^2} = \sqrt{1 - (\vec{x}' \cdot \omega)^2} = \sqrt{1 - \tau^2}$ thus V is constant. If $(\mathbf{t}, \mathbf{n})$ is the Frenet frame of $\vec{X}$−the projected curve, then from Frenet equations we have

$$k\vec{N} = \vec{x}'' = \vec{X}'' = \frac{d}{ds}(\vec{X}') = \frac{d}{ds}(V\mathbf{t}) = V\mathbf{t}' = Vk_{\vec{X}}\mathbf{n},$$

where $k_{\vec{X}}$ is the curvature of the projected curve $\vec{X}$. Equating the right and left hand sides and taking the magnitudes gives $k_{\vec{X}} = k/V = const$. Thus $\vec{X}$ is a circle.

4 Lecture 14

4.1 Surfaces

Defn: A **local surface in** $\mathbb{R}^3$ is a smooth, injective map $\mathbf{x} : D \to \mathbb{R}^3$ with continuous inverse $\mathbf{x}^{-1} : \mathbf{x}(D) \to D$ where D is a domain (open, connected set) in $\mathbb{R}^2$.

$$\mathbf{x} : (u, v) \mapsto \mathbf{x}(u, v)$$

Remarks:

1) The assumption of smoothness implies that $\mathbf{x}$ is somewhat distorted version of D
2) $\mathbf{x}$ is injective points of $\mathbf{x}(D)$ are labelled (coordinated) by corresponding points of D
3) the inverse mapping $\mathbf{x}^{-1}$ is continuous prevents "near self-intersection"

Under these conditions $\mathbf{x}$ is a homeomorphism, i.e. a bijection with continuous inverse $\mathbf{x}^{-1}$.

4.2 Surfaces in $\mathbb{R}^3$

- If local surface $\mathbf{x}$ lies in a set $\Sigma \subset \mathbb{R}^3$ then **$\mathbf{x}$ is a local surface in** Σ.
- **A surface in** $\mathbb{R}^n$ is a subset $\Sigma \subset \mathbb{R}^3$ such that for each point p of Σ there exists a local surface in Σ whose image contains a neighborhood N of p in Σ.

4.3 Examples

- **Sphere:** Let us consider

$$\mathbf{x}(u, v) = \begin{bmatrix} \sin u \cos v \\ \sin u \sin v \\ \cos u \end{bmatrix}, \qquad (u, v) \in D = (0, \pi) \times (0, 2\pi).$$

  The trace of $\mathbf{x}$ is unit sphere excluding semi-circle connecting North and South poles.

  Line $u = const$ is line of latitude and line $v = const$ is line of longitude.

- **Ellipsoid:** Let us consider

$$\mathbf{x}(u, v) = \begin{bmatrix} a \sin u \cos v \\ b \sin u \sin v \\ c \cos u \end{bmatrix}, \qquad (u, v) \in D = (0, \pi) \times (0, 2\pi).$$

  Here a, b, c are non-zero constants. The trace is an ellipsoid with n arc omitted.

- **Graph of function:** Now let $f : D \mapsto \mathbb{R}$ be a smooth function. Consider

$$\mathbf{x}(u,v) = \begin{bmatrix} u \\ v \\ f(u,v) \end{bmatrix}.$$

Note that not all surfaces can be represented as graphs. See the example of unit sphere.

- **Surface of revolution:** Let $t \mapsto \begin{bmatrix} p(t) \\ q(t) \\ 0 \end{bmatrix}$ be a curve with trace in the (x_1, x_2) plane and rotate it about the x_1 axis to get

$$\mathbf{x}(u,v) = \begin{bmatrix} p(u) \\ q(u)\cos v \\ q(u)\sin v \end{bmatrix}.$$

4.4 Regularity

For a surface $\mathbf{x}$, the **partial velocities** at $\mathbf{x}(u,v)$ are tangent vectors to $\mathbb{R}^3$ at $\mathbf{x}(u,v)$ given by

$$\mathbf{x}_u = \begin{bmatrix} \frac{\partial x_1}{\partial u} \\ \frac{\partial x_2}{\partial u} \\ \frac{\partial x_3}{\partial u} \end{bmatrix}, \qquad \mathbf{x}_v = \begin{bmatrix} \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial v} \\ \frac{\partial x_3}{\partial v}. \end{bmatrix}$$

Interpretation: The space curves (lines) $u \mapsto \mathbf{x}(u, v_0)$ and $v \mapsto \mathbf{x}(u_0, v)$, for fixed u_0 and v_0 have velocities $\mathbf{x}_u$ and $\mathbf{x}_v$.

Defn: The local surface is **regular** if $\mathbf{x}_u$ and $\mathbf{x}_v$ are linearly independent at each point of D.

4.5 Displacement and area

Consider a small displacement $\begin{bmatrix} \delta u \\ \delta v \end{bmatrix}$ at each the point $p_0 = (u_0, v_0) \in D$. Then it is mapped by the matrix $[\mathbf{x}_u, \mathbf{x}_v]$ to $[\mathbf{x}_u, \mathbf{x}_v]\begin{bmatrix} \delta u \\ \delta v \end{bmatrix}$. The regularity implies that $\mathbf{x}_u$ and $\mathbf{x}_v$ have distinct directions at each $p_0 \in D$. Hence a small rectangle $(u_0 + \delta u) \times (v_0 + \delta v)$ in (u,v) space is approximately mapped, by $\mathbf{x}$ to the parallelogram with sides $\mathbf{x}_u \delta u, \mathbf{x}_v \delta v$.

If $\mathbf{x}_u, \mathbf{x}_v$ are linearly independent then the matrix $[\mathbf{x}_u, \mathbf{x}_v]$, which approximates $\mathbf{x}$, defines an invertible linear transformation between above rectangle and parallelogram.

The Inverse Function Theorem says that the same holds for the exact map $\mathbf{x}$: Around each point $p_0 \in D$, there exists a neighborhood $\mathbf{u}$ on which $\mathbf{x}$ is a smooth bijection between N and $\mathbf{x}(N)$, where N is a neighborhood of $\mathbf{x}(p_0)$.

4.6 Regularity of previous examples

The sphere is a regular local surface since $\mathbf{x}_u$ and $\mathbf{x}_v$ are linearly independent. Similarly it follows that ellipsoid is regular too.

5 Lecture 15

5.1 Regular implicitly defined surfaces

Defn: Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function and $c \in \mathbb{R}$ be such that the level set $\Sigma = \{\mathbf{x} \in \mathbb{R}^3 : f(\mathbf{x}) = c\}$ is non-empty.

If $\nabla f \neq 0,\ \forall \mathbf{x} \in \Sigma$ then Σ is a **regular implicitly defined surface**.

Theorem: A regular implicitly defined curve surface is a surface in $\mathbb{R}^3$.

Geometric inpterpretation: ∇f is the normal to Σ, if $\frac{\partial x_3}{\partial x_3}$ surface contains no vertical directions surface locally graph of $x_3(x_1, x_2)$.

Examples:

- $f(\mathbf{x}) = |\mathbf{x}|^2 - 1$
- $f(\mathbf{x}) = x_3^2 - x_1^2 - x_2^2$
- Quadric surface $k_1 x_1^2 + k_2 x_2^2 + k_3 x_3^2 = 1$

6 Lecture 16

6.1 Curve in a surface

The curves lying in a surface provide important information about the surface.

Defn: Let Σ be a surface. The curve $\alpha : I \to \mathbb{R}^3$ is a curve in Σ if $\alpha(I) \subset \Sigma$.

When $\Sigma = \mathbf{x}(D)$ is a local surface we can pull back α to D, since it is much easier to work out in D than in $\mathbf{x}(D)$

Lemma Let α be a curve in Σ, Then there exists aunique smooth curve $\mathbf{w} : I \to D$ so that

$$\alpha(t) = \mathbf{x}(\mathbf{w}(t)).$$

Proof: Define $\mathbf{w} = \mathbf{x}^{-1} \circ \alpha : I \to D$, so $\alpha = \mathbf{x} \circ (\mathbf{x}^{-1} \circ \alpha)$.

Example: $D = (0, 2\pi) \times (0, \infty), \mathbf{x}(u, v) = (\cos u, \sin u, v)^T$. Take

- $\mathbf{w}_1(t) = (2t, \pi/t)^T, \mathbf{x}(\mathbf{w}_1(t)) = (\cos 2t, \sin 2t, \pi/4)^T$
- $\mathbf{w}_2(t) = (\pi/2, t)^T, \mathbf{x}(\mathbf{w}_2(t)) = (0, 1, t)^T$
- $\mathbf{w}_3(t) = (2t, t)^T, \mathbf{x}(\mathbf{w}_3(t)) = (\cos 2t, \sin 2t, t)^T$

6.2 Tangent space to surface Σ

Recapitulating: $T_p\mathbb{R}^3$, the tangent space to $\mathbb{R}^3$ at p, is space of all tangent vectors (p, v) to $\mathbb{R}^3$ at p.

Defn: Let $p \in \Sigma$. A tangent vector $\mathbf{v}$ to $\mathbb{R}^3$ at p is a **tangent to** Σ if $\mathbf{v}$ is the velocity at p of some curve in Σ.

The set of all such tangent vectors to Σ at p is the tangent space (plane) to Σ at p, written $T_p\Sigma$.

Next we charcterise of tangent space:

Lemma Let $\mathbf{x}(D)$ be a local surface in Σ and let $p = \mathbf{x}(w_0) \in \mathbf{x}(D)$. Then $T_p\Sigma$ is the subspace of $T_p\mathbb{R}^3$ spanned by $\mathbf{x}_u(\mathbf{w}_0)$ and $\mathbf{x}_v(\mathbf{w}_0)$.

Proof: $\boxed{\text{Step1: } T_p\Sigma \subset \text{Span}(\mathbf{x}_u, \mathbf{x}_v)}$. Let $t \mapsto \mathbf{w}(t)$ be a curve in D, such that $\mathbf{w}(0) = \mathbf{w}_0$. Then $t \mapsto \mathbf{x}(\mathbf{w}(t))$ is a curve in $\mathbf{x}(D)$, passing through p when $t = 0$, ant its velocity at p is

$$\frac{d}{dt}[\mathbf{x}(\mathbf{w}(t))]_{t=0} = \mathbf{x}_u(\mathbf{w}_0)u'(0) + \mathbf{x}_v(\mathbf{w}_0)v'(0).$$

By definition the velocity is in $T_p\Sigma$.

Step2: $\text{Span}(\mathbf{x}_u, \mathbf{x}_v) \subset T_p\Sigma$.
For any $\lambda, \mu \in \mathbb{R}$:

$$\lambda \mathbf{x}_u(\mathbf{w}_0) + \mu \mathbf{x}_v(\mathbf{w}_0) = \frac{d}{dt}[\mathbf{x}(\mathbf{w}_0 + t(\lambda, \mu)^T)]_{t=0}$$

is a velocity of a curve in $\mathbf{x}(D)$.

Summariging $\text{Span}(\mathbf{x}_u, \mathbf{x}_v) = T_p\Sigma$.

7 Lecture 17

7.1 Vector fields on a surface

Defn: A vector field Z on a surface Σ is an assignment to each $p \in \Sigma$ of a tangent vector $\mathbf{Z}(p)$ to $\mathbb{R}^3$ at p.

A unit normal vector field $\mathbf{N}$ is a vector field on Σ such that at each $p \in \Sigma : |\mathbf{N}(p)| = 1, \mathbf{N}(p) \perp T_p\Sigma$
On regular local surface $\mathbf{x}$, a smooth unit normal field is given by

$$\mathbf{N} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}$$

where as for regular implicitly defined surface $\mathbf{N} = \frac{\nabla f}{|\nabla f|}$.

Example: The normal vector field of the sphere $\mathbf{x}(u, v) = \mathbf{N}$ and for the graph of function $z = g(x, y)$ it is $\mathbf{N} = \frac{\nabla f}{|\nabla f|} = \frac{(-g_x, -g_y, 1)}{\sqrt{1+g_x^2+g_y^2}}$.

7.2 First Fundamental form of Surface

Definition: Let $p : \mathbf{x}(\vec{w}_0)$ be a point in a regular surface Σ. **The first fundamental form** is a symmetric bilinear form on $T_p\Sigma$ defined by

$$I_p(\mathbf{X}, \mathbf{Y}) = \mathbf{X} \cdot \mathbf{Y}, \qquad \forall \mathbf{X}, \mathbf{Y} \in T_p\Sigma$$

The matrix that defines this bilinear form is

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

where

$$E = \mathbf{x}_u \cdot \mathbf{x}_u, \qquad F = \mathbf{x}_u \cdot \mathbf{x}_v, \qquad G = \mathbf{x}_v \cdot \mathbf{x}_v.$$

7.3 Examples

- Sphere
- Cylinder
- Graph
- Surface of revolution

8 Lecture 18

8.1 Arclength

Let $\vec{z}(t) = \mathbf{x}(\vec{w}), \vec{w}(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$ be a regular curve lying in Σ. Its velocity is

$$\vec{z}'(t) = (\mathbf{x}_u \mathbf{x}_v) \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix}$$

Hence the arclength of $\vec{z}$ from $\vec{z}(a)$ to $\vec{z}(b)$ is

$$\int_a^b |\vec{z}'(t)|dt = \int_a^b (E(u')^2 + 2Fu'v' + G(v')^2)^{1/2} dt.$$

Example 1: For the sphere $\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2 u \end{bmatrix}$. Take $\mathbf{w}(t) = (\pi - t, 2t), t \in (o, \pi)$. $\mathbf{w}'(t) = (-1, 2)$ hence $s = \int_0^\pi [1 + 4\sin^2 t]^{1/2} dt$.

Example 2: For the cylinder $\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. For $\mathbf{w}(t) = (t,t)^T, t \in (0, 2\pi) \Rightarrow \mathbf{w}'(t) = (1,1)^T$ hence $s = \int_0^{2\pi} \sqrt{1^2 + 1^2} dt = 2\sqrt{2}\pi$.

8.2 Reparametrisation

We can reparametrise as usual: if $s \mapsto \mathbf{z}(s)$, lying in Σ, is arclength parametrised then its unit speed and $I(\mathbf{z}', \mathbf{z}') = 1$

Example: Cone has parametrisation $\mathbf{x}(u,v) = \begin{bmatrix} u\cos v \\ u \sin v \\ u \end{bmatrix}$ and $\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & u^2 \end{bmatrix}$. Let $\vec{w}(t) = (t, \sqrt{2}\ln t), t \in (0, \infty)$. Then $\mathbf{w}' = (1, \frac{\sqrt{2}}{t})$ and hence $s(t) = \int_0^t 2 = 2t$ so arc length parametrisation is $\mathbf{z}_1(s) = \mathbf{x}(\mathbf{w}_1(s))$, where $\mathbf{w}_1(s) = (\frac{s}{2}, \sqrt{2}\ln(\frac{s}{2}))^T$.

8.3 Isometry

Defn: If S_1 and S_2 are surfaces, a smooth map $\phi : S_1 \to S_2$ is called a **local isometry** if it takes any curve in S_1 to a curve of the same length in S_2. If a local isometry ϕ exists we say S_1 and S_2 are locally isomorphic.

Theorem: A smooth $\phi : S_1 \to S_2$ is a local isometry if and only if the matrices of first fundamental forms are equal.

Example: Plane $(u, v, 0)$ and cyliner $(\cos u, \sin u, v)$ are isomorphic sonce $\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

8.4 Geodesics

Motivation Look for analogue in surface Σ of straight lines in plane. Straight lines are characterised by: zero acceleration and distance minimising between two points.

Definition: A curve $\vec{z}$ in $\Sigma \subset \mathbb{R}^3$ is **a** geodesic of Σ if its acceleration $\vec{z}''$ is always normal to Σ.

Since $\vec{z}''$ is normal to Σ we get $\vec{z}'' \cdot \vec{z}'$ (as $\vec{z}'$ is tangent to Σ) thus $\frac{d|\vec{z}'|^2}{dt} = 2\vec{z}'' \cdot \vec{z}' = 0$ implying that $|\vec{z}'| = const$. Hence we conclude that geodesics have constant speed.

9 Lecture 19

Let us find the geodesics of the sphere and cylinder.

Sphere: First consider the sphere S of radius r, $\mathbf{x}\cdot\mathbf{x} = r^2$. If $\vec{z}$ is a geodesic then $\vec{z}''(t)$ is collinear to the normal of S at $p = \vec{z}(t)$ i.e. to $\vec{z}(t)$ since $\vec{z}$ is collinear to the normal. Using this observation we compute

$$\frac{d}{dt}(\vec{z}' \times \vec{z}) = \vec{z}' \times \vec{z}' + \vec{z}'' \times \vec{z}.$$

Clearly $\vec{z}' \times \vec{z}' = 0$. But $\vec{z}''$ is collinear to the normal, by the definition of geodesic, and $\vec{z}$ is collinear to the normal hence $\vec{z}'' \times \vec{z} =$. Combining we get

$$\frac{d}{dt}(\vec{z}' \times \vec{z}) = 0.$$

This identity, in particular, implies that the plane Π passing through 0 and formed by the vectors $\vec{z}'$ and $\vec{z}$ has constant normal $\vec{N}$ (independent of t). Since $\vec{z} \in \Pi$ it implies that

$$\vec{N} \cdot \vec{z}(t) = 0.$$

Thus the geodesics are the great circles.

Cylinder: It is a simple exercise. See also Lecture 20.

9.1 Distance minimisation

Theorem: Let $\Sigma = \mathbf{x}(D)$ be a regular, local surface and $\vec{z} = \mathbf{x} \circ \vec{w}$ a regular curve connecting $A = \vec{z}(a), B = \vec{z}(b)$. Then the arclength of $\vec{z}$

$$J[\vec{z}] = \int_a^b (\vec{z}'(s) \cdot \vec{z}'(s))^{1/2}\, ds$$

is stationary implies that $\vec{z}$ is a geodesic.

We want to show that the acceleration $\vec{z}''$ is collinear to $\mathbf{n}$. Let $\vec{z}$ be a curve for which the minimum is realised. Without loss of generality we may assume that $\vec{z}$ is of speed one, i.e. $\vec{z}'\cdot\vec{z}' = 1$ or equivalently $\sqrt{E(u')^2 + 2Fu'v' + G(v')^2} = 1$ where $\vec{z}(t) = \mathbf{x} \circ \mathbf{w}(t)$ and $\mathbf{w}(t) = (u(t), v(t)), t \in (a, b)$ is a curve in domain D. Introduce $\vec{z}(t, \varepsilon) = \mathbf{x}(\mathbf{w} + \varepsilon\mathbf{h})$ where $\varepsilon > 0$ is small and $\mathbf{h}(a) = \mathbf{h}(b) = 0$ thus $\vec{z}(t, \varepsilon)$ leaves the points A, B on the surface unchanged.

Using Taylor's expansion in ε at $\varepsilon = 0$, it is easy to see that

$$\vec{z}(t, \varepsilon) = \vec{z}(t, 0) + \varepsilon\frac{d\vec{z}}{d\varepsilon}(t, 0) + \varepsilon^2(\dots)$$

By definition $\dfrac{d\vec{z}}{d\varepsilon}(t, \varepsilon) = \dfrac{d}{d\varepsilon}\mathbf{x}(\mathbf{w} + \varepsilon\mathbf{h}) = \dfrac{\partial\mathbf{x}}{\partial u}h_1 + \dfrac{\partial\mathbf{x}}{\partial v}h_2$ where $\mathbf{h} = (h_1, h_2)$. Hence at $\varepsilon = 0$ we have $\dfrac{d\vec{z}}{d\varepsilon}(t, 0) = \mathbf{x}_u(\mathbf{w}(t))h_1(t) + \mathbf{x}_v(\mathbf{w}(t))h_2(t)$- linear combination of partial velocities hence $\dfrac{d\vec{z}}{d\varepsilon}(t, 0) \in T_p\Sigma$. Denote this tangent vector by $\mathbf{T} = \dfrac{d\vec{z}}{d\varepsilon}(t, 0) = \mathbf{x}_u(\mathbf{w}(t))h_1(t) + \mathbf{x}_v(\mathbf{w}(t))h_2(t)$. Thus $\vec{z}(t, \varepsilon) = \vec{z} + \varepsilon\mathbf{T} + \varepsilon^2(\dots)$ and $z'(t, \varepsilon) = \vec{z}'(t) + \varepsilon\mathbf{T}' + \varepsilon^2(\dots)$. It follows that

$$\vec{z}'(t, \varepsilon) \cdot \vec{z}'(t, \varepsilon) = [\vec{z}'(t) + \varepsilon\mathbf{T}' + \varepsilon^2(\dots)] \cdot [\vec{z}'(t) + \varepsilon\mathbf{T}' + \varepsilon^2(\dots)] = \vec{z}'(t) \cdot \vec{z}'(t) + 2\vec{z}(t) \cdot \mathbf{T}' + \varepsilon^2(\cdot)$$

therefore

$$\frac{d}{d\varepsilon}\int_a^b \sqrt{\vec{z}'(\tau,\varepsilon)\cdot\vec{z}'(\tau,\varepsilon)}d\tau \quad = \quad \int_a^b \frac{d}{d\varepsilon}\sqrt{\vec{z}'(\tau,\varepsilon)\cdot\vec{z}'(\tau,\varepsilon)}d\tau = \int_a^b \frac{2\mathbf{T}'\cdot\vec{z}'(\tau,\varepsilon)+\varepsilon(\dots)}{2\sqrt{\vec{z}'(\tau,\varepsilon)\cdot\vec{z}'(\tau,\varepsilon)}}$$

at $\varepsilon = 0$ we get

$$\int_a^b \frac{\mathbf{T}'\cdot\vec{z}'}{\sqrt{\vec{z}'(\tau)\cdot\vec{z}'(\tau)}} = 0$$

Since $\vec{z}$ is of speed one, i.e. $\vec{z}'(\tau)\cdot\vec{z}'(\tau) = 1$, we conclude after integration by parts

$$0 = \int_a^b \mathbf{T}'\cdot\vec{z}'d\tau = [\mathbf{T}(b)\vec{z}'(b) - \mathbf{T}(a)\vec{z}(a)] - \int_a^b \mathbf{T}\cdot\vec{z}''d\tau = -\int_a^b \mathbf{T}\cdot\vec{z}''d\tau$$

for $\mathbf{T}(b) = \mathbf{T}(a) = 0$ (recall that $\mathbf{T} = \mathbf{x}_u(\mathbf{w}(t))h_1(t) + \mathbf{x}_v(\mathbf{w}(t))h_2(t)$ and $\mathbf{h}(a) = \mathbf{h}(b) = 0$).

Decompose vector $\vec{z}'' = \vec{z}''_{\mathbf{n}} + \vec{z}''_T$ where $z_{\mathbf{n}}$ is collinear to normal $\mathbf{n}$ and $\vec{z}''_T \in T_p\Sigma$-the tangential component. Then

$$0 = \int_a^b \mathbf{T}\cdot\vec{z}'' = \int_a^b \mathbf{T}\cdot[\vec{z}''_{\mathbf{n}} + \vec{z}''_T] = \int_a^b \mathbf{T}\cdot\vec{z}''_T$$

Since $\mathbf{T}\in T_p\Sigma$ is an arbitrary tangent vector, for $\mathbf{h}$ is arbitrary, we may choose $\mathbf{T} = \vec{z}''_T$ so that

$$\int_a^b [\vec{z}''_T]^2 d\tau = 0$$

thus the tangential component of $\vec{z}''$ vanishes and thus $\vec{z}''$ is collinear to the normal $\mathbf{n}$.

10 Lecture 20

10.1 Energy

The energy functional of curves in Σ is

$$E[\vec{z}] = \int_a^b g(\vec{z}'(\tau))d\tau = \int_a^b \vec{z}'(\tau)\cdot\vec{z}'(\tau)d\tau$$

where we set $g(\vec{z}'(\tau)) = \vec{z}'(\tau)\cdot\vec{z}'(\tau)$.

The derivation of this statement is similar to the one for the arc length function J (see lecture 19). As above we take $\vec{z}(t,\varepsilon) = \mathbf{x}(\mathbf{w}(t) + \varepsilon\mathbf{h}(t))$. Then

$$\frac{d}{d\varepsilon}\int_a^b g(\vec{z}'(\tau,\varepsilon))d\tau = \int_a^b \nabla_{\vec{z}'} g(\vec{z}'(t,\varepsilon))\cdot\frac{d}{d\varepsilon}\vec{z}(\tau,\varepsilon)d\tau$$

thus at $\varepsilon = 0$ we have that

$$\int_a^b \nabla_{\vec{z}'} g(\vec{z}'(\tau))\cdot\mathbf{T}'d\tau = 0.$$

Noting that $\nabla_{\vec{z}'} g(\vec{z}') = \vec{z}'$ we conclude, after integration by parts that

$$0 = \int_a^b \vec{z}'\cdot\mathbf{T}' = -\int_a^b \vec{z}''\cdot\mathbf{T}$$

hence the tangential component of the acceleration $\vec{z}''$ vanishes and $\vec{z}''$ is collinear to the normal $\mathbf{n}$.

In other words if $E[\vec{z}]$ is stationary on curves then $\frac{d}{d\tau}(\nabla g)$ is normal to Σ. Now $\nabla g = 2\vec{z}'$ hence E stationary also implies that $\vec{z}''$ is normal to Σ i.e. extremals of E are also geodesics of Σ.

Remark: For the arclength we have to impose the constraint $|\vec{z}'| = 1$ otherwise we get the traces of geodesics. Plus the awkward $\sqrt{\ }$ term. For E the constant speed condition comes automatically and no awkward terms are present.

10.2 Surfaces of revolution

The question of finding the geodesics of a given surface may be very complicated. However if some additional properties of surface is known it can be reduced to a system of ODE's which may be solved explicitly. Such example is a surface of revolution. For convenience we take (θ, z) as independent variables

$$\mathbf{x}(\theta, z) = \begin{bmatrix} R(z)\cos\theta \\ R(z)\sin\theta \\ z \end{bmatrix}$$

Here z is the height of the point and $R(z) > 0$ is a given function. The partial velocities and the first fundamental form are

$$\mathbf{x}_\theta(\theta, z) = \begin{bmatrix} -R(z)\sin\theta \\ R(z)\cos\theta \\ 0 \end{bmatrix}, \qquad \mathbf{x}_z(\theta, z) = \begin{bmatrix} R'(z)\cos\theta \\ R'(z)\sin\theta \\ 1 \end{bmatrix}$$

$$E = R^2, \qquad F = 0, \qquad G = (R')^2 + 1$$

hence the energy functional is

$$E = \int_a^b R^2(\theta')^2 + (R'^2 + 1)\ z'^2$$

Consider the Euler-Lagrange equations (remember that the unknown functions are $\theta(t), z(t)$)

$$\frac{d}{dt}(2R^2\theta') = 0$$

$$\frac{d}{dt}(2(R'^2 + 1)z') - \frac{d}{dz}(R^2(\theta')^2 + (R'^2 + 1)\ z'^2) = 0$$

Unlike the first equation the second one is very complicated. Instead of simplifying it we concentrate on the first one and $R^2(\theta')^2 + (R'^2 + 1)\ z'^2 = 1$ (since the speed of geodesic is constant). Thus

$$\frac{d}{dt}(2R^2\theta') = 0$$

$$R^2(\theta')^2 + (R'^2 + 1)\ z'^2 = 1$$

The first equation means that $R^2\theta' = C$, C is a constant.

If $C = 0$ then $\theta = k$-constant giving the meridians as geodesics. Notice that $z = 0$ is not necessarily a geodesic.

If $C \neq 0$ then $\theta' = \frac{C}{R^2}$, then after dividing the second equation by θ'^2 we get

$$R^2 + (R'^2 + 1)\left(\frac{z'}{\theta'}\right)^2 = \frac{1}{(\theta')^2} = \frac{R^4}{C^2}$$

or equivalently

$$\left(\frac{z'}{\theta'}\right)^2 = \left[\frac{R^4}{C^2} - R^2\right]\frac{1}{(R'^2 + 1)}.$$

Now assume that z is a function of θ, i.e. in the domain D of (θ, z)-coordinates, z can be represented as a graph over θ-axis, then by chain rule we have

$$\frac{dz}{dt} = \frac{dz}{d\theta}\frac{d\theta}{dt}$$

or equivalently

$$\frac{z'}{\theta'} = \frac{dz}{d\theta}$$

after substituting into the equation above results

$$\left(\frac{dz}{d\theta}\right)^2 = \left[\frac{R^4}{C^2} - R^2\right]\frac{1}{(R'^2+1)}.$$

This is an ODE with unknown function $z = z(\theta)$ and any solution of this equation represents a geodesic in (θ, z)-plane (the domain D).

10.3 Cylinder

An interesting case is when $R(z) = \rho$-constant which corresponds to the cylinder. Then $R' = 0$ and the ODE is

$$\left(\frac{dz}{d\theta}\right)^2 = \frac{\rho^2}{C^2}\left[\rho^2 - C^2\right] = A$$

Using the first Euler-Lagrange equation we deduce $\theta' = \frac{C}{\rho^2}$-constant thus $\theta(t) = \frac{C}{\rho^2}t + D$ where D is an arbitrary constant. Thereby $z = A\theta + B = \frac{AC}{\rho^2}t + AD + B$ where A, B are constants. Thus

$$\vec{z}(t) = \begin{bmatrix} \rho\cos(\frac{C}{\rho^2}t + D) \\ \rho\sin(\frac{C}{\rho^2}t + D) \\ \frac{AC}{\rho^2}t + AD + B \end{bmatrix}$$

is the general form of the geodesic for cylinder. It is helix if $C \neq 0, A \neq 0$, vertical line if $C = 0, A \neq 0$ and circle if $C \neq 0, A = 0$.