Homework 2

October 27, 2014

Homework assignment 2 (due on Friday 03/10, 2.10pm, before class starts):

1) Consider the inner product space of continuously differentiable functions $C^1[0,1]$ with the inner product

$$\langle f, g\rangle \;=\; \int_0^1 f(x)\overline{g(x)}\,dx \;+\; \int_0^1 f'(x)\overline{g'(x)}\,dx.$$

Show that $\langle f, \cosh\rangle = f(1)\sinh(1)$ for any $f \in C^1[0,1]$ and use this to show that the subspace

$$\{f \in C^1[0,1] \;:\; f(1) = 0\,\}$$

is a closed subspace of $C^1[0,1]$.

Solution: Integration by parts shows that $\int_0^1 f'(x)\sinh(x)dx = -\int_0^1 f(x)\cosh(x)dx + f(1)\sinh(1)$ and hence $\langle f, \cosh\rangle = f(1)\sinh(1)$. Now consider a sequence $\{f_n\} \subset W = \{f \in C^1[0,1] \;:\; f(1) = 0\,\}$, that is, $f_n(1) = 0$ for all n and suppose that $f_n \to f$ in $C^1[0,1]$. Then by the Cauchy-Schwarz inequality

$$|\langle f_n, \cosh\rangle - \langle f, \cosh\rangle| = |\langle f_n - f, \cosh\rangle| \le \|f_n - f\|\,\|\cosh\|$$

which tends to zero as $n \to \infty$. But $\langle f_n, \cosh\rangle = f_n(1)\sinh(1) = 0$ for all n. Hence $\langle f, \cosh\rangle = f(1)\sinh(1) = 0$ and therefore $f \in W$ showing that W is closed.

2) Let $(X, \|\cdot\|)$ be a n.l.s and $\{x_n\}$ a sequence in X such that

$$\sum_{i=1}^{\infty} \|x_{n+1} - x_n\| < \infty.$$

Prove that $\{x_n\}$ is Cauchy sequence. Is the converse statement true?

Solution: For $n > m$ consider

$$\|x_n - x_m\| = \|x_n - x_{n-1} + x_{n-1} - x_{n-2} + \cdots + x_{m+1} - x_m\|$$

and after applying triangle inequality successively we get

$$\|x_n - x_m\| \le \sum_{i=m-1}^{n-1} \|x_{i+1} - x_i\| \to 0$$

as $m, n \to \infty$. Converse statement is not true, for instance take $x_n = \sum_{i=0}^{n} \frac{(-1)^i}{i}$, i.e. partial sums of alternating series (hence convergent). We see that $\sum_{i=1}^{\infty} \|x_{n+1} - x_n\| = \sum_{i=1}^{\infty} \frac{1}{i} = \infty$. Then we have that $x_i \to 0$ but $\sum_{i=1}^{\infty} |x_i -$

3) Let $(C[0,1], \|\cdot\|_2)$ be the n.l.s. with $\|\cdot\|_2$ norm. For $x \in C[0,1]$ define

$$\|x\| = \left(\int_0^1 v(t)[x(t)]^2 dt \right)^{\frac{1}{2}}$$

where $v(t)$ is continuous on $[0,1]$ and $v(t) \geq \frac{1}{\sqrt{2}}$. Prove that $\|\cdot\|$ is equivalent to $\|\cdot\|_2$.

Solution: Let $M = \max_{t \in [0,1]} v(t)$ (this is achieved because v is continuous on $[0,1]$). Thus we have

$$\frac{1}{\sqrt[4]{2}} \left(\int_0^1 [x(t)]^2 dt \right)^{\frac{1}{2}} \leq \left(\int_0^1 v(t)[x(t)]^2 dt \right)^{\frac{1}{2}} \leq \sqrt{M} \left(\int_0^1 [x(t)]^2 dt \right)^{\frac{1}{2}}$$

and the proof follows.