Homework 3

October 27, 2014

(due on Friday 10/17, 2.10pm, before class starts):

1) Suppose that M is a closed subspace of a Hilbert space H.

a) Show that $M = (M^\perp)^\perp$.

b) Is there a similar statement for subspaces M which are not necessarily closed?

Solution: Suppose that M is a closed subspace of a Hilbert

a) Show that $M = (M^\perp)^\perp$.

$\forall x \in M, \forall y \in M^\perp, \langle x, y\rangle = 0 \Rightarrow M \subset (M^\perp)^\perp$. Hence if M is a closed subspace of H, then M is a closed subspace of the Hilbert space $(M^\perp)^\perp$ and so $(M^\perp)^\perp = M \oplus W$ where W is the orthogonal complement of M in $(M^\perp)^\perp$. Hence $W \subset M^\perp \cap (M^\perp)^\perp = \{0\} \Rightarrow W = \{0\}$ and so $M = (M^\perp)^\perp$.

b) Is there a similar statement for subspaces M which are not necessarily closed? $\overline{M} = (M^\perp)^\perp$.

2) Let H be a Hilbert space and $A : H \to H$ be a linear bounded operator such that Ax is defined for every $x \in H$ (in other words the domain of A is the whole space H). Prove that

$$\|A\| = \sup_{x,y\in H, x\neq 0, y\neq 0} \frac{|\langle Ax, y\rangle|}{\|x\|\|y\|}$$

Solution: We have that

$$\sup_{x,y\in H, x\neq 0, y\neq 0} \frac{|\langle Ax, y\rangle|}{\|x\|\|y\|} = \sup_{x,y\in H, x\neq 0, y\neq 0} \frac{|\langle Ax, \frac{y}{\|y\|}\rangle|}{\|x\|} = \sup_{x,y\in H, x\neq 0, y\neq 0} \left|\langle \frac{Ax}{\|x\|}, \frac{y}{\|y\|}\rangle\right|$$

thus we get

$$\sup_{x,y\in H, x\neq 0, y\neq 0} \frac{|\langle Ax, y\rangle|}{\|x\|\|y\|} = \sup_{\xi,\eta\in H, \|\xi\|\leq 1, \|\eta\|\leq 1} |\langle A\xi, \eta\rangle|$$

but

$$\sup_{\xi,\eta\in H, \|\xi\|\leq 1, \|\eta\|\leq 1} |\langle A\xi, \eta\rangle| \geq \sup_{\xi\in H, \|\xi\|\leq 1,} |\langle A\xi, \eta\rangle| \geq \|A\|$$

if we choose $\eta = \frac{A\xi}{\|A\xi\|}$. To get the reversed inequality we note

$$|\langle A\xi, \eta\rangle| \leq \|A\|$$

if $\|\xi\| \leq 1, \|\eta\| \leq 1$, hence the proof follows.