Homework 5

November 11 2014

1. The principal value of $\frac{1}{x}$ is defined as $\mathcal{P}\frac{1}{x}(\phi) = \lim_{\epsilon \to 0} \int_{|x| \geq \epsilon} \frac{\phi(x)}{x} dx$

   - Show that $\mathcal{P}\frac{1}{x}$ defines a distribution
   - Represent $\mathcal{P}\frac{1}{x}(\phi)$ as a double integral.
   - Find the primitive of $\mathcal{P}\frac{1}{x}$ in the sense of distributions.

   Solution:

   For the first part we have to check that the distribution defined in this form is continuous because the linearity is obvious. Thus we want to show that for any compact set $K\mathbb{R}$ there is a positive integer k and a constant $C(K) > 0$ such that the following holds

$$|\langle \phi, \mathcal{P}\frac{1}{x} \rangle| \leq C(K) \sum_{i=0}^{k} \sup_{K} |\phi^{(i)}(x)|.$$

Without loss of generality we will take $K = (-R, R)$ because $\mathbb{R} = \cup_{R>0}(-R, R)$. We have from mean value theorem that the principal value satisfies the following estimates (vp stands for principal value)

$$\begin{aligned} |\langle \phi, \mathcal{P}\frac{1}{x} \rangle| &= \left| vp \int \frac{\phi(x)}{x} dx \right| = \left| vp \int_{-R}^{R} \frac{\phi(0) + \phi'(x_0)x}{x} dx \right| \qquad (1) \\ &= \left| \lim_{\epsilon \to 0} \int_{-R}^{-\epsilon} + \int_{\epsilon}^{R} \frac{\phi(0) + \phi'(x_0)x}{x} dx \right| \\ &\leq \int_{-R}^{R} |\phi'(x_0)| dx \leq 2R \sup |\phi'| \end{aligned}$$

where x_0 is some pony in the interval $|x| < R$. Hence $k = 1$ and $C(K) = 2R$.

As for the second part we note that

$$\int_{-R}^{-\epsilon} + \int_{\epsilon}^{R} \frac{\phi(x)}{x} dx = \int_{-R}^{-\epsilon} + \int_{\epsilon}^{R} \frac{\phi(x) - \phi(0)}{x} dx \tag{2}$$

$$= \int_{-R}^{-\epsilon} + \int_{\epsilon}^{R} \frac{1}{x} \int_{0}^{x} \phi'(y) dy dx$$

$$= \int_{-R}^{-\epsilon} + \int_{\epsilon}^{R} \int_{0}^{1} \phi'(tx) dt dx.$$

Hence

$$\lim_{\epsilon \to 0} \int_{-R}^{-\epsilon} + \int_{\epsilon}^{R} \frac{\phi(x)}{x} dx = \int_{-R}^{R} \int_{0}^{1} \phi'(tx) dt dx \tag{3}$$

Finally the third part follows by direct computation using integration by parts.

2. Let f be a function on $\mathbf{R}$ which is zero for $x < 0$, continuous for $x > 0$ and assume that $\int_0^1 x|f(x)|dx < \infty$. Show that f represents a distribution of order at most 1.

Solution:

Note that $f(x) = 0$ if $x \in (-\infty, 0)$. Hence supp$f \subset [0, +\infty)$. Now take $\phi \in C_0^\infty[0, +\infty)$ (in particular $\phi(0) = 0$). Suppose supp$\phi \subset [0, R]$ for some $R > 1$, we have

$$\left| \int_0^R f(x)\phi(x) dx \right| = \left| \int_0^R f(x) \left(\int_0^x \phi'(t) dt \right) dx \right| \leq$$

$$\leq \sup |\phi'| \left\{ \int_0^1 x|f(x)|dx + \int_1^R x|f(x)|dx \right\}.$$

Note that the second integral in the last line is finite because f is continuous in $[1, R]$ and hanse f is bounded on $[1, R]$.