Homework 1

October 27, 2014

4. **(Homework)** Show that if x and y are two vectors in an inner product space such that $\|x+y\| = \|x\| + \|y\|$, then x and y are linearly dependent.

Solution: We square the equality. We get that

$$\|x+y\|^2 = \|x\|^2 + 2\|x\|\|y\| + \|y\|^2.$$

Expanding left-hand side as an inner product $\langle x+y, x+y\rangle$ we get that

$$\langle x, y\rangle + \langle y, x\rangle = 2\|x\|\|y\|.$$

The left-hand side is equal to $2\mathrm{Re}\langle x, y\rangle \leq 2|\langle x, y\rangle|$. Hence it follows that $\|x\|\|y\| \leq |\langle x, y\rangle|$ which is a reverse of Cauchy-Schwartz inequality. We conclude that equality must hold (so that Cauchy-Schwartz is not violated). But we know that equality in Cauchy-Schwartz inequality only if the vectors are linearly dependent.

5. **(Homework)** Show the following version of polarization identity:

$$\langle x, y\rangle \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} \|x + \mathrm{e}^{i\theta}y\|^2 \mathrm{e}^{i\theta}\, d\theta.$$

Solution:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \|x + \mathrm{e}^{i\theta}y\|^2 \mathrm{e}^{i\theta}\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} [\langle x, x\rangle \mathrm{e}^{i\theta} + \langle y, y\rangle \mathrm{e}^{i\theta}\mathrm{e}^{-i\theta} + \langle y, x\rangle \mathrm{e}^{2i\theta} + \langle x, y\rangle] d\theta = \langle x, y\rangle.$$