Workshop 5

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1. Let $\phi \in \mathcal{D}(\mathbf{R})$ and assume that $\phi(0) = \phi'(0) = \cdots = \phi^{(k)}(0)$. Show that there is $\psi \in \mathcal{D}(\mathbf{R})$ with $\phi(x) = x^{k+1}\psi(x)$.

Solution: Expanding ϕ by its Taylor series around 0 we see that the ratio $\phi(x)/x^{k+1}$ is well defined near the origin. Hence we set $\psi(x) = \frac{\phi(x)}{x^{k+1}}$. It follows from Leibnitz rule that $\psi \in \mathcal{D}(\mathbf{R})$ because x^{k+1} vanishes only at the origin and hence ψ's support is in the support of ϕ.

2. Show that there is a $\psi \in \mathcal{D}(\mathbf{R})$ with $\phi = \psi^{(k)}$ if and only if $\int_{-\infty}^{+\infty} P(x)\phi(x)dx = 0$ for each polynomial P of degree at most $k-1$.

Solution: Use integration by parts. Indeed, let us assume that $\phi = \psi^{(k)}$ and show that the integral vanishes. We have after k times integration by parts

$$\int_{-\infty}^{+\infty} P(x)\phi(x)dx = \int_{-\infty}^{+\infty} P(x)\psi^{(k)}(x)dx \tag{1}$$

$$= (-1)^k \int_{-\infty}^{+\infty} P^{(k)}(x)\psi(x)dx = (-1)^k \int_{-\infty}^{+\infty} 0\psi(x)dx = 0.$$

Now let us prove the converse statement. First assume that $k = 2$ then we want to show that if

$$\int_{-\infty}^{+\infty} (ax+b)\phi(x)dx = 0$$

for any real numbers a, b then there is $\psi \in \mathcal{D}(\mathbf{R})$ such that $\psi'' = \phi$. Let $A > 0$ such that $\text{supp}\phi \subset [-A, A]$. Taking first $a = 0, b = 1$ and then $a = 1, b = 0$ we get

$$\int_{-A}^{+A} \phi(x)dx = 0 \tag{2}$$

$$\int_{-A}^{+A} x\phi(x)dx = 0. \tag{3}$$

Define

$$\psi_1(x) = \begin{cases} \int_{-A}^{x} \phi(t)dt & \text{if } x \geq -A, \\ 0 & \text{otherwise.} \end{cases}$$

Thanks to (2) we see that $\psi_1 \in \mathcal{D}(\mathbf{R})$. Next we define

$$\psi(x) = \begin{cases} \int_{-A}^{x} \psi_1(t)dt & \text{if } x \geq -A, \\ 0 & \text{otherwise.} \end{cases}$$

Thanks to (3) we see that that $\psi \in \mathcal{D}(\mathbf{R})$. Observe that $\psi_1' = \phi, \psi'' = \psi_1' = \phi$ and the proof follows. The general case follows by induction.

3. (Homework problem) The principal value of $\frac{1}{x}$ is defined as $\mathcal{P}\frac{1}{x}(\phi) = \lim_{\epsilon \to 0} \int_{|x|\geq \epsilon} \frac{\phi(x)}{x} dx$

- Show that $\mathcal{P}\frac{1}{x}$ defines a distribution
- Represent $\mathcal{P}\frac{1}{x}(\phi)$ as a double integral.
- Find the primitive of $\mathcal{P}\frac{1}{x}$ in the sense of distributions.

Solution:

For the first part we have to check that the distribution defined in this form is continuous because the linearity is obvious. Thus we want to show that for any compact set $K\mathbb{R}$ there is a positive integer k and a constant $C(K) > 0$ such that the following holds

$$|\langle \phi, \mathcal{P}\frac{1}{x} \rangle| \leq C(K) \sum_{i=0}^{k} \sup_K |\phi^{(i)}(x)|.$$

Without loss of generality we will take $K = (-R, R)$ because $\mathbb{R} = \cup_{R>0}(-R, R)$. We have from mean value theorem that the principal value satisfies the following estimates (vp stands for principal value)

$$\begin{aligned} |\langle \phi, \mathcal{P}\frac{1}{x} \rangle| &= \left| vp \int \frac{\phi(x)}{x} dx \right| = \left| vp \int_{-R}^{R} \frac{\phi(0) + \phi'(x_0)x}{x} dx \right| \qquad (4) \\ &= \left| \lim_{\epsilon \to 0} \int_{-R}^{-\epsilon} + \int_{\epsilon}^{R} \frac{\phi(0) + \phi'(x_0)x}{x} dx \right| \\ &\leq \int_{-R}^{R} |\phi'(x_0)| dx \leq 2R \sup |\phi'| \end{aligned}$$

where x_0 is some pony in the interval $|x| < R$. Hence $k = 1$ and $C(K) = 2R$.

As for the second part we note that

$$\int_{-R}^{-\epsilon} + \int_{\epsilon}^{R} \frac{\phi(x)}{x} dx = \int_{-R}^{-\epsilon} + \int_{\epsilon}^{R} \frac{\phi(x)-\phi(0)}{x} dx \tag{5}$$
$$= \int_{-R}^{-\epsilon} + \int_{\epsilon}^{R} \frac{1}{x} \int_{0}^{x} \phi'(y) dy dx$$
$$= \int_{-R}^{-\epsilon} + \int_{\epsilon}^{R} \int_{0}^{1} \phi'(tx) dt dx.$$

Hence

$$\lim_{\epsilon \to 0} \int_{-R}^{-\epsilon} + \int_{\epsilon}^{R} \frac{\phi(x)}{x} dx = \int_{-R}^{R} \int_{0}^{1} \phi'(tx) dt dx \tag{6}$$

Finally the third part follows by direct computation using integration by parts.

4. Find all $f \in \mathcal{D}'(\mathbf{R})$ with $xf(x) = 1$.

Solution: If $x \neq 0$ then $f(x) = \frac{1}{x}$. Extending $\frac{1}{x}$ around the origin and using the previous problem we will get that $f(x) = \mathcal{P}\frac{1}{x}$.

5. Compute the following limits in $\mathcal{D}'(\mathbf{R})$.

- (a) $\lim\limits_{t \to \infty} t^2 x \cos tx$
- (b) $\lim\limits_{t \to \infty} t^2 |x| \cos tx$
- (c) $\lim\limits_{t \to \infty} \frac{\sin tx}{x}$
- (d) $\lim\limits_{t \to \infty} (\cos tx) vp(1/x)$
- (e) $\lim\limits_{t \to \infty} t \sin(t|x|)$

Solution: The main idea is to use integration by parts. In what follows $\phi \in C_0^\infty(\mathbf{R})$.

(a)

$$\lim_{t\to\infty} \int t^2 x \cos tx \phi(x) dx = \lim_{t\to\infty} \int tx (\sin tx)' \phi(x) dx = -\lim_{t\to\infty} \int \sin tx (x\phi'(x) + \phi(x)) t dx$$
$$= \lim_{t\to\infty} \int (\cos tx)'(x\phi'(x) + \phi(x)) = -\lim_{t\to\infty} \int \cos tx (x\phi''(x) + 2\phi'(x)) dx =$$
$$= -\lim_{t\to\infty} \int \frac{1}{t} (\sin tx)'(x\phi''(x) + 2\phi'(x)) = \lim_{t\to\infty} \int \frac{1}{t} (\sin tx)(x\phi'''(x) + 3\phi''(x)) dx \to 0$$

(b)

$$\lim_{t\to\infty}\int_{-\infty} t^2|x|\cos tx\phi(x)dx = \lim_{t\to\infty}\left\{-\int_{-\infty}^{0} t^2 x\cos tx\phi(x) + \int_0^\infty t^2 x\cos tx\phi(x)dx\right\}$$

Using partial integration as in the previous problem we get that the limit is $-2\phi(0)$. Hence the limit in $\mathcal{D}'(\mathbf{R})$ is -2δ.

(c)

Recall that the Dirichlet integral is $\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$. Then we have

$$\int_{-\infty}^{\infty}\frac{\sin tx}{x}\phi(x)dx = \phi(0)\left\{\int_{-\infty}^{0}\frac{\sin tx}{x}dx + \int_0^\infty \frac{\sin tx}{x}dx\right\} + \int_{-\infty}^{\infty}[\phi(x)-\phi(0)]\frac{\sin tx}{x}dx$$

$$= \pi\phi(0) + \int_{-\infty}^{\infty}[\phi(x)-\phi(0)]\frac{\sin tx}{x}dx = \pi\phi(0) + \int_{-\infty}^{\infty}[\phi(y/t)-\phi(0)]\frac{\sin y}{y}dy \to \pi\phi(0)$$

Thus the limit in $\mathcal{D}'(\mathbf{R})$ is $\pi\delta$.

(d) Let $\epsilon > 0$ be fixed and small, then

$$\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\frac{1}{x}\cos(tx)\phi(x)dx = \int_{\epsilon}^{\infty}\cos tx\frac{\phi(x)-\phi(-x)}{x}dx \to 0$$

where the last bit follows from Riemann's lemma because $\frac{\phi(x)-\phi(-x)}{x}$ is absolutely integrable.

(e)

$\lim_{t\to\infty} t\sin(t|x|) = 0$ in $\mathcal{D}'(\mathbf{R})$. Hint: consider the integral over $(-\infty, 0)$ and $(0, +\infty)$ respectively and then use partial integration.

6. Compute in $\mathcal{D}'(\mathbf{R}^2 \setminus \{(0,0)\})$:

$$\lim_{t\to\infty} t\sin(t|x^2+y^2-1|)$$

Does this limit exist in $\mathcal{D}'(\mathbf{R}^2)$?

Let $f_t = t\sin(t|x^2+y^2-1|)$. If $\phi \in C_0^\infty(R^2 \setminus 0)$ and $\phi(x,y) = \Phi(r,\theta)$ with polar coordinates then

$$\langle f_t, \phi\rangle = t\int_0^{2\pi}\int_0^\infty \sin(t|r^2-1|)\Phi(r,\theta)rdrd\theta = t\int_0^{2\pi}\int_{-1}^{\infty}\sin(t|s|)\Phi(\sqrt{s+1},\theta)rdrd\theta/2$$

Integration by parts for $s > 0, s < 0$ gives the limit $\int \Phi(1,\theta)d\theta$, that is, $f_t \to ds$, the arc length measure on the unit circle. If we take instead $\phi = \phi(x^2+y^2)$ where $\phi \in C_0^\infty((-1,1))$, then

$$\langle f_t, \phi\rangle = 2\pi t\int_0^1 \sin(t(1-r^2))\phi(r^2)rdr = \pi t\int_0^1 \sin(ts)\phi(1-s)ds$$

$$= \pi[-cos(ts)\phi(1-s)]_0^1 - \pi\int_0^1 \cos(ts)(\phi'(1-s)ds = -\pi(\phi(0)\cos t + o(1).$$

If $\phi(0) \neq 0$ the oscillation as $t \to 0$ shows that there is no limit in $\mathcal{D}'(\mathbf{R}^2)$.

7. Is there a distribution on $\mathbf{R}$, the restriction of which to $(0, \infty)$ equals $e^{1/x}$?

Solution: The answer is no. To see this let us argue towards the contradiction. If such distribution, say f, exists then we must have

$$\langle f, \phi \rangle = \int e^{\frac{1}{x}} \phi(x) dx.$$

Pick $\phi_0 \in \mathcal{D}$ such that such that $\phi(x) = 0$ if $x < 1$ or $x > 2$, $\phi_0 \geq 0$, $\int_1^2 \phi_0 = 1$.

Define the sequence

$$\phi_k(x) = e^{-\frac{k}{2}} k \phi_0(kx) \to 0$$

as $k \to \infty$ in $\mathcal{D}$. Then we have

$$\langle f, \phi_k \rangle = \int e^{\frac{1}{x}} \phi_k(x) dx = \int_1^2 e^{k(\frac{1}{y} - \frac{1}{2})} \phi_0(y) dy \geq \int_1^2 \phi_0(y) dy = 1 \tag{7}$$

Which means that f is not continuous in $\mathcal{D}'$, hence contradiction.

8. Is there a distribution on $\mathbf{R}$, the restriction of which to $(0, \infty)$ equals $e^{1/x} \exp(ie^{1/x})$?

Solution:

The answer is yes. Denote $f = e^{\frac{1}{x}}, g = fe^{if}$ and define the distribution u as follows

$$\langle u, \phi \rangle = i \int_0^{\infty} e^{if(x)} \left(\frac{f(x)\phi(x)}{f'(x)} \right)' dx$$

where $\phi \in \mathcal{D}(\mathbf{R})$. Clearly u and g agree on $(0, +\infty)$. Computing the derivative explicitly $\left(\frac{f(x)\phi(x)}{f'(x)} \right)'$ one can easily see that the integral is convergent around 0 and hence u is continuous in $\mathcal{D}'(\mathbf{R})$. Indeed, note that $f' = -\frac{1}{x^2} e^{\frac{1}{x}}$, $\frac{f}{f'} = -x^2$ and $\frac{ff''}{(f')^2} = 1 + 2x$. Combining these we get

$$\left(\frac{f(x)\phi(x)}{f'(x)} \right)' = \phi + \phi' \frac{f}{f'} - \frac{ff''}{(f')^2} = \phi - x^2 \phi' - (1 + 2x)$$

and the continuity of u follows.

9. (Homework problem) Let f be a function on $\mathbf{R}$ which is zero for $x < 0$, continuous for $x > 0$ and assume that $\int_0^1 x|f(x)| dx < \infty$. Show that f represents a distribution of order at most 1.

Solution:

Note that $f(x) = 0$ if $x \in (-\infty, 0)$. Hence $\text{supp} f \subset [0, +\infty)$. Now take $\phi \in C_0^\infty[0, +\infty)$ (in particular $\phi(0) = 0$). Suppose $\text{supp}\phi \subset [0, R]$ for some $R > 1$, we have

$$\begin{aligned} \left| \int_0^R f(x)\phi(x)dx \right| &= \left| \int_0^R f(x) \left(\int_0^x \phi'(t)dt \right) dx \right| \leq \\ &\leq \sup |\phi'| \left\{ \int_0^1 x|f(x)|dx + \int_1^R x|f(x)|dx \right\}. \end{aligned}$$

Note that the second integral in the last line is finite because f is continuous in $[1, R]$ and hanse f is bounded on $[1, R]$.

10. Solve the following equations in $\mathcal{D}'(\mathbf{R})$:

(a) $xf'(x) = \delta(x)$,

(b) $xf'(x) + f(x) = 0$.

Solution:

(a) Consider $xf'(x) = \delta(x)$. We have that

$\phi(0) = \langle f'(x)x, \phi(x) \rangle = \langle f'(x), x\phi(x) \rangle = -\langle f(x), x\phi'(x) + \phi(x) \rangle$

Therefore $f = -\delta$ is a solution.

(b) Observe that $xf'(x) + f(x) = (xf(x))'$ hence we must have that $f = c\mathcal{P}\frac{1}{x}$ where c is any constant.