Workshop 2

October 28, 2014

1. Let $(X, \|\cdot\|)$ be a normed linear space and let $W \subset X$ be a proper linear subspace. Show that W is not open.

2. Consider the following subspace of the vector space of all infinite sequences of complex numbers:

$$X \;=\; lin\{e_n : n \in \mathbf{N}\}.$$

Here e_j denotes the element whose jth component is 1 and all other components are zero.

i) Describe the vectors in X in terms of their components.

ii) Note that one can consider X as a subspace of ℓ^∞ or ℓ^2. Show that in either case, X is not a closed subspace.

iii) As a subspace of ℓ^2, what is the closure of X? As a subspace of ℓ^∞, what is the closure of X?

3. Let $X = C[0,1]$ and $W = \{f \in X : f(0) = 0\}$. With respect to $\|\cdot\|_\infty$, what is the closure of W? With respect to $\|\cdot\|_1$, what is the closure of W? Give reasons for your answers.

4. **Homework assignment** Consider the inner product space of continuously differentiable functions $C^1[0,1]$ with the inner product

$$\langle f, g\rangle \;=\; \int_0^1 f(x)\overline{g(x)}\,dx \;+\; \int_0^1 f'(x)\overline{g'(x)}\,dx.$$

Show that $\langle f, \cosh \rangle = f(1)\sinh(1)$ for any $f \in C^1[0,1]$ and use this to show that the subspace

$$\{f \in C^1[0,1] \,:\, f(1) = 0\,\}$$

is a closed subspace of $C^1[0,1]$.

Additional problems for self-study:

5. Consider the open unit ball $B = \{x : \|x\| < 1\}$ in a nls $(X, \|\cdot\|)$. Show that the closure of B is

$$\{x \in X \,:\, \|x\| \leq 1\,\}.$$

6. Let X be a normed linear space and Y a closed proper subspace. Prove that for all $\epsilon > 0$, there is an $x \in X$ with $\|x\| = 1$ and such that $\|x - y\| \geq 1 - \epsilon$ for all $y \in Y$.

7. Suppose X is an infinite dimensional nls. Use exercise 6 to construct an infinite sequence of unit vectors, $\{x_j\}$, in X such that $\|x_j - x_k\| \geq 1/2$ for all $j \neq k$ and from this, deduce that in any infinite dimensional normed linear spaces, the closed unit ball $\{x : \|x\| \leq 1\}$ is not compact.