Advanced Partial Differential Equations 1

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Lecture 5

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Let U be an open subset of $\mathbb{R}^n$.

Definition (Elliptic operator)

The linear partial differential operator $L : \mathbb{R}^n \to \mathbb{R}$ given by

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} \tilde{b}^i(x)u_{x_i} + c(x)u, \tag{1}$$

or

$$Lu = -\sum_{i,j=1}^{n} (a^{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u, \tag{2}$$

is *uniformly elliptic* if there is a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \qquad \text{for a.e. } x \in U \text{ and all } \xi \in \mathbb{R}^n. \tag{3}$$

If the inequality holds for $\theta = 0$, the operator is *elliptic*.

Remark

If the coefficients a^{ij} are C^1 we clearly see that the different forms are related through

$$\tilde{b}^i = b^i - \sum_{j=1}^{n} a^{ij}_{x_j}.$$

We will also assume the symmetry $a^{ij} = a^{ji}$.

Boundary value problem

We will study the boundary value problem

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U, \end{cases} \tag{4}$$

where U is open, and bounded. (Dirichlet boundary condition.)

Lejeune Dirichlet (1805 - 1859)

Weak solutions

General assumptions

Assume $a^{ij}, b^i, c \in L^\infty(U)$, and $f \in L^2(U)$.

If u is smooth, we can multiply the PDE with $v \in C_c^\infty(U)$, and integrate over U. After an integration by parts we get

$$\int_U \sum_{i,j=1}^n a^{ij}(x) u_{x_i} v_{x_j} + \sum_{i=1}^n b^i(x) u_{x_i} v + c(x) uvdx = \int_U fvdx.$$

By approximation this extends to $v \in H_0^1(U)$.
The identity then makes sense if $u \in H_0^1(U)$.
The boundary condition $u = 0$ on ∂U makes us choose $H_0^1(U)$ instead of $H^1(U)$.

Definition

The bilinear form $B[\ ,\ ]$ associated with the divergence form elliptic operator L is

$$B[u,v] = \int_U \sum_{i,j=1}^{n} a^{ij}(x) u_{x_i} v_{x_j} + \sum_{i=1}^{n} b^i(x) u_{x_i} v + c(x) uv dx$$

for $u, v \in H_0^1(U)$.

Definition

We say that $u \in H_0^1(U)$ is a weak solution of the boundary value problem (4) if

$$B[u,v] = (f,v) \quad \text{for all } v \in H_0^1(U),$$

where $(\ ,\ )$ denotes the $L^2(U)$ inner product.

Weaker boundary value problem

As $v \in H_0^1(U)$, we can let $f \in H^{-1}(U)$ instead of $L^2(U)$.
Recall that if $f \in H^{-1}(U)$, then there exist functions $f^0, f^1, \ldots, f^n \in L^2(U)$ such that

$$\langle f, v \rangle = \int_U f^0 v + \sum_{i=1}^{n} f^i v_{x_i} dx \qquad \forall v \in H_0^1(U). \tag{5}$$

We informally write this $f = f^0 - \sum_{i=1}^n f^i_{x_i}$.
Hence, we can consider the boundary value problem

$$\begin{cases} Lu = f^0 - \sum_{i=1}^n f^i_{x_i} & \text{in } U \\ u = 0 & \text{on } \partial U, \end{cases} \tag{6}$$

with $f^i \in L^2(U)$.

Definition

We say that $u \in H_0^1(U)$ is a weak solution of the boundary value problem (6) if

$$B[u, v] = \langle f, v \rangle \quad \text{for all } v \in H_0^1(U),$$

where $f = f^0 - \sum_{i=1}^{n} f^i_{x_i} \in H^{-1}$.

Other boundary values

Assume ∂U is C^1, then we can study the boundary value problem

$$\begin{cases} Lu = f & \text{in } U \\ u = g & \text{on } \partial U, \end{cases} \tag{7}$$

with $f \in H^{-1}$ and g is the trace on ∂U of some $w \in H^1$.
The bilinear form identity will be unchanged because $v \in H_0^1(U)$.
We get that $\tilde{u} = u - w \in H_0^1(U)$, and is a weak solution to

$$\begin{cases} L\tilde{u} = \tilde{f} & \text{in } U \\ \tilde{u} = 0 & \text{on } \partial U, \end{cases} \tag{8}$$

with $\tilde{f} = f - Lw \in H^{-1}(U)$.

Existence of weak solutions

To prove existence of weak solutions, we will use tools from linear functional analysis. In particular we study bilinear forms.

Theorem (Lax-Milgram Theorem)

Let H be a Hilbert space, and assume that $B : H \times H \to \mathbb{R}$ is a bilinear mapping, for which there exist constants $\alpha, \beta > 0$ such that

$$|B[u,v]| \leq \alpha \|u\| \|v\| \qquad \forall u, v \in H,$$
$$\beta \|u\|^2 \leq B[u,u] \qquad \forall u \in H.$$

Let $f : H \to \mathbb{R}$ be a bounded linear functional on H.
Then there exist a unique element $u \in H$ such that

$$B[u,v] = \langle f, v \rangle \qquad \forall v \in H.$$

Proof

Fix an element $u \in H$. Since the mapping $v \mapsto B[u, v]$ is a bounded linear functional on H, the Riesz Representation Theorem gives a unique element $w \in H$ such that

$$B[u, v] = (w, v) \qquad \forall v \in H.$$

This works for any u, so we can write $Au = w$ and get

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We want to show that $A : H \to H$ is a bounded linear operator.
Let $\lambda_1, \lambda_2 \in \mathbb{R}$ and $u_1, u_2, v \in H$ be arbitrary. Then we have

$$\begin{aligned}(A(\lambda_1 u_1 + \lambda_2 u_2), v) &= B[\lambda_1 u_1 + \lambda_2 u_2, v] = \lambda_1 B[u_1, v] + \lambda_2 B[u_2, v] \\ &= \lambda_1 (Au_1, v) + \lambda_2 (Au_2, v).\end{aligned}$$

For the boundedness we find

$$\|Au\|^2 = (Au, Au) = B[u, Au] \leq \alpha \|u\| \|Au\| \qquad \forall u \in H.$$

Hence, we have the bound $\|Au\| \leq \alpha \|u\|$.

Next we want to show

$$\begin{cases} A \textbf{ is one-to-one}, \\ R(A) \textbf{ is closed in } H, \end{cases}$$

where $R(A)$ denotes the range of A.
We can make the estimate

$$\beta \|u\|^2 \leq B[u,u] = (Au, u) \leq \|Au\| \|u\|.$$

Hence, $\beta\|u\| \leq \|Au\|$. This gives $Au = 0 \Rightarrow u = 0$ (injectivity).
Also if $\{Au_k\}_{k=1}^{\infty}$ is a Cauchy sequence, then $\{u_k\}_{k=1}^{\infty}$ will be a Cauchy sequence, i.e. $u_k \to u$ and $Au_k \to Au$. Hence, $R(A)$ is closed.

Now we want to show $R(A) = H$:
Since $R(A)$ is closed, we have $H = R(A) \oplus R(A)^{\perp}$. But if $w \in R(A)^{\perp}$ we get $\beta\|w\|^2 \leq B[w, w] = (Aw, w) = 0$. Hence, $w = 0$ and we can conlude $R(A)^{\perp} = \{0\}$.

The Riesz Representation Theorem gives us an element $w \in H$ such that

$$\langle f, v \rangle = (w, v) \qquad \forall v \in H.$$

As A is bijective, we can find a unique $u \in H$ such that $Au = w$. Then

$$B[u, v] = (Au, v) = (w, v) = \langle f, v \rangle \qquad \forall v \in H.$$

The choice of $u \in H$ is unique:
Assume $u, \tilde{u} \in H$, such that $B[u, v] = \langle f, v \rangle = B[\tilde{u}, v] \quad \forall v \in H$. Then $B[u - \tilde{u}, v] = 0 \quad \forall v \in H$. Set $v = u - \tilde{u}$ to find $\beta\|u - \tilde{u}\|^2 \leq B[u - \tilde{u}, u - \tilde{u}] = 0$. □

- Consider again the bilinear form $B[\ ,\ ]$ associated with the elliptic operator L:

$$B[u,v] = \int_U \sum_{i,j=1}^{n} a^{ij}(x) u_{x_i} v_{x_j} + \sum_{i=1}^{n} b^i(x) u_{x_i} v + c(x) uv dx$$

- We will use the Lax-Milgram Theorem, to prove existence and uniqueness of weak solutions to our elliptic problem.
- However, as the operator might have a non-trivial kernel, we will modify the operator a little bit.
- But first we need energy estimates, so we can use the Lax-Milgram Theorem.

Theorem (Energy estimates)

There exist constants $\alpha, \beta > 0$ *and* $\gamma \geq 0$ *such that*

$$|B[u,v]| \leq \alpha \|u\|_{H^1(U)} \|v\|_{H^1(U)},$$

and

$$\beta \|u\|^2_{H^1(U)} \leq B[u,u] + \gamma \|u\|^2_{L^2(U)}$$

for all $u, v \in H^1_0(U)$.

Proof

Using Hölders inequality, we can make the estimate

$$
\begin{aligned}
|B[u,v]| \leq & \sum_{i,j=1}^{n} \|a^{ij}\|_{L^\infty} \int_U |Du||Dv|dx + \sum_{i=1}^{n} \|b^i\|_{L^\infty} \int_U |Du||v|dx \\
& + \|c\|_{L^\infty} \int_U |u||v|dx \\
\leq & \, \alpha \|u\|_{H^1(U)} \|v\|_{H^1(U)},
\end{aligned}
$$

for some constant α depending on the coefficients a^{ij}, b^i, c but not on u or v.

From the uniform ellipticity condition we get

$$\begin{aligned}\theta \int_U |Du|^2 dx &\leq \int_U \sum_{i,j=1}^n a^{ij} u_{x_i} u_{x_j} dx \\ &= B[u,u] - \int_U \sum_{i=1}^n b^i u_{x_i} u + cu^2 dx \\ &\leq B[u,u] + \sum_{i=1}^n \|b^i\|_{L^\infty} \int_U |Du||u| dx + \|c\|_{L^\infty} \int_U u^2 dx.\end{aligned}$$

The Cauchy inequality with ϵ gives for $\epsilon > 0$

$$\int_U |Du||u| dx \leq \epsilon \int_U |Du|^2 dx + \frac{1}{4\epsilon} \int_U u^2 dx$$

Choose ϵ small enough so

$$\epsilon \sum_{i=1}^{n} \|b^i\|_{L^\infty} < \frac{\theta}{2}.$$

Together, we get

$$\theta \int_U |Du|^2 dx \leq B[u,u] + \frac{\theta}{2} \int_U |Du|^2 dx + \Big(\frac{1}{4\epsilon} \sum_{i=1}^{n} \|b^i\|_{L^\infty} + \|c\|_{L^\infty} \Big) \int_U u^2 dx.$$

Hence, there is a constant C such that

$$\frac{\theta}{2} \int_U |Du|^2 dx \leq B[u,u] + C \int_U u^2 dx.$$

From the Poincaré inequality, we also have

$$\|u\|_{L^2(U)} \leq C\|Du\|_{L^2(U)}.$$

It therefore follows that we can find constants $\beta > 0$ and $\gamma \leq 0$ such that

$$\beta\|u\|^2_{H^1(U)} \leq B[u,u] + \gamma\|u\|^2_{L^2(U)}.$$

□

Example

Let $Lu = -\Delta u$. Then we get

$$B[u, v] = \int_U Du \cdot Dvdx.$$

The Poincaré inequality then directly gives

$$\|u\|^2_{H^1(U)} = \|u\|^2_{L^2(U)} + \|Du\|^2_{L^2(U)} \leq C\|Du\|^2_{L^2(U)} = CB[u, u].$$

Hence, in this case the estimate holds for $\gamma = 0$.

Theorem (First Existence Theorem for weak solutions)

There is a number $\gamma \geq 0$ such that for each $\mu \geq \gamma$, and each collection $f^0, \ldots, f^n \in L^2(U)$, there exists a unique weak solution $u \in H_0^1(U)$ of the boundary value problem

$$\begin{cases} Lu + \mu u = f^0 - \sum_{i=1}^n f^i_{x_i} & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

Proof

Use the previous theorem to obtain $\alpha, \beta > 0$ and $\gamma \geq 0$. Let $\mu \geq \gamma$ and define the bilinear form

$$B_\mu[u, v] = B[u, v] + \mu(u, v)_{L^2(U)} \qquad \forall u, v \in H_0^1(U).$$

This corresponds to the operator $L_\mu u = Lu + \mu u$. Now, $B_\mu[\ ,\ ]$ satisfies the hypotheses of the Lax-Milgram Theorem because

$$|B_\mu[u, v]| \leq \alpha \|u\|_{H^1(U)} \|v\|_{H^1(U)} + \mu \|u\|_{L^2(U)} \|v\|_{L^2(U)} \leq \tilde{\alpha} \|u\|_{H^1(U)} \|v\|_{H^1(U)},$$

$$\beta \|u\|^2_{H^1(U)} \leq B_\mu[u, u] + (\gamma - \mu) \|u\|^2_{L^2(U)} \leq B_\mu[u, u].$$

Fix $f = f^0 - \sum_{i=1}^n f^i_{x_i} \in H^{-1}(U)$.
Then $\langle f, v\rangle = \int_U f^0 v + \sum_{i=1}^n f^i v_{x_i} dx$ is a bounded linear functional on $H^1_0(U)$.
Now, the Lax-Milgram Theorem gives us a unique function $u \in H^1_0(U)$ satisfying

$$B_\mu[u, v] = \langle f, v\rangle \qquad \forall v \in H^1_0(U).$$

Consequently u is a weak solution to the boundary value problem. □

Remark

Note that $L_\mu = L + \mu I$ maps $H^1_0(U)$ to $H^{-1}(U)$, and the theorem tells us that we can find an inverse. Hence, $L_\mu : H^1_0(U) \to H^{-1}(U)$ is an isomorphism.

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$$
\begin{aligned}
B[u,v] &= \int_U \sum_{i,j=1}^{n} a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} v + cuvdx \\
&= \int_U \sum_{i,j=1}^{n} a^{ij} v_{x_j} u_{x_i} - \sum_{i=1}^{n} (b^i v_{x_i} + b^i_{x_i} v) u + cvudx \\
&= B^*[v,u]
\end{aligned}
$$

Definition

1. The operator L^*, the formal adjoint of L, is

$$L^*v = -\sum_{i,j=1}^{n}(a^{ij}v_{x_j})_{x_i} - \sum_{i=1}^{n} b^i v_{x_i} + (c - \sum_{i=1}^{n} b^i_{x_i})v$$

   provided $b^i \in C^1(\bar{U})$.

2. The adjoint bilinear form

$$B^* : H_0^1(U) \times H_0^1(U) \to \mathbb{R}$$

   is defined by

$$B^*[v,u] = B[u,v] \qquad \forall u, v \in H_0^1(U).$$

Definition (Adjoint problem)

We say that $v \in H_0^1(U)$ is a weak solution of the adjoint problem with $f \in L^2(U)$

$$\begin{cases} L^* v = f & \text{in } U, \\ v = 0 & \text{on } \partial U, \end{cases}$$

provided

$$B^*[v, u] = (f, u) \qquad \forall u \in H_0^1(U).$$

Theorem (Second Existence theorem for weak solutions)

1. *Exactly one of the following statements holds: (The Fredholm Alternative)*
   either
   $$\begin{cases} \text{for each } f \in L^2(U) \text{ there exists a unique weak} \\ \text{solution } u \text{ to the boundary value problem} \\ \begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \end{cases}$$
   or else
   $$\begin{cases} \text{there exists a non-trivial weak solution } u \\ \text{of the homogeneous problem} \\ \begin{cases} Lu = 0 & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \end{cases}$$

2. Furthermore, should the second case hold, the dimension of the subspace $N \subset H_0^1(U)$ of weak solutions of the homogeneous problem is finite and equals the dimension of the subspace $N^* \subset H_0^1(U)$ of weak solutions of

$$\begin{cases} L^* v = 0 & \text{in } U \\ v = 0 & \text{on } \partial U \end{cases}$$

3. Finally, the first boundary problem has a weak solution if and only if

$$(f, v) = 0 \qquad \forall v \in N^*.$$

Ivar Fredholm (1866-1927)

Definition

A bounded linear operator $K : X \to Y$ is called compact if for every bounded sequence $\{u_k\}_{i=1}^{\infty} \subset X$, the sequence $\{Ku_k\}_{i=1}^{\infty}$ is precompact in Y, i.e it has a subsequence that converges in Y.

Important properties

Assume that H is a Hilbert space, and $K : H \to H$ is compact. Then we have

- K maps weakly convergent sequences to convergent sequences.
- The adjoint $K^* : H \to H$ is also compact.

Theorem (Fredholm Alternative)

Let $K : H \to H$ be a compact linear operator. Then

1. $N(I - K)$ *is finite dimensional,*
2. $R(I - K)$ *is closed,*
3. $R(I - K) = N(I - K^*)^\perp$
4. $N(I - K) = 0$ *if and only if* $R(I - K) = H$*,*
5. $\dim N(I - K) = \dim N(I - K^*)$.