Workshop 2

April 30, 2015

1. Let $(X, \|\cdot\|)$ be a normed linear space and let $W \subset X$ be a proper linear subspace. Show that W is not open.

Solution: Suppose that $x \in X$ and there is $r > 0$ such that $B_r(x) \subset W$. Take any $z \in X, z \notin W$ and let $y = \frac{r}{2}\frac{z-x}{\|z-x\|}$. Clearly $y \in B_r(x) \subset W$. From here we get $y\frac{2\|z-x\|}{r} + x = z$, i.e. z is a linear combination of two elements on W, which are y and x. Since W is a subspace then $z \in W$ which is a contradiction.

2. Consider the following subspace of the vector space of all infinite sequences of complex numbers:

$$X = lin\{e_n : n \in \mathbf{N}\}.$$

Here e_j denotes the element whose jth component is 1 and all other components are zero.

i) Describe the vectors in X in terms of their components.

ii) Note that one can consider X as a subspace of ℓ^∞ or ℓ^2. Show that in either case, X is not a closed subspace.

iii) As a subspace of ℓ^2, what is the closure of X? As a subspace of ℓ^∞, what is the closure of X?

Solution: Exercise.

3. Let $X = C[0,1]$ and $W = \{f \in X : f(0) = 0\}$. With respect to $\|\cdot\|_\infty$, what is the closure of W? With respect to $\|\cdot\|_1$, what is the closure of W? Give reasons for your answers.

Solution: W is closed w.r.t. $\|\cdot\|_\infty$ hence $\overline{W} = W$. Indeed, let if the sequence $f_n \in W$ converges to f w.r.t. $\|\cdot\|_\infty$ then for any $\epsilon > 0$ there is $N > 0$ such that $\sup_{t\in[0,1]} |f_n(t) - f(t)| < \epsilon$ whenever $n > N$. We know that f is continues thanks to the uniform convergence. In particular at $t = 0$ we must have that $|f_n(0) - f(0)| < \epsilon$. But since $f_n(0) = 0$ it implies that $|f(0)| < \epsilon$. Since ϵ is arbitrary the result follows.

For L^1 norm it is not, e.g. consider the functions $f_n(t) = t^{\frac{1}{n}}, n \to \infty$. Then $0 \le t^{\frac{1}{n}} \le 1$ for $t \in [0,1]$ and hence

$$\int_0^1 |t^{\frac{1}{n}} - 1| dt = \int_0^1 (1 - t^{\frac{1}{n}}) dt = 1 - \frac{1}{1+\frac{1}{n}} = \frac{\frac{1}{n}}{1+\frac{1}{n}} \to 0$$

as $n \to \infty$. Clearly, the constant function $f(t) = 1, t \in [0,1]$ fails to be in W. The closure w.r.t. $\|\cdot\|_1$ gives L^1.

4. **Homework assignment** Consider the inner product space of continuously differentiable functions $C^1[0,1]$ with the inner product

$$\langle f, g\rangle \;=\; \int_0^1 f(x)\overline{g(x)}\, dx \;+\; \int_0^1 f'(x)\overline{g'(x)}\, dx.$$

Show that $\langle f, \cosh\rangle = f(1)\sinh(1)$ for any $f \in C^1[0,1]$ and use this to show that the subspace

$$\{f \in C^1[0,1] \;:\; f(1) = 0\,\}$$

is a closed subspace of $C^1[0,1]$.

Solution:

Integration by parts shows that $\int_0^1 f'(x)\sinh(x) dx = -\int_0^1 f(x)\cosh(x) dx + f(1)\sinh(1)$ and hence $\langle f, \cosh\rangle = f(1)\sinh(1)$. Now consider a sequence $\{f_n\} \subset W = \{f \in C^1[0,1] \;:\; f(1) = 0\,\}$, that is, $f_n(1) = 0$ for all n and suppose that $f_n \to f$ in $C^1[0,1]$. Then by the Cauchy-Schwarz inequality

$$|\langle f_n, \cosh\rangle - \langle f, \cosh\rangle| = |\langle f_n - f, \cosh\rangle| \le \|f_n - f\|\,\|\cosh\|$$

which tends to zero as $n \to \infty$. But $\langle f_n, \cosh\rangle = f_n(1)\sinh(1) = 0$ for all n. Hence $\langle f, \cosh\rangle = f(1)\sinh(1) = 0$ and therefore $f \in W$ showing that W is closed.

Additional problems for self-study:

5. Consider the open unit ball $B = \{x : \|x\| < 1\}$ in a nls $(X, \|\cdot\|)$. Show that the closure of B is

$$\{x \in X \,:\, \|x\| \leq 1\}.$$

Solution: Hint: show that any point in $\{x \in X \,:\, \|x\| \leq 1\}$. is a limit of point from B.

6. Let X be a normed linear space and Y a closed proper subspace. Prove that for all $\epsilon > 0$, there is an $x \in X$ with $\|x\| = 1$ and such that $\|x - y\| \geq 1 - \epsilon$ for all $y \in Y$.

7. Suppose X is an infinite dimensional nls. Use exercise 6 to construct an infinite sequence of unit vectors, $\{x_j\}$, in X such that $\|x_j - x_k\| \geq 1/2$ for all $j \neq k$ and from this, deduce that in any infinite dimensional normed linear spaces, the closed unit ball $\{x : \|x\| \leq 1\}$ is not compact.

Solution: Exercise.