Stability of implicit and implicit–explicit multistep methods for nonlinear parabolic equations in Hilbert spaces

Georgios Akrivis

Department of Computer Science & Engineering

University of Ioannina, Greece

Modern numerical discretisations of p.d.e's and adaptivity

Scottish Numerical Methods Network 2018

University of Strathclyde, Glasgow

Outline

1. Nonlinear parabolic equations
   Quantification of the non-self-adjointness of linear elliptic operators
2. Implicit multistep methods
3. The stability result
4. Implicit–explicit multistep methods
5. Key ingredients in the stability analysis
6. An example

- A.: IMA J. Numer. Anal. (2018)
- Math. Comp. (2013, 2016)
- M. Zlámal: Math. Comp. (1975)

Key issues

1. Choice of suitable assumptions
2. Choice of stability technique
   - Energy method
   - Semigroup technique
   - Spectral technique
   - Fourier technique (= Parseval's identity)[1]
   - Perturbation arguments
   - Discrete maximal parabolic regularity
   - $\cdots$
3. An assumption may be suitable for a stability technique but not suitable for another stability technique.

[1] C. Lubich: Numer. Math. (1988, 1991)

Key issues

1. Choice of suitable assumptions
2. Choice of stability technique
   - Energy method
   - Semigroup technique
   - Spectral technique
   - Fourier technique (= Parseval's identity)[1]
   - Perturbation arguments
   - Discrete maximal parabolic regularity
   - ...
3. An assumption may be suitable for a stability technique but not suitable for another stability technique.

[1] C. Lubich: Numer. Math. (1988, 1991)

Key issues

1. Choice of suitable assumptions
2. Choice of stability technique
   - Energy method
   - Semigroup technique
   - Spectral technique
   - Fourier technique (= Parseval's identity)[1]
   - Perturbation arguments
   - Discrete maximal parabolic regularity
   - ...
3. An assumption may be suitable for a stability technique but not suitable for another stability technique.

[1] C. Lubich: Numer. Math. (1988, 1991)

1. Nonlinear parabolic equations

Consider the initial value problem

$$\begin{cases} u'(t) + A(t)u(t) = B(t, u(t)), \quad 0 < t < T, \\ u(0) = u^0, \end{cases}$$

in a usual triple $V \subset H = H' \subset V'$ of separable complex Hilbert spaces, with V densely and continuously embedded in H.

Here

- $A(t) : V \to V'$ uniformly coercive and bounded linear operator,
- $B(t, \cdot) : V \to V',\ t \in [0, T]$, possibly nonlinear, "small".

For instance: in the case of second order parabolic equations subject to homogeneous Dirichlet b.c., we have

$$H = L^2(\Omega), \quad V = H_0^1(\Omega), \quad V' = H^{-1}(\Omega).$$

Notation:

- $|\cdot|, \|\cdot\|, \|\cdot\|_\star$ norms on $H, V,$ and $V',$ respectively.
- $(\cdot,\cdot)$ inner product on H and duality pairing between V' and V.
- $A_s(t) := \frac{1}{2}\big[A(t) + A(t)^\star\big], \quad A_a(t) := \frac{1}{2}\big[A(t) - A(t)^\star\big]$
- $\boxed{A(t) = A_s(t) + A_a(t)}$

Quantification of the non-self-adjointness of $A(t)$

Consider the bounded linear operator $\mathcal{A}(t) : H \to H$ and its anti-self-adjoint part $\mathcal{A}_a(t)$,

$$\boxed{\mathcal{A}(t) := A_s^{-1/2}(t)A(t)A_s^{-1/2}(t), \quad \mathcal{A}_a(t) = A_s^{-1/2}(t)A_a(t)A_s^{-1/2}(t),}$$

$\mathcal{A}(t) = I + \mathcal{A}_a(t)$. We have $|\mathcal{A}(t)|^2 = 1 + |\mathcal{A}_a(t)|^2$ and

$$\boxed{\forall v \in V \quad (A(t)v, v) \in S_\varphi \iff |\mathcal{A}(t)| \le \frac{1}{\cos\varphi} \iff |\mathcal{A}_a(t)| \le \tan\varphi,}$$

for any angle $\varphi < 90^\circ$, and the sector

$$S_\varphi := \{z \in \mathbb{C} : z = \rho\, \mathrm{e}^{\mathrm{i}\,\psi}, \rho \ge 0, |\psi| \le \varphi\}.$$

The smallest half-angle of a sector $S_{\varphi(t)}$[2] as well as the norms of $\mathcal{A}_a(t)$ or $\mathcal{A}(t)$ are exact measures of the non-self-adjointness of $A(t)$.

[2] G. Savaré: Numer. Math. (1993)

Comparison to the commonly used estimate

Let

$$\|A(t)v\|_\star \le \nu(t)\|v\| \quad \forall v \in V$$

and

$$\mathrm{Re}(A(t)v, v) \ge \kappa(t)\|v\|^2 \quad \forall v \in V.$$

Then

$$|\mathcal{A}(t)| \le \frac{\nu(t)}{\kappa(t)}.$$

The ratio $\nu(t)/\kappa(t)$ is an estimate of the non-self-adjoindness of $A(t)$. Since it depends on the choice of the specific norm $\|\cdot\|$ on V, it may be a crude one![3]

The ratio $\nu(t)/\kappa(t)$ attains its minimal value, namely $|\mathcal{A}(t)|$, if we endow V with the time-dependent norm $\|\cdot\|_t$,

$$\|v\|_t := (A_s(t)v, v)^{1/2} \quad \forall v \in V.$$

[3] A.: SINUM (2015), A., Lubich: Numer. Math. (2015)

Assumptions on A and B

1. Uniform boundedness of $\boxed{\mathcal{A}_a(t) := A_s^{-1/2}(t)A_a(t)A_s^{-1/2}(t)}$:

$$\forall t \in [0,T]\ \forall v \in H \quad |\mathcal{A}_a(t)v| \le \lambda_1(t)|v|$$

   with a stability function $\lambda_1(t)$.

2. Local Lipschitz condition on B:[4]

   Let $T_u := \{v \in V : \min_t \|v - u(t)\| \le 1\}$ and assume that

$$|A_s^{-1/2}(t)\big(B(t,v) - B(t,\tilde{v})\big)| \le \lambda_2(t)|A_s^{1/2}(t)(v-\tilde{v})| + \mu_2(t)|v-\tilde{v}|,$$

   for $v, \tilde{v} \in T_u$, with a "small" stability function $\lambda_2(t)$.

3. "Weak" Lipschitz (or bounded variation) conditions on $A(t)$ and $B(t,\cdot)$ in time.

---

[4] A., Crouzeix, Makridakis: Numer. Math. (1999)

Parabolic equations satisfying our assumptions

1. Reaction–diffusion equation
   $u_t - \Delta u = f(u).$
2. Quasi-linear parabolic equations.
3. Cahn–Hilliard equation
   $u_t + \nu u_{xxxx} - (u^3 - u)_{xx} = 0.$
4. Kuramoto–Sivashinsky equation (with low-order dispersion)
   $u_t + \nu u_{xxxx} + \delta u_{xxx} + u_{xx} + u u_x = 0.$
5. Topper-Kawahara equation.
6. Systems of Kuramoto–Sivashinsky-type equations.
7. Parabolic equations of the form
   $u_t - \sum_{i,j=1}^{d} \big((a_{ij}(x,t) + \tilde{a}_{ij}(x,t)) u_{x_j}\big)_{x_i} = B(t,u)$
   with positive definite and Hermitian, and anti-Hermitian matrices, respectively, with smooth entries $a_{ij}(x,t)$ and $\tilde{a}_{ij}(x,t)$, respectively, and $B(t,\cdot)$ suitable, possibly nonlinear, operators.

2. Implicit multistep schemes

- (α, β) an implicit q-step scheme generated by the polynomials

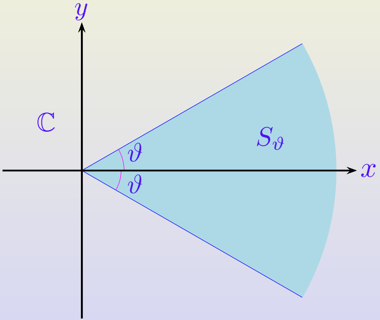
$$\alpha(\zeta) = \sum_{i=0}^{q} \alpha_i \zeta^i, \quad \beta(\zeta) = \sum_{i=0}^{q} \beta_i \zeta^i.$$

- Let $N \in \mathbb{N}$, $k := T/N$, and $t^n := nk,\ n = 0, \dots, N.$
- Let $U^0, \dots, U^{q-1} \in V$ be given starting approximations.
- Define approximations U^m to $u^m := u(t^m)$, $m = q, \dots, N$, by

$$\boxed{\sum_{i=0}^{q} \big[\alpha_i I + k\beta_i A(t^{n+i})\big] U^{n+i} = k \sum_{i=0}^{q} \beta_i B(t^{n+i}, U^{n+i}),}$$

$n = 0, \dots, N - q.$

Assumption: The scheme (α, β) is (strongly) $A(\vartheta)$-stable, i.e., for $z \in S_\vartheta$, $\chi(z;\cdot) = \alpha(\cdot) + z\beta(\cdot)$ satisfies the root condition, and the roots of β are (strictly) less than 1 in modulus.

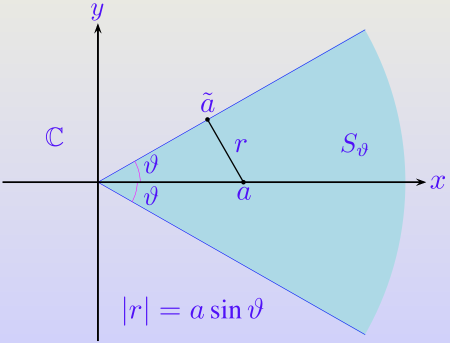


An important constant

Let

$$K_{(\alpha,\beta)} := \sup_{x>0} \max_{\zeta\in\mathcal{K}} \frac{|x\beta(\zeta)|}{|\alpha(\zeta)+x\beta(\zeta)|} = \frac{1}{\sin\vartheta},$$

with $\mathcal{K}$ the unit circle, $\mathcal{K} := \{\zeta \in \mathbb{C} : |\zeta| = 1\}$, and ϑ as large as possible s.t. the scheme (α,β) is $A(\vartheta)$-stable.



Example: BDF methods

Let

$$\alpha(\zeta) = \sum_{j=1}^{q} \frac{1}{j}\zeta^{q-j}(\zeta-1)^j, \quad \beta(\zeta) = \zeta^q, \quad q = 1, \dots, 6.$$

(α, β) is the q-step BDF scheme; its order is q.

The q-step BDF scheme is strongly $A(\vartheta_q)$-stable with

$$\vartheta_1 = \vartheta_2 = 90^\circ, \;\; \vartheta_3 = 86.03^\circ, \;\; \vartheta_4 = 73.35^\circ, \;\; \vartheta_5 = 51.84^\circ, \;\; \vartheta_6 = 17.84^\circ.$$

3. The stability result

Let $V^m \in T_u$ satisfy the perturbed equations

$$\sum_{i=0}^{q} \big[\alpha_i I + k\beta_i A(t^{n+i})\big] V^{n+i} = k \sum_{i=0}^{q} \beta_i B(t^{n+i}, V^{n+i}) + kE^n.$$

Theorem

Let $\vartheta^m := V^m - U^m$. *If*

$$\boxed{(\cot\vartheta)\lambda_1(t) + K_{(\alpha,\beta)}\lambda_2(t) < 1 \quad \forall t \in [0,T],}$$

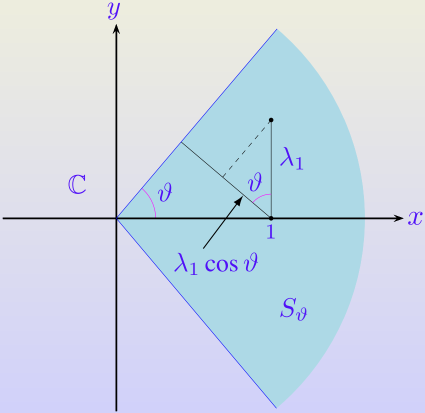
then we have the stability estimate

$$|\vartheta^n|^2 + k\sum_{\ell=q}^{n} \|\vartheta^\ell\|^2 \le C \sum_{j=0}^{q-1} \big(|\vartheta^j|^2 + k\|\vartheta^j\|^2\big) + Ck \sum_{\ell=0}^{n-q} \|E^\ell\|_\star^2,$$

$n = q, \dots, N$, *with a constant* C *independent of* k, n, U^m, V^m *and* E^m.

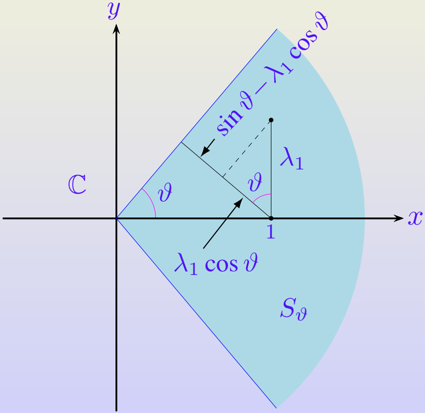
Geometric interpretation of the stability condition

$$(\cot\vartheta)\lambda_1(t) + K_{(\alpha,\beta)}\lambda_2(t) < 1 \iff (\cos\vartheta)\lambda_1(t) + \lambda_2(t) < \sin\vartheta$$

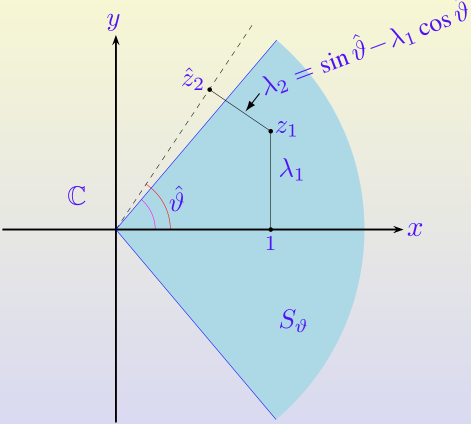


Sharpness of the stability condition

$$(\cot\vartheta)\lambda_1(t) + K_{(\alpha,\beta)}\lambda_2(t) < 1 \iff \lambda_2(t) < \sin\vartheta - (\cos\vartheta)\lambda_1(t)$$



$$\lambda_2 = \sin\hat\vartheta - (\cos\hat\vartheta)\lambda_1, \quad \vartheta < \hat\vartheta < 90^\circ$$



The method (α, β) is unstable for the equation

$$u' + \hat z_2 A_s u = u' + A_s u + \mathrm{i}\,\lambda_1 A_s u - (z_1 - \hat z_2)A_s u = 0.$$

Alternative forms of the sufficient stability condition

Uniform boundedness of $\mathcal{A}_a(t) := A_s^{-1/2}(t)A_a(t)A_s^{-1/2}(t)$:

$$\forall t \in [0,T]\ \forall v \in H \quad |\mathcal{A}_a(t)v| \le \lambda_1(t)|v|$$

with a stability function $\lambda_1(t)$.
Sufficient stability condition: $(\cos\vartheta)\lambda_1(t) + \lambda_2(t) < \sin\vartheta$

Alternative assumptions: 1. Uniform boundedness of $\mathcal{A}(t) := A_s^{-1/2}(t)A(t)A_s^{-1/2}(t)$:

$$\forall t \in [0,T]\ \forall v \in H \quad |\mathcal{A}(t)v| \le \tilde{\lambda}_1(t)|v|$$

with a stability function $\tilde{\lambda}_1(t)$.
Since $|\mathcal{A}(t)|^2 = 1 + |\mathcal{A}_a(t)|^2$, we may assume that $\tilde{\lambda}_1(t)^2 = 1 + \lambda_1(t)^2$.
Then, the stability condition reads

$$\boxed{(\cos\vartheta)\sqrt{\tilde{\lambda}_1(t)^2 - 1} + \lambda_2(t) < \sin\vartheta}$$

2. Let $\varphi(t)$ be the smallest half-angle of a sector containing the numerical range of $A(t)$,

$$(A(t)v, v) \in S_{\varphi(t)} \quad \forall v \in V \ \ \forall t \in [0, T].$$

Then,

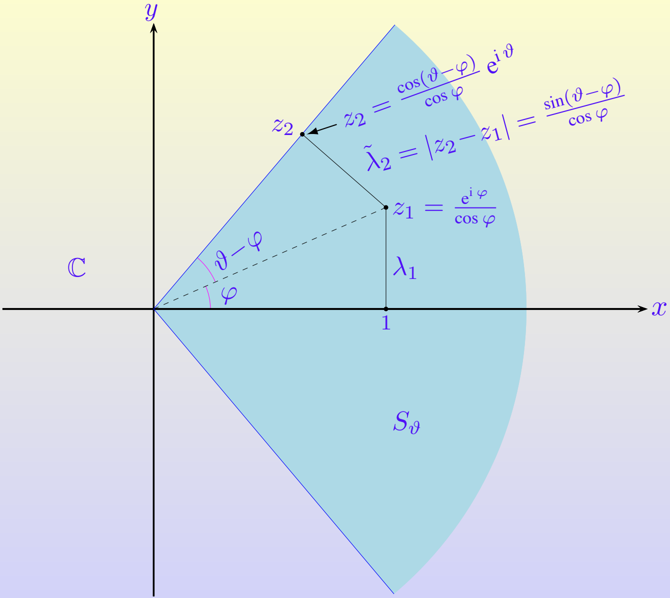
$$\forall t \in [0, T] \ \forall v \in H \quad |\mathcal{A}_a(t)v| \leq \lambda_1(t)|v|$$

with $\lambda_1(t) = \tan \varphi(t)$.
Then, the sufficient stability takes the form

$$\boxed{(\cos \vartheta) \tan \varphi(t) + \lambda_2(t) < \sin \vartheta}$$

which can also be written as

$$\boxed{\lambda_2(t) < \frac{\sin(\vartheta - \varphi(t))}{\cos \varphi(t)}}$$


y
$z_2 = \frac{\cos(\vartheta-\varphi)}{\cos\varphi}\,\mathrm{e}^{\mathrm{i}\,\vartheta}$
z_2
$\tilde\lambda_2 = |z_2 - z_1| = \frac{\sin(\vartheta-\varphi)}{\cos\varphi}$
$z_1 = \frac{\mathrm{e}^{\mathrm{i}\,\varphi}}{\cos\varphi}$
$\vartheta-\varphi$
$\mathbb{C}$
λ_1
φ
x
1
S_ϑ

3. Let

$$\|A(t)v\|_\star \le \nu(t)\|v\| \quad \forall v \in V$$

and

$$\operatorname{Re}(A(t)v, v) \ge \kappa(t)\|v\|^2 \quad \forall v \in V.$$

Then

$$|\mathcal{A}(t)| \le \frac{\nu(t)}{\kappa(t)}.$$

We may assume that $\tilde{\lambda}_1(t) \le \frac{\nu(t)}{\kappa(t)}$ and the stability condition reads

$$\boxed{(\cos\vartheta)\sqrt{\frac{\nu(t)^2}{\kappa(t)^2} - 1} + \lambda_2(t) < \sin\vartheta}$$

Comparison to the energy technique

For $q \in \{1, \dots, 5\}$, stability results for BDF schemes have also been established via energy techniques under the sufficient stability condition

$$\boxed{\hat{\eta}_q \tilde{\lambda}_1(t) + (1 + \hat{\eta}_q)\lambda_2(t) < 1 \quad \forall t \in [0, T]}$$

with

$$\hat{\eta}_1 = \hat{\eta}_2 = 0, \quad \hat{\eta}_3 = 1/13 = 0.07692, \quad \hat{\eta}_4 = 0.2878, \quad \hat{\eta}_5 = 0.80973.$$

For $q = 3, 4, 5$, since $\hat{\eta}_q > \cos\vartheta_q$ and $1 + \hat{\eta}_q > 1/\sin\vartheta_q$, this is not a best possible linear stability condition.

- Nevanlinna, Odeh: Numer. Funct. Anal. Optim. (1981)
- Lubich, Mansour, Venkataraman: IMA J. Numer. Anal. (2013)
- A., Lubich: Numer. Math. (2015)
- A.: SINUM (2015)
- A., Katsoprinakis: Math. Comp. (2016)

4. Implicit–explicit multistep schemes

- (α, β) an implicit q-step scheme

  (α, γ) an explicit q-step scheme,

$$\alpha(\zeta) = \sum_{i=0}^{q} \alpha_i \zeta^i, \quad \beta(\zeta) = \sum_{i=0}^{q} \beta_i \zeta^i, \quad \gamma(\zeta) = \sum_{i=0}^{q-1} \gamma_i \zeta^i.$$

- Let $N \in \mathbb{N}$, $k := T/N$, and $t^n := nk$, $n = 0, \dots, N$.
- Let $U^0, \dots, U^{q-1} \in V$ be given starting approximations.
- Define approximations U^m to $u^m := u(t^m), m = q, \dots, N$, by

$$\boxed{\sum_{i=0}^{q} \big[\alpha_i I + k\beta_i A(t^{n+i})\big] U^{n+i} = k \sum_{i=0}^{q-1} \gamma_i B(t^{n+i}, U^{n+i}).}$$

- M. Crouzeix: Numer. Math. (1980)

The stability result

Let

$$K_{(\alpha,\beta,\gamma)} := \sup_{x>0} \max_{\zeta\in\mathcal{K}} \frac{|x\gamma(\zeta)|}{|\alpha(\zeta)+x\beta(\zeta)|}.$$

Example: Implicit–explicit BDF methods

$$\alpha(\zeta)=\sum_{j=1}^{q}\frac{1}{j}\zeta^{q-j}(\zeta-1)^j,\quad \beta(\zeta)=\zeta^q,\quad \gamma(\zeta)=\zeta^q-(\zeta-1)^q.$$

(α,γ) the unique explicit q-step scheme of order q.

In this case $\boxed{K_{(\alpha,\beta,\gamma)}=|\gamma(-1)|=2^q-1}$ [5]

Let $V^m\in T_u$ satisfy the perturbed equations

$$\sum_{i=0}^{q}\big[\alpha_i I+k\beta_i A(t^{n+i})\big]V^{n+i}=k\sum_{i=0}^{q-1}\gamma_i B(t^{n+i},V^{n+i})+kE^n.$$

[5] A., Crouzeix, Makridakis: Numer. Math. (1999)

Theorem

Let $\vartheta^m := V^m - U^m$. *If*

$$\boxed{(\cot\vartheta)\lambda_1(t) + K_{(\alpha,\beta,\gamma)}\lambda_2(t) < 1 \quad \forall t \in [0,T],}$$

then we have the stability estimate

$$|\vartheta^n|^2 + k\sum_{\ell=q}^{n}\|\vartheta^\ell\|^2 \le C\sum_{j=0}^{q-1}\big(|\vartheta^j|^2 + k\|\vartheta^j\|^2\big) + Ck\sum_{\ell=0}^{n-q}\|E^\ell\|_\star^2,$$

$n = q,\dots,N$, *with a constant* C *independent of* k, n, U^m, V^m *and* E^m.

In this case the stability condition is:

- Best possible linear sufficient stability condition.
- Sharp if the implicit scheme is A-stable.

5. Key ingredients in the stability analysis

1. Stability technique: Combination of spectral and Fourier techniques

2. Advantageous decomposition of the linear operator
   Rewrite $u'(t) + A(t)u(t) = B(t, u(t))$ in the form

   $$u'(t) + \widehat{A}_s(t)u(t) + \widetilde{A}(t)u(t) = B(t, u(t))$$

   with

   $$\widehat{A}_s(t) := (1 + \eta)A_s(t), \quad \widetilde{A}(t) := A_a(t) - \eta A_s(t),$$

   with η a nonnegative quantity that may depend on $\lambda_1(t)$ and $\lambda_2(t)$.

3. Time independent operators
   Choose $\eta := (\tan \vartheta)\lambda_1$ and apply a known stability result.

4. Time dependent operators
   Freeze the time, use the previous stability estimate and employ a discrete perturbation argument.

$$
\begin{cases}
\dfrac{1}{\alpha(\zeta)+x\beta(\zeta)} = \displaystyle\sum_{\ell=q}^{\infty} e(\ell,x)\,\zeta^{-\ell}, & |\zeta|\ge 1,\\[2ex]
\dfrac{\beta(\zeta)}{\alpha(\zeta)+x\beta(\zeta)} = \displaystyle\sum_{\ell=0}^{\infty} f(\ell,x)\,\zeta^{-\ell}, & |\zeta|\ge 1.
\end{cases}
$$

Now, with $b^\ell := B(V^\ell) - B(U^\ell)$, let

$$
\vartheta_i^n := \begin{cases}
-k\displaystyle\sum_{\ell=0}^{n} f(n-\ell,kA_s)A_a\vartheta^\ell, & i=1,\\[2ex]
k\displaystyle\sum_{\ell=0}^{n} f(n-\ell,kA_s)b^\ell, & i=2,\\[2ex]
k\displaystyle\sum_{\ell=0}^{n-q} e(n-\ell,kA_s)E^\ell, & i=3,
\end{cases}
$$

and

$$\vartheta_4^n := \vartheta^n - \vartheta_1^n - \vartheta_2^n - \vartheta_3^n,$$

$n = 0, \dots, N.$
Then, we have

$$\sum_{i=0}^{q} (\alpha_i I + k\beta_i A_s)\vartheta_2^{n+i} = k \sum_{i=0}^{q} \beta_i b^{n+i}, \quad n = 0, \dots, N - q.$$

Claim:

$$k \sum_{\ell=0}^{n} \|\theta_2^\ell\|^2 \le K_{(\alpha,\beta)}^2 \; k \sum_{\ell=0}^{n} \|b^\ell\|_\star^2, \quad n = 0, \dots, N.$$

It suffices to show this estimate for $b^\ell = 0$ for $\ell \ge n$, and n replaced by ∞.

We introduce $\hat{b}$ and $\hat{\theta}_2$ by

$$\boxed{\hat{b}(t) = \sum_{\ell=0}^{\infty} b^\ell \, \mathrm{e}^{2\,\mathrm{i}\,\pi \ell t}, \quad \hat{\theta}_2(t) = \sum_{\ell=0}^{\infty} \theta_2^\ell \, \mathrm{e}^{2\,\mathrm{i}\,\pi \ell t}\,.}$$

From the definition of θ_2, we deduce

$$\hat{\theta}_2(t) = k\beta(\mathrm{e}^{-2\,\mathrm{i}\,\pi\ell t})\big\{\alpha(\mathrm{e}^{-2\,\mathrm{i}\,\pi\ell t})I + \beta(\mathrm{e}^{-2\,\mathrm{i}\,\pi\ell t})kA_s\big\}^{-1}\hat{b}(t).$$

Therefore, $\|\hat{\theta}_2(t)\| \le K_{(\alpha,\beta)}\|\hat{b}(t)\|_\star$, whence, using Parseval's identity,

$$\sum_{\ell=0}^{\infty} \|\theta_2^\ell\|^2 = \int_0^1 \|\hat{\theta}_2(t)\|^2 \; dt \le K_{(\alpha,\beta)}^2 \int_0^1 \|\hat{b}(t)\|_\star^2 \; dt = K_{(\alpha,\beta)}^2 \; \sum_{\ell=0}^{\infty} \|b^\ell\|_\star^2.$$

6. An example

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\Omega$, and consider the following initial and boundary value problem

$$\begin{cases} u_t - \sum_{i,j=1}^{d} \big((a_{ij}(x,t) + \tilde{a}_{ij}(x,t))u_{x_j}\big)_{x_i} = B(t,u) & \text{in} \quad \Omega \times [0,T], \\ u = 0 & \text{on} \quad \partial\Omega \times [0,T], \\ u(\cdot,0) = u^0 & \text{in} \quad \Omega, \end{cases}$$

with T positive and $u^0 : \Omega \to \mathbb{C}$ a given initial value. Here, $\mathcal{A}, \widetilde{\mathcal{A}} : \Omega \times [0,T] \to \mathbb{C}^{d,d}$ are uniformly positive definite and Hermitian, and anti-Hermitian matrices, respectively, with smooth entries $a_{ij}(x,t)$ and $\tilde{a}_{ij}(x,t)$, respectively, and $B(t,\cdot)$ are suitable, possibly nonlinear, operators.

Consider the antihermitian matrices

$$S(x,t) := \mathcal{A}^{-1/2}(x,t)\widetilde{\mathcal{A}}(x,t)\mathcal{A}^{-1/2}(x,t).$$

The boundedness condition is then satisfied with

$$\lambda_1(t) := \max_{x\in\overline{\Omega}} \rho\big(S(x,t)\big), \quad t \in [0,T],$$

with $\rho(\cdot)$ the spectral radius.

Two special cases

First case: Let

$$\widetilde{\mathcal{A}}(x,t) = \mathrm{i}\, a(x,t)\mathcal{A}(x,t), \quad x \in \Omega, \ \ 0 \le t \le T,$$

with a a smooth real-valued function, $a : \overline{\Omega} \times [0,T] \to \mathbb{R}.$ Then, $S(x,t) = \mathrm{i}\, a(x,t) I_d,$ whence

$$\lambda_1(t) = \max_{x \in \overline{\Omega}} |a(x,t)| \quad \forall t \in [0,T].$$

Second case: $d=2.$

It is well known that

$$\mathcal{A}^{1/2}=\frac{1}{\sqrt{\operatorname{tr}\mathcal{A}+2\sqrt{\det\mathcal{A}}}}\big(\mathcal{A}+\sqrt{\det\mathcal{A}}\,I_2\big),$$

and

$$\mathcal{A}^{-1/2}=\frac{1}{\sqrt{\det\mathcal{A}}\,\sqrt{\operatorname{tr}\mathcal{A}+2\sqrt{\det\mathcal{A}}}}\Big(\big(\operatorname{tr}\mathcal{A}+\sqrt{\det\mathcal{A}}\,\big)I_2-\mathcal{A}\Big),$$

with $\operatorname{tr}\mathcal{A}:=a_{11}+a_{22}$ the trace of $\mathcal{A}$.
Therefore,

$$\boxed{S=\frac{1}{\det\mathcal{A}\,\big(\operatorname{tr}\mathcal{A}+2\sqrt{\det\mathcal{A}}\,\big)}\big(c_{\mathcal{A}}^2\widetilde{\mathcal{A}}-c_{\mathcal{A}}(\mathcal{A}\widetilde{\mathcal{A}}+\widetilde{\mathcal{A}}\mathcal{A})+\mathcal{A}\widetilde{\mathcal{A}}\mathcal{A}\big),}$$

with the constant $c_{\mathcal{A}}:=\operatorname{tr}\mathcal{A}+\sqrt{\det\mathcal{A}}.$

Thank you very much!