Advanced Partial Differential Equations 1

Aram Karakhanyan

The University of Edinburgh

October 8 2015

Lectures 1-2[1]

http://www.maths.ed.ac.uk/~aram/advancedpde.html/

[1] These notes are partly based on the slides by Thomas Bäckdahl

Overview PDE 1 course

- Not much about structure of hyperbolic PDEs, but only focusing on elliptic and parabolic equations.
- Not much about solution generating techniques. E.g. we will not discuss Perron's method but will touch upon the energy method
- Much about the function spaces they act on.
- Much about existence and uniqueness of solutions.

Operator picture

$$\mathcal{A} : X \to Y$$

$\mathcal{A}$ is an operator – comes from the PDE.
X, Y are function spaces.

$\mathcal{A}f = g$ in $U \subset \mathbb{R}^n$
Given $g \in Y$,
prove existence of $f \in X$.

Operator picture

$$\mathcal{A} : X \to Y$$

$\mathcal{A}$ is an operator – comes from the PDE.
X, Y are function spaces.

$\mathcal{A}f = g$ in $U \subset \mathbb{R}^n$ and $f = h$ on ∂U.
Given $g \in Y$, $h \in Z$
prove existence of $f \in X$.

Operator picture

$$\mathcal{A} : X \to Y$$

$\mathcal{A}$ is an operator – comes from the PDE.
X, Y are function spaces.

$\mathcal{A}f = g$ in $U \subset \mathbb{R}^n$ and $f = h$ on ∂U.
Given $g \in Y$, $h \in Z$
prove existence of $f \in X$.

We can use the powerful tools of functional analysis to study solvability of equations involving $\mathcal{A}$.
The main work is to find the right spaces X, Y and the abstract operator $\mathcal{A}$, not the functional analysis parts itself.

Second order PDEs

The theory about the function spaces we study in this course can be used to study basically any PDE, but here we will mainly study $\mathbb{R}$ valued second order linear PDEs on $\mathbb{R}^n$ or on a bounded domain U. We have three main types

- Elliptic (Laplace eq. $\sum_{i=1}^{n} u_{x_i x_i} = 0$)
- Parabolic (Heat eq. $u_t - \sum_{i=1}^{n} u_{x_i x_i} = 0$)
- Hyperbolic (Wave eq. $u_{tt} - \sum_{i=1}^{n} u_{x_i x_i} = 0$)

We will study the elliptic and hyperbolic case in detail, but not the parabolic case.

Definition

Let U be an open subset of $\mathbb{R}^n$. The linear partial differential operator $Lu : \mathbb{R}^n \to \mathbb{R}$ given by

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u, \tag{1}$$

is *uniformly elliptic* if there is a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \qquad \text{for a.e. } x \in U \text{ and all } \xi \in \mathbb{R}^n. \tag{2}$$

If the inequality holds for $\theta = 0$, the operator is *degenerate elliptic*.

Remark

a^{ij} is always **symmetric**.

Other example is the divergence form equation

$$\text{div}[a(x)\nabla u(x)] = 0.$$

Definition

Let U be an open subset of $\mathbb{R}^n$. The linear partial differential operator $Lu : \mathbb{R}^n \to \mathbb{R}$ given by

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u, \tag{1}$$

is *uniformly elliptic* if there is a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \qquad \text{for a.e. } x \in U \text{ and all } \xi \in \mathbb{R}^n. \tag{2}$$

If the inequality holds for $\theta = 0$, the operator is *degenerate elliptic*.

Remark

a^{ij} is always **symmetric***.*

Other example is the divergence form equation

$$\text{div}[a(x)\nabla u(x)] = 0. \Rightarrow \partial_i[a^{ij}u_j] = 0.$$

Definition

The operator $\frac{\partial}{\partial t} + L$ with L as in (1) but a, b and c are allowed to depend on both x and t is *uniformly parabolic* if (2) is satisfied for $0 < t \leq T$ for some $T > 0$.

Definition

The operator $\frac{\partial}{\partial t} + L$ with L as in (1) but a, b and c are allowed to depend on both x and t is *uniformly parabolic* if (2) is satisfied for $0 < t \leq T$ for some $T > 0$.

Definition

The operator $\frac{\partial^2}{\partial t^2} + L$ with L as in (1) but a, b and c are allowed to depend on both x and t is *uniformly hyperbolic* if (2) is satisfied for $0 < t \leq T$ for some $T > 0$.

Definition

The operator $\frac{\partial}{\partial t} + L$ with L as in (1) but a, b and c are allowed to depend on both x and t is *uniformly parabolic* if (2) is satisfied for $0 < t \leq T$ for some $T > 0$.

Definition

The operator $\frac{\partial^2}{\partial t^2} + L$ with L as in (1) but a, b and c are allowed to depend on both x and t is *uniformly hyperbolic* if (2) is satisfied for $0 < t \leq T$ for some $T > 0$.

Remark

More general definitions which does not explicitly separate the variable t are possible. Divergence forms are also useful.

Notation

For partial derivatives we will use the notation $u_{x_i} = \partial_{x_i} u$ etc. We will also use multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

$$D^\alpha u = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u.$$

U, V and W usually denote open subsets of $\mathbb{R}^n$. $B^0(x, r)$ the open ball centred at x with radius r, and $B(x, r)$ the corresponding closed ball.
V is compactly contained in U (denoted $V \Subset U$ or $V \subset\subset U$) if $V \subset \overline{V} \subset U$ and $\overline{V}$ is compact.

Weak maximum principle

Recall $Lu = -\sum_{i,j=1}^n a^{ij}(x)u_{x_ix_j} + \sum_{i=1}^n b^i(x)u_{x_i} + c(x)u$, where a^{ij}, b^i and c are continuous functions and a^{ij} is uniformly elliptic. In the two theorems below we assume that U is bounded.

Theorem (Weak maximum principle)

Assume $u \in C^2(U) \cap C^1(\bar{U})$ and

$$c = 0 \quad \text{in } U.$$

1. *If $Lu \leq 0$ in U, then*

$$\max_{\bar{U}} u = \max_{\partial U} u.$$

2. *If $Lu \geq 0$ in U, then*

$$\min_{\bar{U}} u = \min_{\partial U} u.$$

Weak maximum principle $c \geq 0$

Theorem (Weak maximum principle $c \geq 0$)

Assume $u \in C^2(U) \cap C^1(\bar{U})$ *and*

$$c \geq 0 \quad \text{in } U.$$

1. *If* $Lu \leq 0$ *in* U, *then*
$$\max_{\bar{U}} u \leq \max_{\partial U} u^+.$$
2. *If* $Lu \geq 0$ *in* U, *then*
$$\min_{\bar{U}} u \geq -\min_{\partial U} u^-.$$
3. *In particular if* $Lu = 0$ *in* U *then*
$$\max_{\bar{U}} |u| = \max_{\partial U} |u|.$$

Strong maximum principle.

- The weak maximum principle: A subsolution attains its maximum on the boundary.

Strong maximum principle.

- The weak maximum principle: A subsolution attains its maximum on the boundary.
- The strong maximum principle: A subsolution can not attain its maximum at an interior point unless it is constant. (On a connected region.)

Strong maximum principle.

- The weak maximum principle: A subsolution attains its maximum on the boundary.
- The strong maximum principle: A subsolution can not attain its maximum at an interior point unless it is constant. (On a connected region.)
- We show this by analysing the outer normal derivative $\frac{\partial u}{\partial \nu}$ at a boundary maximum point.

Lemma (Hopf's Lemma)

Assume $u \in C^2(U) \cap C^1(\bar{U})$ and $c = 0$ in U. Suppose further $Lu \leq 0$ in U, and there exists a point $x^0 \in \partial U$ such that

$$u(x^0) > u(x) \text{ for all } x \in U. \tag{3}$$

Assume finally that U satisfies the interior ball condition at x^0; that is, there exists an open ball $B \subset U$ with $x^0 \in \partial B$.

1. *Then*

$$\frac{\partial u}{\partial \nu}(x^0) > 0,$$

   where ν is the outer unit normal to B at x^0.

2. *If $c \geq 0$ in U, the same conclusion holds provided*

$$u(x^0) \geq 0.$$

Remark

The importance of point 1 is the **strict** *inequality. Observe that* $\frac{\partial u}{\partial \nu}(x^0) \geq 0$ *is obvious because*

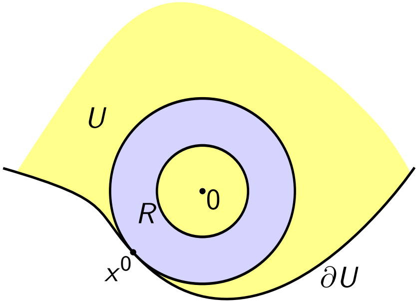
$$\frac{\partial u}{\partial \nu}(x^0) = \lim_{x \to x_0} \frac{u(x_0) - u(x)}{|x_0 - x|}.$$

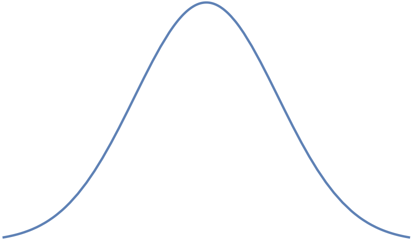
Note that the interior ball condition automatically holds if ∂U *is* C^2*.*

Proof

Assume $c \geq 0$ and $u(x^0) \geq 0$. We may as well further assume $B = B^0(0, r)$ for some radius $r > 0$. For $\lambda > 0$ define

$$v(x) = e^{-\lambda|x|^2} - e^{-\lambda r^2} \;\; (x \in B(0, r)).$$





Then using the uniform ellipticity condition, we compute:

$$\begin{aligned} Lv &= -\sum_{i,j=1}^{n} a^{ij} v_{x_i x_j} + \sum_{i=1}^{n} b^i v_{x_i} + cv \\ &= e^{-\lambda|x|^2} \sum_{i,j=1}^{n} a^{ij}(-4\lambda^2 x_i x_j + 2\lambda\delta_{ij}) - e^{-\lambda|x|^2} \sum_{i=1}^{n} b^i 2\lambda x_i + c(e^{-\lambda|x|^2} - e^{-\lambda r^2}) \\ &\leq e^{-\lambda|x|^2} \left(-4\theta\lambda^2|x|^2 + 2\lambda \; TrA + 2\lambda|b||x| + c\right) , \end{aligned}$$

for $A = ((a^{ij})), b = (b^1, \ldots, b^n)$. Consider next the open annular region $R = B^0(0, r) - B(0, r/2)$. If we choose $\lambda > 0$ large enough, we get

$$Lv \leq e^{-\lambda|x|^2}(-\theta\lambda^2 r^2 + 2\lambda \; TrA + 2\lambda|b|r + c) \leq 0 \text{ in } R. \tag{4}$$

From (3) we get that there exists a constant $\epsilon > 0$ so small that

$$u(x^0) \geq u(x) + \epsilon v(x) \quad (x \in \partial B(0, r/2)). \tag{5}$$

In addition note

$$u(x^0) \geq u(x) + \epsilon v(x) \quad (x \in \partial B(0, r)), \tag{6}$$

since $v = 0$ on $\partial B(0, r)$.
As u and v are subsolutions we find

$$L(u + \epsilon v - u(x^0)) \leq -cu(x^0) \leq 0 \text{ in } R, \tag{7}$$

and from (5), (6) we observe $u + \epsilon v - u(x^0) \leq 0$ on ∂R.
The weak maximum principle gives $u + \epsilon v - u(x^0) \leq 0$ in R.
But $u(x^0) + \epsilon v(x^0) - u(x^0) = 0$, and so

$$\frac{\partial u}{\partial \nu}(x^0) + \epsilon \frac{\partial v}{\partial \nu}(x^0) \geq 0.$$

Consequently

$$\frac{\partial u}{\partial \nu}(x^0) \geq -\epsilon \frac{\partial v}{\partial \nu}(x^0) = -\frac{\epsilon}{r} Dv(x^0) \cdot x^0 = 2\lambda \epsilon r e^{-\lambda r^2} > 0,$$

as required. □

Theorem (Strong maximum principle.)

Assume $u \in C^2(U) \cap C(\bar{U})$ and $c = 0$ in U. Suppose also U is connected and open.

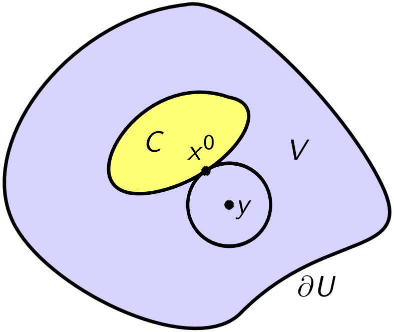
1. *If $Lu \leq 0$ in U and u attains its maximum over $\bar{U}$ at an interior point, then u is constant within U.*
2. *Similarly, if $Lu \geq 0$ in U and u attains its minimum over $\bar{U}$ at an interior point, then u is constant within U.*

Proof.

Write $M = \max_{\bar{U}} u$ and $C = \{x \in U | u(x) = M\}$.
Then if $u \not\equiv M$, set

$$V = \{x \in U | u(x) < M\}.$$

Choose a point $y \in V$ satisfying $\operatorname{dist}(y, C) < \operatorname{dist}(y, \partial U)$, and let $B = B(r, y)$ with largest r such that $B^0(r, y) \subset V$. Then there exists some point $x^0 \in C$, with $x^0 \in \partial B$.
Clearly V satisfies the interior ball condition at x^0.
Hopf's Lemma point 1, gives $\frac{\partial u}{\partial \nu}(x^0) > 0$. But this is a contradiction: since u attains its maximum at $x^0 \in U$, we have $Du(x^0) = 0$. □



Theorem (Strong maximum principle with $c \geq 0$.)

Assume $u \in C^2(U) \cap C(\bar{U})$ and $c \geq 0$ in U. Suppose also U is connected.

1. *If $Lu \leq 0$ in U and u attains a nonnegative maximum over $\bar{U}$ at an interior point, then u is constant within U.*
2. *Similarly, if $Lu \geq 0$ in U and U attains a nonpositive minimum over $\bar{U}$ at an interior point, then u is constant within U.*

The proof is like that above, except that we use point 2 in Hopf's Lemma.

A priori estimates

Theorem

Let $u \in C(\overline{U}) \cap C^2(U)$ solve $a^{ij}(x)u_{ij} + b^i(x)u_i + c(x)u \geq f$ in the bounded domain U, a^{ij} is elliptic (i.e. $a^{ij}\xi^i\xi^j \geq \lambda|\xi|^2$, $\lambda > 0$) and $c \leq 0$. Then

$$\sup_U u \leq \sup_{\partial U} u^+ + C\frac{\sup_U |f^-|}{\lambda}$$

where the constant depends on diam$U$ and $\beta := \frac{\sup |b|}{\lambda}$. In particular if $a^{ij}(x)u_{ij} + b^i(x)u_i + c(x)u = f$ then

$$\sup_U |u| \leq \sup_{\partial U} |u| + C\frac{\sup_U |f|}{\lambda}.$$

Here $u^\pm$ are the positive and negative parts of u respectively defined by $u^+ = \max(u, 0)$, $u^- = -\min(u, 0)$, hence $u = u^+ - u^-$. In particular, if U lies between two parallel planes a distance d apart, then the inequality above is satisfied with $C = e^{(\beta+1)d} - 1$.

Proof

Let U lie in the slab $0 < x_1 < d$, and set $L_0 u = a^{ij} u_{ij} + b^i u_i$. For $\alpha \geq \beta + 1$ we have

$$L_0 e^{\alpha x_1} = (\alpha^2 a^{11} + \alpha b^1) e^{\alpha x_1} \geq \lambda(\alpha^2 - \alpha\beta) e^{\alpha x_1} \geq \lambda.$$

Let

$$v(x) = \sup_{\partial U} u^+ (e^{\alpha d} - e^{\alpha x_1}) \frac{\sup_U |f^-|}{\lambda}.$$

Then, since $L_0 v + cv \leq -\lambda \sup_U (|f^-|/\lambda)$, consequently

$$L_0(v - u) + c(v - u) \leq -\lambda \left(\frac{\sup_U |f^-|}{\lambda} + \frac{f}{\lambda} \right) \leq 0 \quad \text{in } U$$

and $v - u \geq 0$ on ∂U. Hence, for $C = e^{d\alpha} - 1$ and $\alpha \geq \beta + 1$ we obtain the desired result for the case $L_0 u + cu \geq f$.
Replacing u by $-u$, we obtain the result for the case $L_0 u + cu = f$.

Bernstein's method

Theorem

Let u be harmonic in $B_1 = \{x \in \mathbb{R}^n \text{ st } |x| < 1\}$ and assume that $u \in C^3(\overline{B_1})$ then there is a constant C depending only on the dimension n such that

$$|Du(0)| \leq C \sup_{B_1} |u|.$$

Proof

Introduce the auxiliary function $w(x) = \zeta^2|Du|^2 + \lambda|u|^2$, where $\lambda > 0$ is a constant to be fixed below.
We have that $\Delta u = 0$ and differentiating both sides of this equation in $x_i, i = 1, 2 \ldots, n$ it follows that $\Delta u_i = 0$. Then we have

$$
\begin{aligned}
\Delta w &= |Du|^2 \Delta(\zeta)^2 + \zeta^2 \left[2u_i \Delta u_i + 2 \sum_{ij} u_{ij}^2 \right] + 8\zeta\zeta_i u_j u_{ij} + 2\lambda |Du|^2 + 2\lambda u \Delta u \\
&= |Du|^2 \left[2\lambda + \Delta(\zeta^2) \right] + 2\zeta^2 \sum_{ij} u_{ij}^2 + 8[\zeta_i u_j][\zeta u_{ij}] \geq \\
&\geq |Du|^2 \left[2\lambda + \Delta(\zeta^2) - 8|D\zeta|^2 \right]
\end{aligned}
$$

where we used the fact that $\Delta u = 0, \Delta u_i = 0$ and Cauchy-Schwartz inequality for the last line. Choosing λ large enough so that $2\lambda \geq \sup |\Delta(\zeta^2)| + 8|D\zeta|^2$ we infer that

$$\Delta w \geq 0.$$

Applying the weak maximum principle we infer that

$$\sup_{B_1} w \leq \sup_{\partial B_1} w = \lambda \sup_{\partial B_1} |u|^2.$$

On the other hand $\sup_{B_1} w \geq \sup_{B_1} \zeta^2 |Du|^2 \geq \zeta^2(0)|Du(0)|^2$. Thus the result follows. Now we give two applications:

- let u be harmonic in B_r then

$$|Du(0)| \leq \frac{C}{r} \sup_{B_r} |u|.$$

  Let $v(x) = u(rx), x \in B_1$ then $Dv(x) = r(Du)(rx)$. Applying previous theorem the claim follows.

- let u be harmonic in B_r, then

$$|D^k u(0)| \leq \frac{C^{|k|}}{r^{|k|}} \sup_{B_r} |u|, k = (k_1, \ldots, k_n), |k| = k_1 + \cdots + k_n.$$

To see this one has to iterate the previous estimate and note that $\Delta(D^k u) = 0$ for any multi index k. For instance for the second order derivative $u_i, i = 1, \ldots, n$ we have

$$|Du_i(0)| \leq \frac{C}{r} \sup_{B_r} |Du| \leq \frac{C^2}{r^2} \sup_{B_r} |u|$$

Since i is arbitrary it follows that

$$|D^2 u(0)| \leq \frac{C^2}{r^2} \sup_{B_r} |u|$$