Workshop 5

November 11 2014

1. Let $\phi \in \mathcal{D}(\mathbf{R})$ and assume that $\phi(0) = \phi'(0) = \cdots = \phi^{(k)}(0)$. Show that there is $\psi \in \mathcal{D}(\mathbf{R})$ with $\phi(x) = x^{k+1}\psi(x)$.

2. Show that there is a $\psi \in \mathcal{D}(\mathbf{R})$ with $\phi = \psi^{(k)}$ if and only if $\int_{-\infty}^{+\infty} P(x)\phi(x)dx = 0$ for each polynomial P of degree at most $k-1$.

3. (Homework problem) The principal value of $\frac{1}{x}$ is defined as $\mathcal{P}\frac{1}{x}(\phi) = \lim_{\epsilon \to 0} \int_{|x| \geq \epsilon} \frac{\phi(x)}{x} dx$

- Show that $\mathcal{P}\frac{1}{x}$ defines a distribution
- Represent $\mathcal{P}\frac{1}{x}(\phi)$ as a double integral.
- Find the primitive of $\mathcal{P}\frac{1}{x}$ in the sense of distributions.

4. Find all $f \in \mathcal{D}'(\mathbf{R})$ with $xf(x) = 1$.

5. Compute the following limits in $\mathcal{D}'(\mathbf{R})$.

- (a) $\lim_{t \to \infty} t^2 x \cos tx$
- (b) $\lim_{t \to \infty} t^2 |x| \cos tx$
- (c) $\lim_{t \to \infty} \frac{\sin tx}{x}$
- (d) $\lim_{t \to \infty} (\cos tx) vp(1/x)$
- (e) $\lim_{t \to \infty} t \sin(t|x|)$

6. Compute in $\mathcal{D}'(\mathbf{R}^2 \setminus \{(0,0)\})$:

$$\lim_{t\to\infty} t\sin(t|x^2+y^2-1|)$$

Does this limit exist in $\mathcal{D}'(\mathbf{R}^2)$?

7. Is there a distribution on $\mathbf{R}$, the restriction of which to $(0,\infty)$ equals $e^{1/x}$?

8. Is there a distribution on $\mathbf{R}$, the restriction of which to $(0,\infty)$ equals $e^{1/x}\exp(ie^{1/x})$?

9. (Homework problem) Let f be a function on $\mathbf{R}$ which is zero for $x<0$, continuous for $x>0$ and assume that $\int_0^1 x|f(x)|dx<\infty$. Show that f represents a distribution of order at most 1.

10. Solve the following equations in $\mathcal{D}'(\mathbf{R})$:
   (a) $xf'(x)=\delta(x)$,
   (b) $xf'(x)+f(x)=0$.