Homework 4

October 27, 2014

(due on Tuesday November 4, 2.10pm, before class starts):

1. Let $L : C[0,1] \to \mathbf{C}$ denote the linear functional defined by

$$L(f) \;=\; f(0).$$

a) Show $L \in (C[0,1], \|\cdot\|_\infty)^*$.

b) Show $L \notin (C[0,1], \|\cdot\|_2)^*$.

Solution: We check that L is bounded as an operator with domain $(C[0,1], \|\cdot\|_\infty)$. Indeed,

$$\|L\| = \sup_{\|f\|_\infty \le 1} |Lf| = \sup_{\|f\|_\infty \le 1} |f(0)| \le \sup_{\|f\|_\infty \le 1} \|f\|_\infty = 1.$$

It follows that $\|L\| \le 1$.

On the other hand if we consider $(C[0,1], \|\cdot\|_2)$ we want to show that L is unbounded. Consider functions that are $f_n(x) = A_n(1/n - x)$ on $[0, 1/n]$ and 0 elsewhere. Such functions are continuous with norm

$$\|f_n\|_2^2 = A_n^2 \int_0^{1/n} (x - 1/n)^2 dx = A_n/(3n^3).$$

Choose $A_n = 3n^3$, then $\|f_n\|_2 = 1$. If follows that

$$\|L\| = \sup_{\|f\|_2 \le 1} |Lf| = \sup_{\|f\|_2 \le 1} |f(0)| \ge |f_n(0)| = 3n^3/n^2 = 3n.$$

As n can be taken arbitrary large it follows that $\|L\| = \infty$, hence L is an unbounded operator.

2 Let H be a Hilbert space and $A : H \to H$ linear self-adjoint operator defined for any $x \in H$. Show that A is continuous. [Hint:] Use Banach-Steinhaus theorem.

Solution: We have for all $x \in H, \|x\| \le 1$

$$|\langle Ax, y\rangle| = |\langle x, Ay\rangle| \le \|Ay\|.$$

and for all $y \in H$. From Riesz representation theorem we have that for all $x \in H, \|x\| \leq 1$ we have $|L(Ax)| \leq C(L)$ for any $L \in H^*$ with some constant $C(L)$ depending only on L. Thus from Banach-Steinhaus theorem A^* is bounded, but A is self-adjoint hence A is bounded and, in addition, continuous.