Workshop 1

September 23, 2014

1. Show that on $M_n(\mathbf{C}) = (n \times n$ complex matrices) the prescription $\langle A, B\rangle =$ trace B^*A defines an inner product. Describe the corresponding norm.

2. For any $f \in C[0,1]$, show that

$$\Big|\int_0^1 f(x)\sin(\pi x)dx\Big| \le \frac{1}{\sqrt{2}}\Big[\int_0^1 |f(x)|^2 dx\Big]^{1/2}.$$

3. Let X be the vector space of complex polynomials.
   a) Find the inner product on X which gives rise to the norm

$$\|f\| = \Big[\int_{-1}^1 |x||f(x)|^2 + 3|f'(x)|^2 dx\Big]^{1/2}.$$

   b) Show that for any complex polynomial $f \in X$,

$$\Big|\int_{-1}^1 |x|^3 f(x) + 6xf'(x)dx\Big| \le \frac{5}{\sqrt{3}}\Big[\int_{-1}^1 |x||f(x)|^2 + 3|f'(x)|^2 dx\Big]^{1/2}.$$

4. **(Homework)** Show that if x and y are two vectors in an inner product space such that $\|x+y\| = \|x\| + \|y\|$, then x and y are linearly dependent.

5. **(Homework)** Show the following version of polarization identity:

$$\langle x, y\rangle \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} \|x + \mathrm{e}^{i\theta}y\|^2 \mathrm{e}^{i\theta}\, d\theta.$$

Additional problems for self-study

6. Show that the normed linear space $(C([0,1]), \|\cdot\|_1)$ does not satisfy the parallelogram law. Hence deduce that the norm $\|\cdot\|_1$ does not arise from an inner product.

7. Fix a positive integer N, put $\omega = \mathrm{e}^{2\pi i/N}$, prove the orthogonality relations

$$\frac{1}{N}\sum_{n=1}^{N} \omega^{nk} = 1, \quad \text{if} \quad k = 0, \qquad \text{and} \qquad \frac{1}{N}\sum_{n=1}^{N} \omega^{nk} = 0, \quad \text{if} \quad 1 \leq k \leq N-1$$

and use them to derive the identities

$$\langle x, y \rangle \;=\; \frac{1}{N}\sum_{n=1}^{N} \|x + \omega^n y\|^2 \omega^n$$

which hold in any inner product space if $N \geq 3$.

8. By a *trigonometric polynomial* we mean a function of the form

$$f(x) \;=\; \sum_{n=1}^{k} a_n \, e^{i\lambda_n x}$$

for some positive integer k and some collection of complex coefficients $\{a_n\}$ and real frequencies $\{\lambda_n\}$. Let TP denote the collection of all trigonometric polynomials (TP forms a vector space with the usual vector addition and scalar multiplication of adding and scalar multiplying functions). Show that

$$\langle f, g \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(x)\overline{g(x)} dx$$

defines an inner product on TP.

9. (For the ambitious) If $(X, \|\cdot\|)$ is a normed linear space for which the parallelogram law holds, show that the norm $\|\cdot\|$ arises from an inner product. (Hint: Try it first with real scalars and let the polarization identity guide you. First do additivity, then scalar multiplicativity.)