Workshop 3

October 28, 2014

1. Let H be a Hilbert space and $x_0 \in H$. If M be a closed subspace of H, show that

$$\min\{\|x - x_0\| : x \in M\} = \max\{|\langle x_0, y\rangle| : y \in M^\perp, \|y\| = 1\}.$$

Solution Decomposing $x_0 = Px_0 + Qx_0$ we get

$$\|x - x_0\|^2 = \|(x - Px_0) - Qx_0\|^2 = \|(x - Px_0)\|^2 + \|Qx_0\|^2$$

thus the minimum is achieved for $x = Px_0$ and is equal $\|Qx_0\|^2$. Now let us explore the maximum on the right hand side. Again, by decomposing x_0 we have

$$|\langle x_0, y\rangle| = |\langle Px_0 + Qx_0, y\rangle| = |\langle Qx_0, y\rangle|$$

hence the maximum is achieved for $y = \frac{Qx_0}{\|Qx_0\|}$ and is equal to $\|Qx_0\|$. Comparing max and min the result follows.

2. Compute

$$\min_{a,b,c\in\mathbf{R}} \int_{-1}^{1} |x^3 - a - bx - cx^2|^2 dx$$

and find $\max \int_{-1}^{1} x^3 g(x)dx$ where $g \in C[-1,1]$ is subject to the restrictions

$$\int_{-1}^{1} g(x)dx = \int_{-1}^{1} xg(x)dx = \int_{-1}^{1} x^2 g(x)dx = 0; \quad \int_{-1}^{1} |g(x)|^2 dx = 1.$$

Solution Hint; Take $H = L^2[-1,1]$, $f(x) = x^3$ and $q(x) = a + bx + cx^2$ and consider $\min_g \|f - g\|_{L^2}$ when $g \in H$ and $g \in M$ where $M \subset H$ is defined by

$$\int_{-1}^{1} g(x)dx = \int_{-1}^{1} xg(x)dx = \int_{-1}^{1} x^2 g(x)dx = 0; \quad \int_{-1}^{1} |g(x)|^2 dx = 1.$$

Homework Assignment

3. Suppose that M is a closed subspace of a Hilbert space H.

a) Show that $M = (M^{\perp})^{\perp}$.

b) Is there a similar statement for subspaces M which are not necessarily closed?

4. Consider $C[0,1]$ with two different norms, $\|\cdot\|_1$ and $\|\cdot\|_\infty$ (recall $\|f\|_1 = \int_0^1 |f(x)|dx$ and $\|f\|_\infty = \sup_{0\le x\le 1} |f(x)|$). Show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are not equivalent norms on $C[0,1]$.

Solution Hint: consider the functions $f_n(t) = t^{\frac{1}{n}}$ as $n \to \infty$.

Additional problem for self-study

5. Compute

$$\min_{a,b,c\in\mathbf{R}} \int\limits_0^\infty |x^3 - a - bx - cx^2|^2 \mathrm{e}^{-x} dx.$$

State and solve the corresponding maximum problem, as in Exercise 1.

6. Let L be a continuous linear functional on a Hilbert space H (i.e., $L \in H^*$). If $L \not\equiv 0$ and $M = \{x \in H : Lx = 0\}$, show that $\dim M^{\perp} = 1$.

Solution Hint: Use Riesz representation theorem and the homework problem above on $(M^{\perp})^{\perp} = M$

7. Let H be a Hilbert space, M a closed subspace of H and let $P : H \to M$ be the orthogonal projection onto M. Prove that P is bounded and calculate its norm. Observe that there are two cases, $M = \{0\}$ and $\dim M \ge 1$.

Solution $\|P\| = 1$ it follows from Pythagoras theorem.

8. Let H be a Hilbert space and let $L \in H^*$. Show that there exists an $x \in H$, $\|x\| = 1$ such that $|L(x)| = \|L\|$.

Solution Hint: Use Riesz representation theorem