Workshop 4

October 28, 2014

Homework Assignment

1. Let $L : C[0,1] \to \mathbf{C}$ denote the linear functional defined by

$$L(f) \;=\; f(0).$$

a) Show $L \in (C[0,1], \|\cdot\|_\infty)^*$.

b) Show $L \notin (C[0,1], \|\cdot\|_2)^*$.

Additional problems for self-study

2. Let $\{u_n\}_{n=1}^\infty$ be an orthonormal family in a Hilbert space H. Show that for $x \in H$,

$$\#\{n : |\langle x, u_n\rangle| > \frac{1}{m}\} \;\leq\; m^2\|x\|^2.$$

3. Let H denote the Hilbert space of almost periodic functions. For $s \in \mathbf{R}$, set $u_s(t) = \mathrm{e}^{ist}$. Show $\{u_s\}_{s\in\mathbf{R}}$ is a maximal orthnornormal family and hence deduce that H is not separable.

4. Let H be a Hilbert space, M a closed subspace of H and let $P : H \to M$ be the orthogonal projection onto M.

a) Show that P is self-adjoint.

b) Show that P is compact if and only if M is finite dimensional.

5. Let $H = L^2[0,1]$ and consider $T \in \mathcal{L}(H)$ defined on continuous functions by

$$Tf(s) = \int_0^1 e^{s-t} f(t)dt.$$

a) Show that T is Hilbert-Schmidt.

b) Show that

$$\|T\| = \|T\|_{HS} = \sinh 1.$$

Recall that $\sinh 1 = (e - e^{-1})/2$.

6. Let $T \in \mathcal{L}(L^2[0,1])$ be an integral operator $Tf(x) = \int_0^1 K(x,y)f(y)dt$.
When $K(x,y) = \begin{cases} 1, & \text{if } 0 \leq y \leq x \leq 1; \\ 0, & \text{if } 0 \leq x < y \leq 1. \end{cases}$
show that T has no eigenvalues.

7. Let H be the Hilbert space of 2π-periodic functions equipped with the inner product

$$\langle f, g \rangle = \int_0^{2\pi} f(x)\overline{g(x)}dx.$$

For a real number s consider the shift operator T_s defined by

$$T_s f(x) = f(s + x).$$

a) Is T_s a bounded operator on $\mathcal{L}(H)$? If so calculate its norm.
b) Is T_s a Hilbert-Schmidt operator? If so, calculate its Hilbert-Schmidt norm.
c) Show that the adjoint operator T_s^* is just T_{-s}.
d) Use the result of part (c) to show that T_π is a self-adjoint operator.
e) Is T_π a compact operator?