Mock exam

10 December 2014

1. State the definition of inner product and give two examples of inner product space.
2. Write down the polarisation identity and prove it.
3. Prove Bessel's inequality.
4. Let $X = C[0,1]$ and $W = \{f \in X : f(0) = 0\}$. With respect to $\|\cdot\|_\infty$, what is the closure of W? With respect to $\|\cdot\|_1$, what is the closure of W? Give reasons for your answers.
5. Consider the inner product space of continuously differentiable functions $C^1[0,1]$ with the inner product
$$\langle f, g\rangle \;=\; \int_0^1 f(x)\overline{g(x)}\,dx \;+\; \int_0^1 f'(x)\overline{g'(x)}\,dx.$$
Show that $\langle f, \cosh\rangle = f(1)\sinh(1)$ for any $f \in C^1[0,1]$ and use this to show that the subspace
$$\{f \in C^1[0,1] \;:\; f(1) = 0\,\}$$
is a closed subspace of $C^1[0,1]$.
6. Let $x = (x_1, x_2, x_3, x_4, \dots)$ be an element of ℓ^2. Define the left shit operator as $Sx = (x_2, x_3, x_4, \dots)$. Find the adjoint of S.
7. Prove that ℓ^2 is complete.
8. Give an example of self-adjoint operator on $L^2(0,1)$.
9. Is a finite rank operator compact? Justify your answer.
10. Define $T \in \mathcal{L}(L^2[0,1])$ by
$$Tf(x) \;=\; \int_0^1 K(x,y)\,f(y)\,dy \qquad \text{when} \quad f \in C[0,1].$$

When

$$K(x,y) = \begin{cases} 1, & \text{if } 0 \le y \le x \le 1; \\ 0, & \text{if } 0 \le x < y \le 1, \end{cases}$$

show that T has no eigenvalues.

11. Let H be the Hilbert space of 2π periodic functions with equipped with the inner product $\langle f, g\rangle = \int_0^{2\pi} f(x)\overline{g(x)}dx$. Consider the shift operator $T_s f(x) = f(x+s)$ where $s > 0$. Is T_s Hilbert-Schmidt operator? Justify your answer.

12. Let H be a Hilbert space and $A : H \to H$ linear self-adjoint operator defined for any $x \in H$. Show that A is continuous. [Hint: Use Banach-Steinhaus theorem.]

13. Let H be a Hilbert space, M a closed subspace of H and let $P : H \to M$ be the orthogonal projection onto M.

    a) Show that P is self-adjoint.

    b) Show that P is compact if and only if M is finite dimensiona

14. Let $\phi \in \mathcal{D}(\mathbf{R})$ and assume that $\phi(0) = \phi'(0) = \cdots = \phi^{(k)}(0)$. Show that there is $\psi \in \mathcal{D}(\mathbf{R})$ with $\phi(x) = x^{k+1}\psi(x)$.

15. Find all $f \in \mathcal{D}'(\mathbf{R})$ with $xf(x) = 1$.

16. Compute the limit $\lim\limits_{t\to\infty} \frac{\sin tx}{x}$ in $\mathcal{D}'(\mathbf{R})$.

17. Compute in $\mathcal{D}'(\mathbf{R}^2 \setminus \{(0,0)\})$:

$$\lim_{t\to\infty} t\sin(t|x^2+y^2-1|)$$

Does this limit exist in $\mathcal{D}'(\mathbf{R}^2)$?