Workshop 3

October 28, 2014

1. Let H be a Hilbert space and $x_0 \in H$. If M be a closed subspace of H, show that

$$\min\{\|x - x_0\| : x \in M\} = \max\{|\langle x_0, y\rangle| : y \in M^{\perp}, \|y\| = 1\}.$$

2. Compute

$$\min_{a,b,c\in\mathbf{R}} \int_{-1}^{1} |x^3 - a - bx - cx^2|^2 dx$$

and find $\max \int_{-1}^{1} x^3 g(x)dx$ where $g \in C[-1,1]$ is subject to the restrictions

$$\int_{-1}^{1} g(x)dx = \int_{-1}^{1} xg(x)dx = \int_{-1}^{1} x^2 g(x)dx = 0; \quad \int_{-1}^{1} |g(x)|^2 dx = 1.$$

Homework Assignment

3. Suppose that M is a closed subspace of a Hilbert space H.
   a) Show that $M = (M^{\perp})^{\perp}$.
   b) Is there a similar statement for subspaces M which are not necessarily closed?

4. Consider $C[0,1]$ with two different norms, $\|\cdot\|_1$ and $\|\cdot\|_\infty$ (recall $\|f\|_1 = \int_0^1 |f(x)|dx$ and $\|f\|_\infty = \sup_{0\le x\le 1} |f(x)|$). Show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are not equivalent norms on $C[0,1]$.

Additional problem for self-study

5. Compute

$$\min_{a,b,c\in\mathbf{R}} \int_{0}^{\infty} |x^3 - a - bx - cx^2|^2 \mathrm{e}^{-x} dx.$$

State and solve the corresponding maximum problem, as in Exercise 1.

6. Let L be a continuous linear functional on a Hilbert space H (i.e., $L \in H^*$). If $L \not\equiv 0$ and $M = \{x \in H : Lx = 0\}$, show that $\dim M^\perp = 1$.

7. Let H be a Hilbert space, M a closed subspace of H and let $P : H \to M$ be the orthogonal projection onto M. Prove that P is bounded and calculate its norm. Observe that there are two cases, $M = \{0\}$ and $\dim M \geq 1$.

8. Let H be a Hilbert space and let $L \in H^*$. Show that there exists an $x \in H$, $\|x\| = 1$ such that $|L(x)| = \|L\|$.