Approximation of Time Domain Boundary Integral Equations

Dugald Duncan

Heriot-Watt University, Edinburgh

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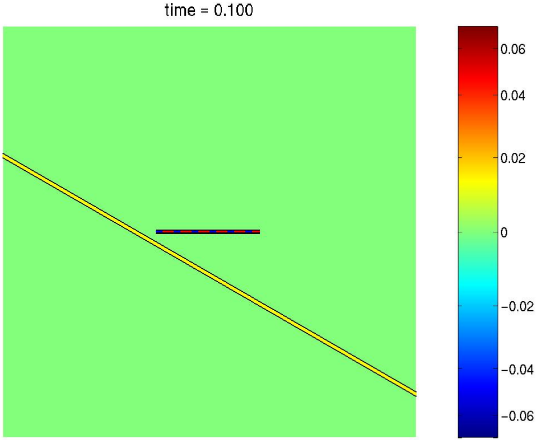
Joint work with Penny Davies (Strathclyde)

Motivation: scattering



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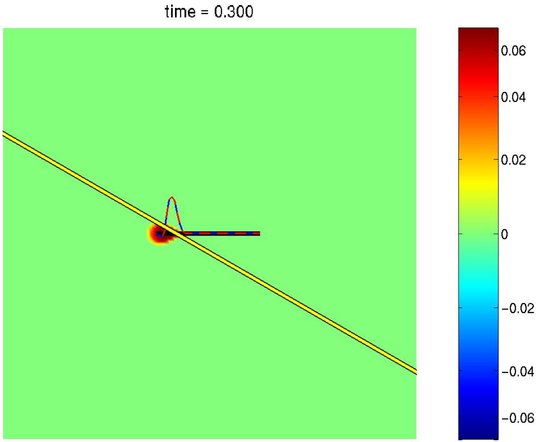

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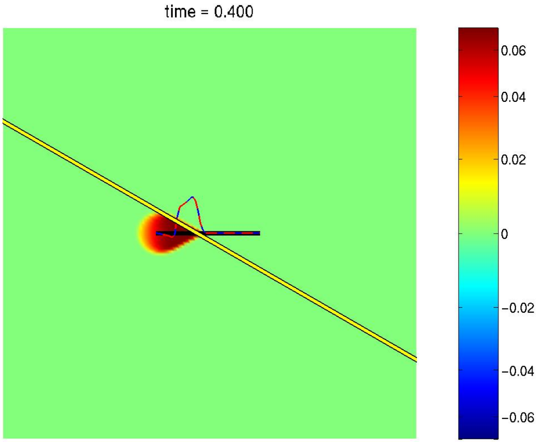

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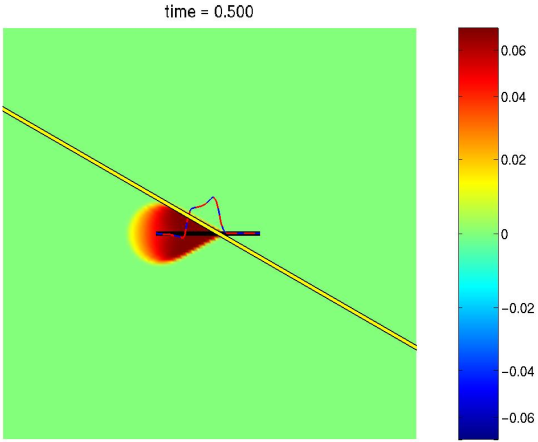

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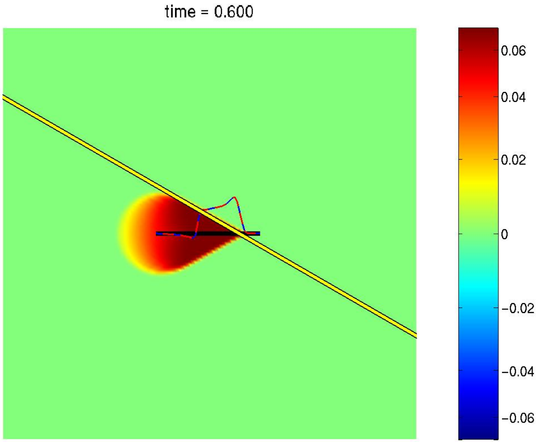

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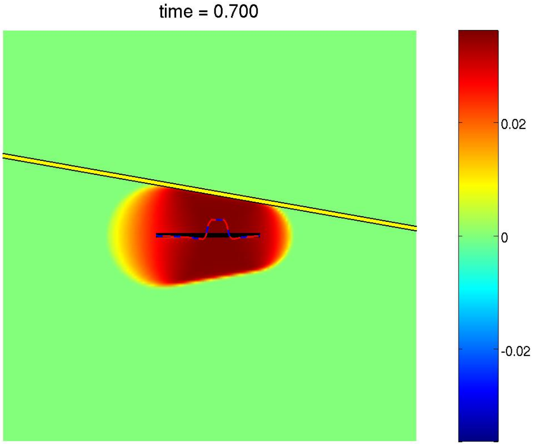

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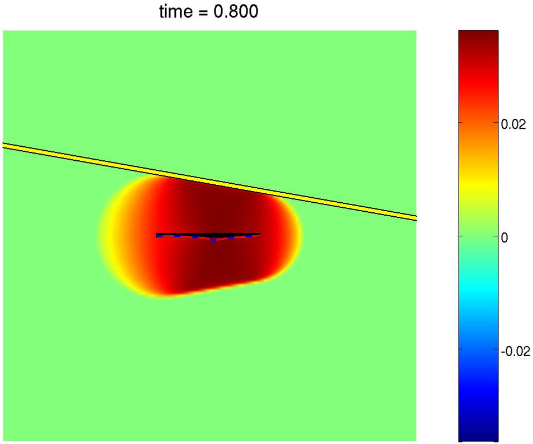

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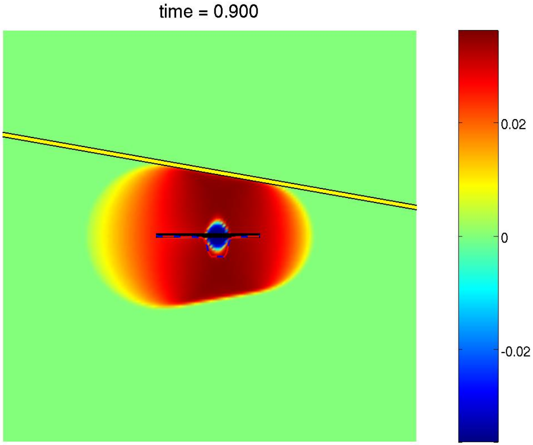

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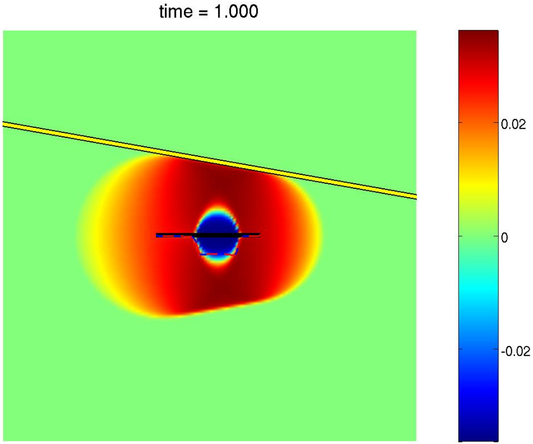

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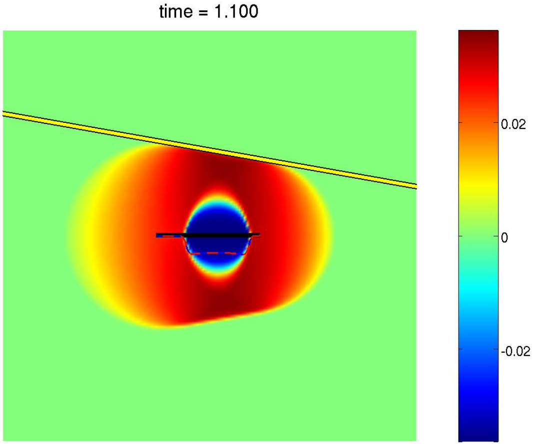

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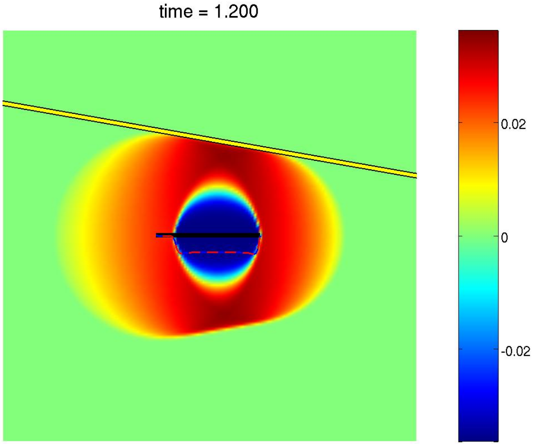

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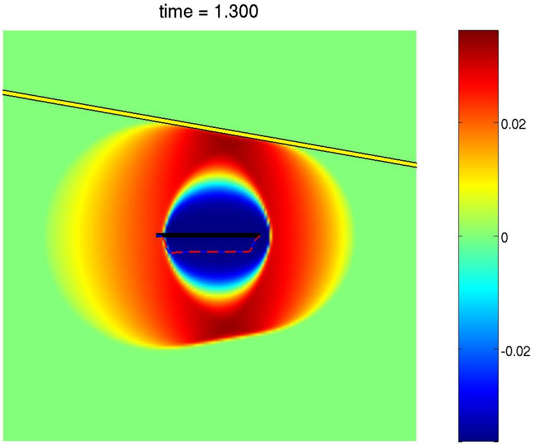

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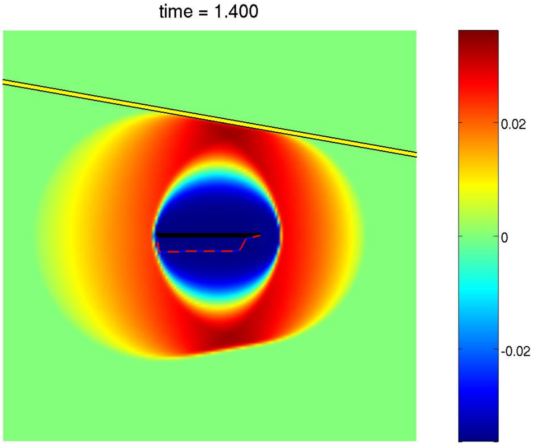

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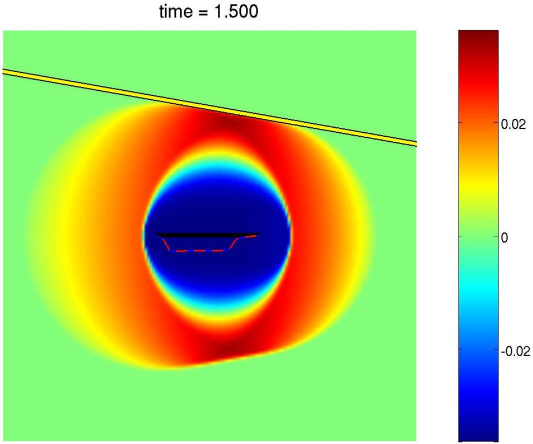

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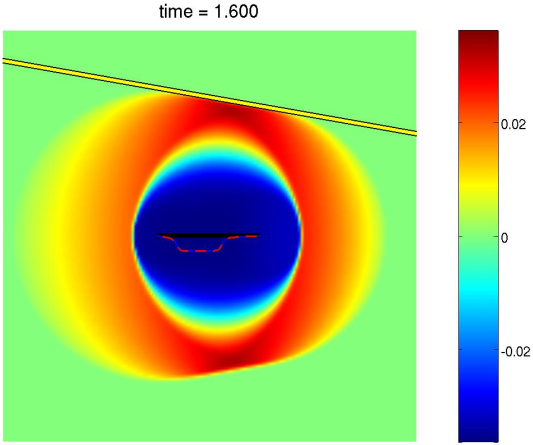

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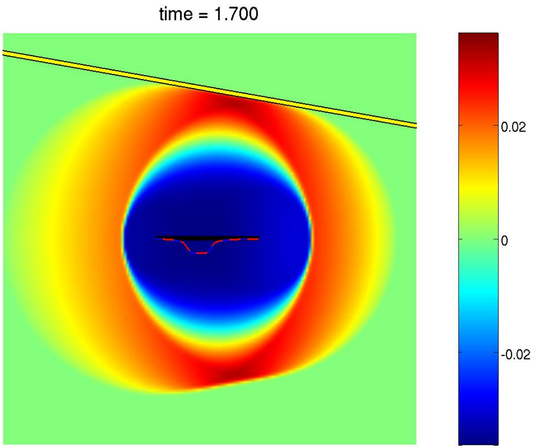

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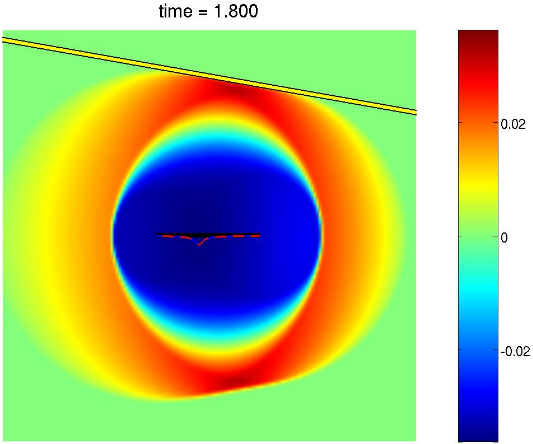

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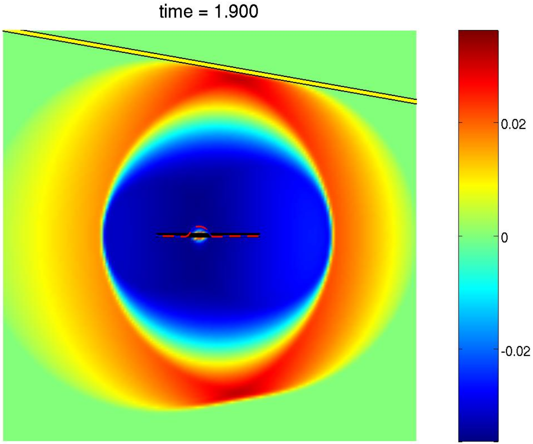

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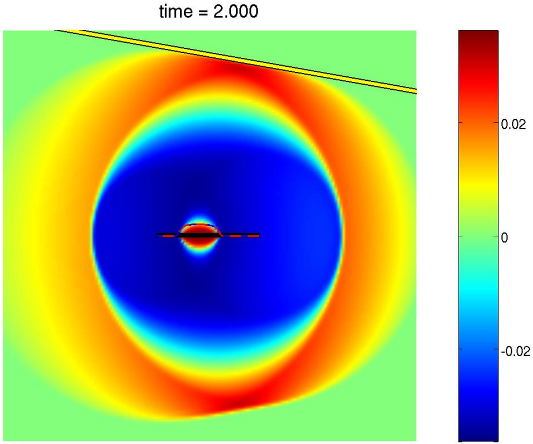

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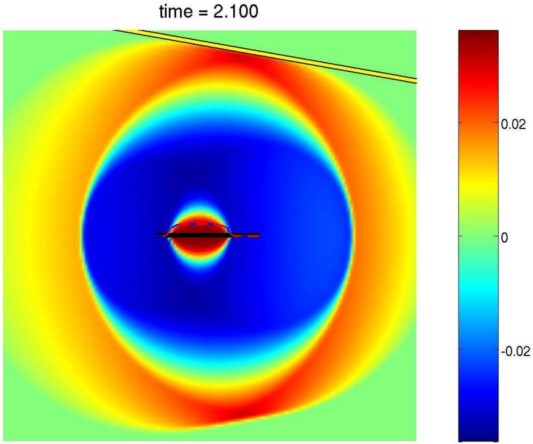

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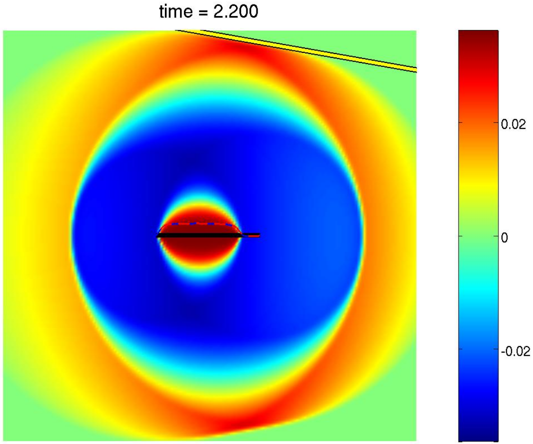

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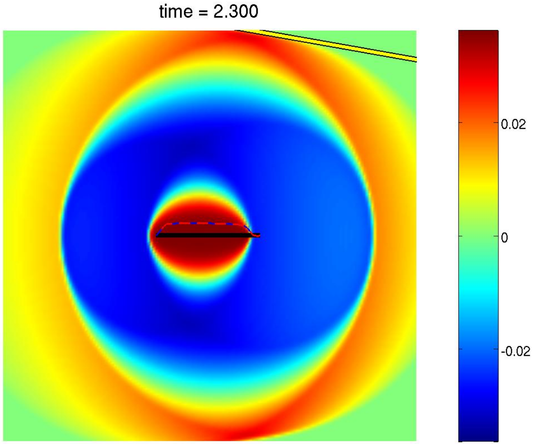

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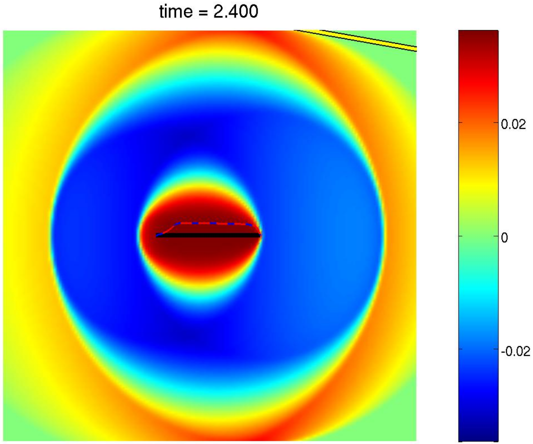

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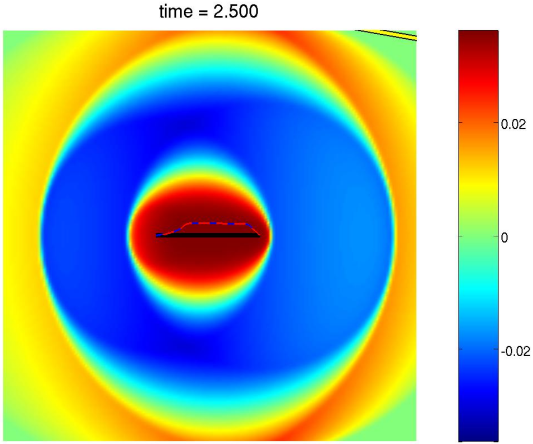

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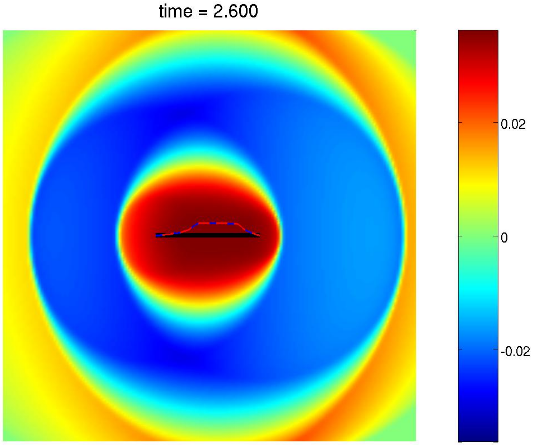

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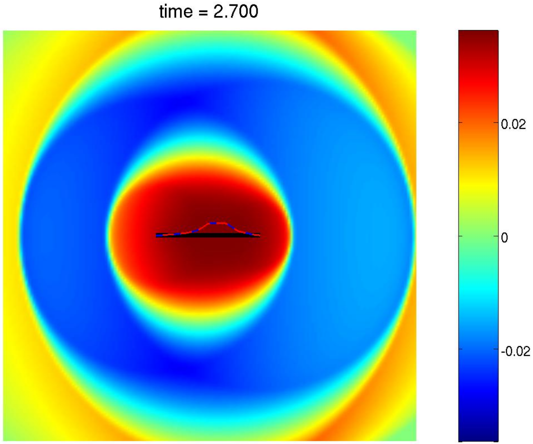

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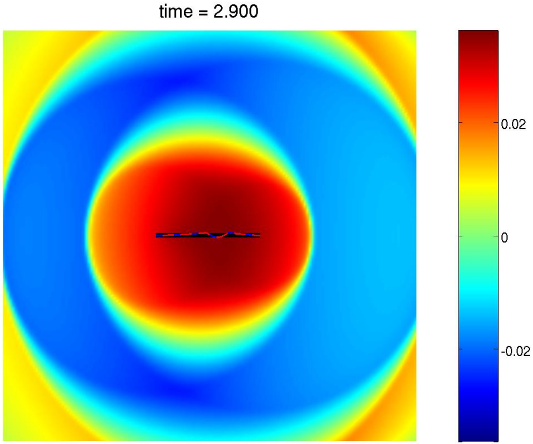

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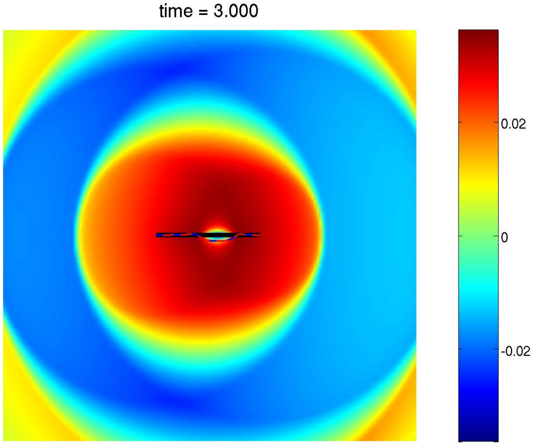

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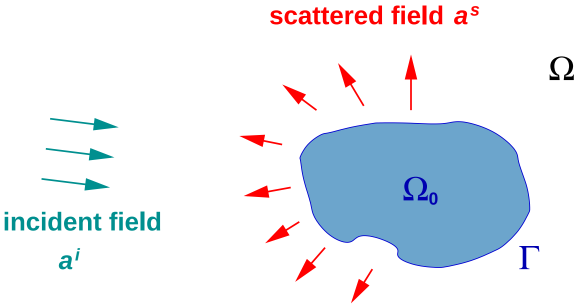
Outline

- Define the time domain boundary integral equation (TDBIE) acoustic scattering problem
- Summarise methods and costs
- Galerkin variational formulations of TDBIE
- Drop space, and concentrate on time stepping illustrated by Volterra integral equation
- Connections to backwards-in-time collocation
- Results

Motivation: acoustic scattering

Problem: $\boldsymbol{a^i(x,t)}$ is incident on Γ for $t > 0$ – find the scattered field $\boldsymbol{a^s(x,t)}$



- PDE: $\boldsymbol{a^s_{tt} = \Delta a^s}$ in Ω (wave speed is $c = 1$);
- BC: $\boldsymbol{a^s} + \boldsymbol{a^i} = 0$ on Γ
- TDBIE: $\boldsymbol{a^s}$ can be obtained from surface potential u:

$$\frac{1}{4\pi}\int_\Gamma \frac{u(\boldsymbol{x}', t-|\boldsymbol{x}'-\boldsymbol{x}|)}{|\boldsymbol{x}'-\boldsymbol{x}|}\, d\sigma_{\boldsymbol{x}'} = -\boldsymbol{a^i(x,t)} \quad \boldsymbol{x}\in\Gamma,\ t>0$$

Motivation: acoustic scattering

Problem: $\boldsymbol{a^i(x,t)}$ is incident on Γ for $t > 0$ – find the scattered field $\boldsymbol{a^s(x,t)}$

- Solve TDBIE for surface potential u:

$$\frac{1}{4\pi}\int_\Gamma \frac{u(\mathbf{x}', t-|\mathbf{x}'-\mathbf{x}|)}{|\mathbf{x}'-\mathbf{x}|}\, d\sigma_{\mathbf{x}'} = -\boldsymbol{a^i(x,t)} \quad \mathbf{x}\in\Gamma,\ t>0$$

- Use surface potential u to compute (in the exterior):

$$\boldsymbol{a^s(x,t)} = \frac{1}{4\pi}\int_\Gamma \frac{u(\mathbf{x}', t-|\mathbf{x}'-\mathbf{x}|)}{|\mathbf{x}'-\mathbf{x}|}\, d\sigma_{\mathbf{x}'} \quad \mathbf{x}\in\Omega,\ t>0$$

- Both steps easier said than done!
- Gives all frequencies simultaneously by Fourier transform in time of $\boldsymbol{a^s(x,t)}$ - multiscale!

Approximate solution methods for TDBIE

Find u given a^i from

$$(Su)(\boldsymbol{x}, t) := \frac{1}{4\pi} \int_\Gamma \frac{u(\boldsymbol{x}', t - |\boldsymbol{x}' - \boldsymbol{x}|)}{|\boldsymbol{x}' - \boldsymbol{x}|} \, d\sigma_{\boldsymbol{x}'} = -\boldsymbol{a^i(x, t)} \quad \boldsymbol{x} \in \Gamma,\ t > 0$$

- **Convolution Quadrature** in time (based on Laplace transform techniques) and coupled with Galerkin in space. Needs a talk by itself! Lübich and then many subsequent papers, including by Banjai on a version based on RK methods, as well as a proper fast method.
- **Full space-time Galerkin**. Bamberger and Ha Duong. Full version has theoretical backing. A simplified version is usually used and usually works, but lacks theory to back it up. Space mesh adaptation recently by Gimperlein and Stark.
- **Collocation** in space and time - usually fails.
- **Collocation** in time with Galerkin in space - can work (EM example).
- **Backwards-in-time collocation** with Galerkin in space - usually works, no theory.

Approximate solution methods for TDBIE

Find u given $f = -\boldsymbol{a}^{\boldsymbol{i}}$ (switch notation from now on) from

$$(Su)(\boldsymbol{x},t) := \frac{1}{4\pi}\int_\Gamma \frac{u(\boldsymbol{x}', t-|\boldsymbol{x}'-\boldsymbol{x}|)}{|\boldsymbol{x}'-\boldsymbol{x}|}\, d\sigma_{\boldsymbol{x}'} = f(\boldsymbol{x},t) \quad \boldsymbol{x}\in\Gamma,\ t>0$$

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Computational costs for TDBIE approximation – surface in 3D space

- Time step and space mesh size about the same – $\mathcal{O}(1/N)$
- Surface area of scatterer – $\mathcal{O}(N_S)$ elements
- Number of time steps – $\mathcal{O}(N_T)$
- Explicit time stepping (marching on in time) schemes with local time basis functions – cost $\mathcal{O}(N_T N_S^2) = \mathcal{O}(N^5)$
- Space-time Galerkin schemes with local time basis functions – cost $\mathcal{O}(N^5)\times$ number of iterations to solve linear systems
- Explicit time stepping (marching on in time) schemes with global time basis functions – cost $\mathcal{O}(N_T^{3/2} N_S^2) = \mathcal{O}(N^{11/2})$
- Fast methods (Banjai for acoustics, Michiellsen for EM) can reduce the $\mathcal{O}(N^5)$ costs.
- Compare with PDE in 3D scattering domain – $C\ N^4$ where C is a big constant depending on the size of the domain.

Energy in scattered field

$$(Su)(\boldsymbol{x},t) := \frac{1}{4\pi}\int_\Gamma \frac{u(\boldsymbol{x}',t-|\boldsymbol{x}'-\boldsymbol{x}|)}{|\boldsymbol{x}'-\boldsymbol{x}|}\,d\sigma_{\boldsymbol{x}'} = f(\boldsymbol{x},t) \quad \boldsymbol{x}\in\Gamma,\ t>0$$

- Scattered field energy can be calculated from the surface potential u

$$E(u;t) = \int_0^t\int_\Gamma u(\boldsymbol{x},\tau)(S\dot{u})(\boldsymbol{x},\tau)d\sigma_{\boldsymbol{x}}\,d\tau \geq 0$$

- Ha Duong's results concern its **time integral** and give a coercivity and stability result:

$$\alpha\|u\|^2_{\mathcal{H}^-} \leq \int_0^T E(u;t)dt \leq \beta\|u\|_{\mathcal{H}^-}\|(T-t)\dot{f}\|_{\mathcal{H}^+} \quad \Rightarrow \quad \|u\|_{\mathcal{H}^-} \leq \frac{\beta}{\alpha}\|(T-t)\dot{f}\|_{\mathcal{H}^+}$$

- Note: basic calculus gives:

$$\int_0^T E(u;t)\,dt = \int_0^T \boldsymbol{(T-t)} \int_\Gamma u(\boldsymbol{x},t)\,(S\dot{u})(\boldsymbol{x},t)\,d\sigma_{\boldsymbol{x}}\,dt$$

- Ha Duong uses $\mathcal{H}^+ = H^{1/2,1/2}_{00}$ (Lions & Magenes) in PhD thesis, and $\mathcal{H}^-$ is its dual.

Galerkin variational formulation

- Approx solution in terms of unknowns U_k^n:

$$u(\boldsymbol{x},t) \approx u_h(\boldsymbol{x},t) := \sum_{n=1}^{N_T} \sum_{k=1}^{N_S} U_k^n \, \psi_k(\boldsymbol{x}) \, \phi_n(t) \in V_h, \qquad u(\boldsymbol{x},0) = u_h(\boldsymbol{x},0) = 0$$

- The energy expressions suggest using the **time differentiated TDBIE**

$$S\dot{u} = \dot{f} \quad \textbf{not} \quad Su = f,$$

and one or other of

$$\text{Find } u_h \in V_h \text{ s.t.} \quad \int_0^T \int_\Gamma q_h \, S\dot{u}_h \, d\sigma_{\boldsymbol{x}} \, dt = \int_0^T \int_\Gamma q_h \, \dot{f} \, d\sigma_{\boldsymbol{x}} \, dt \quad \forall q_h \in V_h$$

$$\text{Find } u_h \in V_h \text{ s.t.} \quad \int_0^T (T-t) \int_\Gamma q_h \, S\dot{u}_h \, d\sigma_{\boldsymbol{x}} \, dt = \int_0^T (T-t) \int_\Gamma q_h \, \dot{f} \, d\sigma_{\boldsymbol{x}} \, dt \quad \forall q_h \in V_h$$

Galerkin variational formulation - (lack of) theory

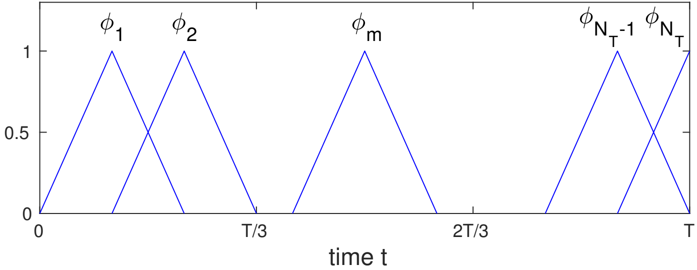
- No theory for the standard Galerkin formulation – no coercivity to work with.
  Find u_h such that
  $$\int_0^T \int_\Gamma q_h \, S \dot{u}_h \, d\sigma_{\boldsymbol{x}} \, dt = \int_0^T \int_\Gamma q_h \, \dot{f} \, d\sigma_{\boldsymbol{x}} \, dt$$
  for each $q_h = \psi_j(\boldsymbol{x}) \, \phi_m(t) \in V_h$.
- But, on finite time intervals Ha Duong proves stability results about the following.
  Find u_h such that
  $$\int_0^T (T - t) \int_\Gamma q_h \, S \dot{u}_h \, d\sigma_{\boldsymbol{x}} \, dt = \int_0^T (T - t) \int_\Gamma q_h \, \dot{f} \, d\sigma_{\boldsymbol{x}} \, dt$$
  for each $q_h = \psi_j(\boldsymbol{x}) \, \phi_m(t) \in V_h$.

We will return to this later.

Galerkin is **not** usually a time-marching scheme

It is when ϕ_m are **piecewise constants** in time, but not in general.

Example: $\phi_m(t) = B_1(t/h - m)$ – 1st order B-splines (hat functions)



- N_T time basis functions.
- $\phi_0(t)$ is not needed since solution $u(\boldsymbol{x}, 0) = 0$.
- $\phi_{N_T}(t)$ is not a “complete” basis function. Time integral is $\int_0^T \cdots dt$.

Galerkin is **not** usually a time-marching scheme

- **Example:** $\phi_m(t) = B_1(t/h - m)$ – 1st order B-splines (hat functions)
- Resulting linear system for the $\boldsymbol{U}^n \in \mathbb{R}^{N_S}$ (N_S space degrees of freedom):

$$\boldsymbol{U}^0 = 0\,, \quad Q^\star\, \boldsymbol{U}^{n+1} + \sum_{m=0}^{n} Q^m\, \boldsymbol{U}^{n-m} = \boldsymbol{f}^n\,, \quad n = 1 : N_T - 1$$

$$\sum_{m=0}^{N_T} P^m\, \boldsymbol{U}^{N_T - m} = \boldsymbol{f}^{N_T}\,, (n = N_T) \text{ from "incomplete" } \phi_{N_T}$$

When $N_T = 4$ (P, Q are sparse block $N_S \times N_S$ matrices):

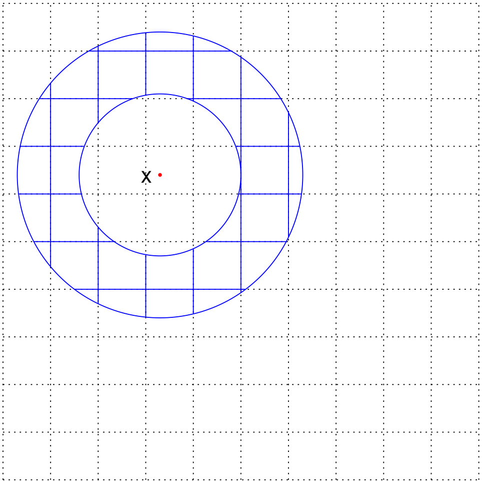
$$\begin{pmatrix} Q^0 & Q^* & 0 & 0 \\ Q^1 & Q^0 & Q^* & 0 \\ Q^2 & Q^1 & Q^0 & Q^* \\ P^3 & P^2 & P^1 & P^0 \end{pmatrix} \begin{pmatrix} \boldsymbol{U}^1 \\ \boldsymbol{U}^2 \\ \boldsymbol{U}^3 \\ \boldsymbol{U}^4 \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}^1 \\ \boldsymbol{f}^2 \\ \boldsymbol{f}^3 \\ \boldsymbol{f}^4 \end{pmatrix}$$

Galerkin matrix assembly hard

- Fix $\boldsymbol{x}$ and t and evaluate

$$\int_\Gamma \frac{\psi_j(\boldsymbol{x}')\dot{\phi}_n(t-|\boldsymbol{x}'-\boldsymbol{x}|)}{|\boldsymbol{x}'-\boldsymbol{x}|}\, d\sigma_{\boldsymbol{x}'}$$

  for each j where it is non-zero.
- Inner/outer circles show $\sup \phi_n(t-|\boldsymbol{x}'-\boldsymbol{x}|)$.
- Intersections of (square) space mesh elements $\sup \phi_n$ are complicated.
- Now multiply by $\phi_m(t)\psi_k(\boldsymbol{x})$ and evaluate $\int_0^T \int_\Gamma \cdots d\sigma_{\boldsymbol{x}} dt$.
- 5D integrals with weird shapes.
- Maischak (Brunel) developed quadrature code.



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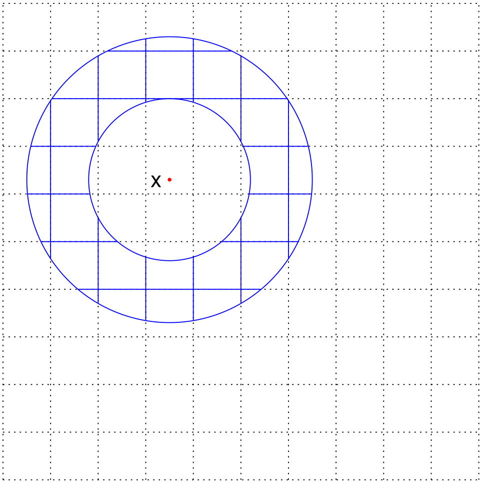

Galerkin matrix assembly hard

- Fix $\boldsymbol{x}$ and t and evaluate

$$\int_\Gamma \frac{\psi_j(\boldsymbol{x}')\dot{\phi}_n(t-|\boldsymbol{x}'-\boldsymbol{x}|)}{|\boldsymbol{x}'-\boldsymbol{x}|}\, d\sigma_{\boldsymbol{x}'}$$

  for each j where it is non-zero.
- Inner/outer circles show $\sup \phi_n(t - |\boldsymbol{x}' - \boldsymbol{x}|)$.
- Intersections of (square) space mesh elements $\sup \phi_n$ are complicated.
- Now multiply by $\phi_m(t)\psi_k(\boldsymbol{x})$ and evaluate $\int_0^T \int_\Gamma \cdots d\sigma_{\boldsymbol{x}}\, dt$.
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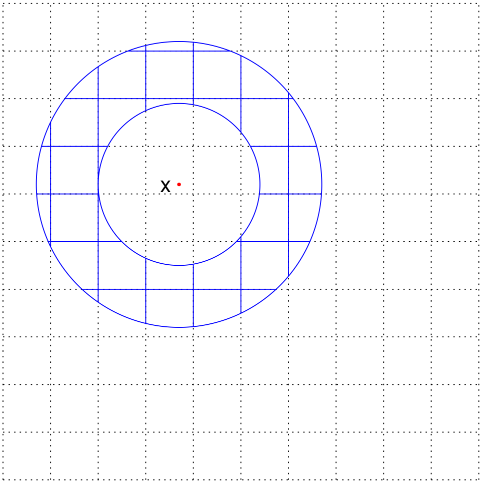

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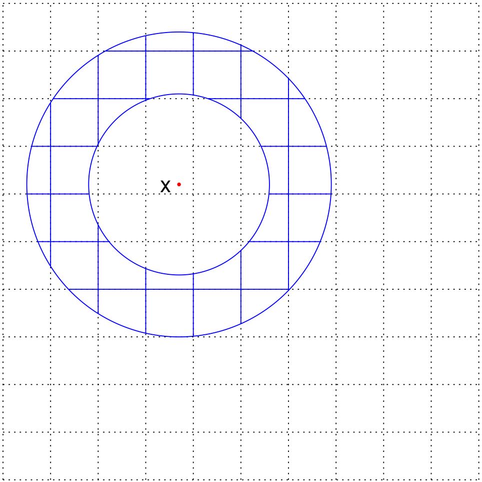

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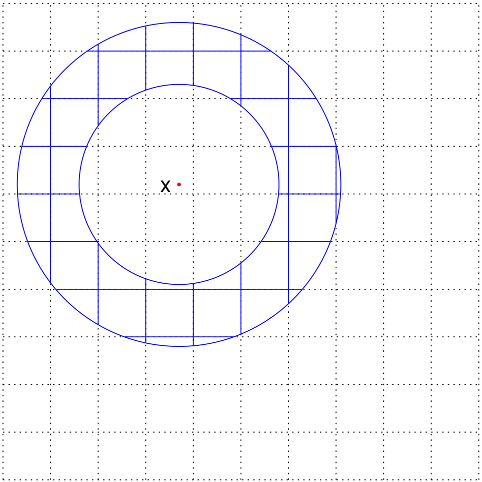

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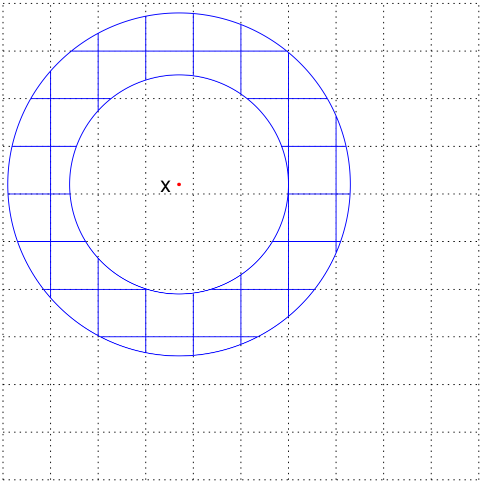


Summary of Galerkin for TDBIE

- No theory unless Ha Duong's more complicated variational form used.
- Matrix assembly hard because of complicated 5D integral regions.
- Does not produce a marching on in time (MOT) scheme - more like a 2 point BVP in time.

Summary of Galerkin for TDBIE

- No theory unless Ha Duong's more complicated variational form used.
  So let's use it.
- Matrix assembly hard because of complicated 5D integral regions.
  Use time basis functions that are globally smooth enough extended by 0 to do simple quadrature based on the space elements.
- Does not produce a marching on in time (MOT) scheme - more like a 2 point BVP in time.
  Modify variational formulation to keep theoretical properties and to produce a MOT scheme.

Illustrate the time-stepping parts using 1st kind Volterra integral equations.

TDBIE connection with 1st kind Volterra integral equations

If Γ is an infinite flat plane, separation of variables in

$$\frac{1}{4\pi} \int_\Gamma \frac{u(\boldsymbol{x}', t - |\boldsymbol{x}' - \boldsymbol{x}|)}{|\boldsymbol{x}' - \boldsymbol{x}|}\, d\sigma_{\boldsymbol{x}'} = f(\boldsymbol{x}, t) \quad \boldsymbol{x} \in \Gamma,\ t > 0$$

gives

$$\int_0^t J_0(|\boldsymbol{\omega}|\tau)\hat{u}(\boldsymbol{\omega}, t - \tau) d\tau = \hat{f}(\boldsymbol{\omega}, t)$$

where $\hat{u}, \hat{f}$ are Fourier transforms of u, f in space over the 2D plane with frequency vector $\boldsymbol{\omega}$.

$J_0 =$ 1st kind Bessel function of order 0.

TDBIE connection with 1st kind Volterra integral equations

If Γ is a sphere surface, separation of variables into spherical harmonics in

$$\frac{1}{4\pi}\int_\Gamma \frac{u(\boldsymbol{x}', t - |\boldsymbol{x}' - \boldsymbol{x}|)}{|\boldsymbol{x}' - \boldsymbol{x}|}\, d\sigma_{\boldsymbol{x}'} = f(\boldsymbol{x}, t) \quad \boldsymbol{x} \in \Gamma,\ t > 0$$

gives step-kernel VIE problem for each $u_{\ell,m}$:

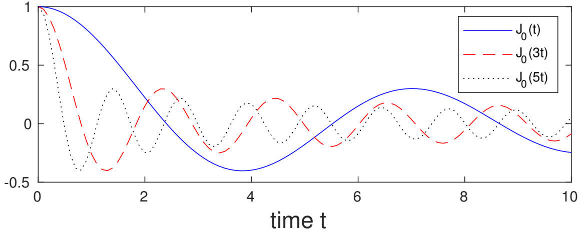
$$\int_0^t K_\ell(\tau)\, u_{\ell,m}(t - \tau) = 2 f_{\ell,m}(1, t)\,, \quad K_\ell(t) = \begin{cases} P_\ell(1 - t^2/2), & t \le 2 \\ 0, & t > 2 \end{cases}$$

for the unit sphere. Note that it takes 2 time units to travel the diameter of sphere.

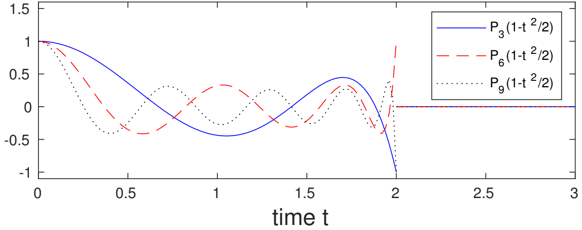
P_ℓ is Legendre polynomial and the indices ℓ, m refer to the order of the spherical harmonics.

VIE kernels

Flat plate Bessel $J_0(|\omega|t)$ kernel:



Sphere surface Legendre $P_\ell(1-t^2/2)$ kernel:



A model problem for time discretistaion

- Use convolution Volterra integral equation VIE (K, f given, find u)

$$\int_0^t K(\tau)\, u(t-\tau)d\tau = f(t), \quad t > 0$$

  as a model to illustrate time discretisation.
- Causal – solution $u(t)$ depends on K, f, u from past, not future.
- Note that when $u, f \equiv 0$ for all $t \leq 0$,

$$\int_0^t K(\tau)\, u(t-\tau)d\tau = \int_0^\infty K(\tau)\, u(t-\tau)\, d\tau, \quad t > 0.$$

A model problem for time discretistaion

- Use convolution Volterra integral equation VIE (K, f given, find u)

$$(K * u)(t) := \int_0^t K(\tau)\, u(t-\tau) d\tau = f(t), \quad t \in (0, T]$$

  as a model to illustrate time discretisation.
- Lots of good methods for the approximate solution of this problem, e.g. convolution quadrature, DG, backward in time collocation. These have a marching on in time (MOT) format. DG perhaps best, but no good for TDBIEs.
- Standard Galerkin is not regarded as a good way to approximate this problem! But we'll use it anyway because of its role in TDBIEs.

Galerkin for VIE $K * u = f$

- Use convolution Volterra integral equation VIE (K, f given, find u)

$$\int_0^t K(\tau)\, u(t-\tau)d\tau = f(t), \quad t \in (0, T].$$

- Ha Duong Galerkin formulation: find $u_h \in V_h$ s.t. $\forall q_h \in V_h$

$$\int_0^T \mathbf{(T-t)} q_h(t) \int_0^t K(\tau)\dot{u}_h(t-\tau)d\tau\, dt = \int_0^T \mathbf{(T-t)} q_h(t)\dot{f}(t)dt.$$

Note that $u_h, q_h \in V_h \quad \Rightarrow \quad u_h(0) = q_h(0) = 0.$

- Rearranged Ha Duong:

$$\int_0^T K(\tau) \int_\tau^T \mathbf{(T-t)} q_h(t)\dot{u}_h(t-\tau)dt\, d\tau = \int_0^T \mathbf{(T-t)} q_h(t)\dot{f}(t)dt.$$

Galerkin for VIE $K * u = f$

- Rearranged Ha Duong:

$$\int_0^T K(\tau) \int_\tau^T \mathbf{(T - t)} q_h(t) \dot{u}_h(t-\tau)\, dt\, d\tau = \int_0^T \mathbf{(T - t)} q_h(t) \dot{f}(t) dt.$$

- Use $u_h(t) = \sum_{n=1}^{N_T} u_n \phi_n(t)$, $q_h(t) = \phi_m(t)$ for each $m = 1, \ldots, N_T$

$$\sum_{n=1}^{N_T} u_n \underbrace{\int_0^T K(\tau) \int_\tau^T \mathbf{(T - t)} \phi_m(t) \dot{\phi}_n(t-\tau)\, dt\, d\tau}_{C_{m,n}} = \int_0^T \mathbf{(T - t)} \phi_m(t) \dot{f}(t) dt.$$

- $C_{m,n}$ looks complicated, and we might expect to have to compute $\mathcal{O}(N_T^2)$ different quantities to set up linear system, ...
- ...but it actually has a lot of structure when the basis functions are splines, and we only need $\mathcal{O}(N_T)$ different quantities.

Galerkin for VIE $K * u = f$

The resulting linear system is

$$(DA + h\hat{A})\boldsymbol{U} = D\boldsymbol{f} + h\hat{\boldsymbol{f}}, \quad D = \mathrm{diag}(T-h, T-2h, \dots, 2h, h, 0).$$

Comes from $\boldsymbol{(T-t) = (T-mh) + (mh-t)}$ for each $m = 1, \dots, N_T$

$$\sum_{n=1}^{N_T} u_n \int_0^T K(\tau) \underbrace{\int_\tau^T \boldsymbol{(T-t)} \phi_m(t)\dot{\phi}_n(t-\tau)dt}\, d\tau = \int_0^T \boldsymbol{(T-t)} \phi_m(t)\dot{f}(t)dt.$$

$$= \boldsymbol{(T-mh)} \int_\tau^T \phi_m(t)\dot{\phi}_n(t-\tau)dt + \int_\tau^T \boldsymbol{(mh-t)} \phi_m(t)\dot{\phi}_n(t-\tau)dt$$

Assemble equations for $m = 1 : N_T$:

$$(D)_{m,m} = \boldsymbol{(T-mh)}, \quad (A)_{m,n} = \int_0^T K(\tau) \int_\tau^T \phi_m(t)\dot{\phi}_n(t-\tau)dt\, d\tau$$

Galerkin for VIE $K * u = f$

The resulting linear system is

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Assemble equations for $m = 1 : N_T$:

$$(h\hat{A})_{m,n} = \int_0^T K(\tau) \int_\tau^T \boldsymbol{(mh-t)}\phi_m(t)\dot{\phi}_n(t-\tau)dt\, d\tau$$

Galerkin for VIE $K * u = f$

The resulting linear system is

$$(DA + h\hat{A})\boldsymbol{U} = D\boldsymbol{f} + h\hat{\boldsymbol{f}}, \quad D = \mathrm{diag}(T-h, T-2h, \ldots, 2h, h, 0).$$

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$$= \boldsymbol{(T-mh)} \int_\tau^T \phi_m(t)\dot{\phi}_n(t-\tau)dt + \int_\tau^T \boldsymbol{(mh-t)}\phi_m(t)\dot{\phi}_n(t-\tau)dt$$

Assemble equations for $m = 1 : N_T$:

$$(\boldsymbol{f})_m = \int_0^T \phi_m(t)\dot{f}(t)dt, \quad (h\hat{\boldsymbol{f}})_m = \int_0^T \boldsymbol{(mh-t)}\phi_m(t)\dot{f}(t)dt$$

Galerkin for VIE $K * u = f$

- Ha Duong Galerkin formulation: find $u_h \in V_h$ s.t. $\forall q_h \in V_h$

$$\int_0^T \boldsymbol{(T - t)} q_h(t) \int_0^t K(\tau) \dot{u}_h(t-\tau) d\tau \, dt = \int_0^T \boldsymbol{(T - t)} q_h(t) \dot{f}(t) dt$$

gives linear system

$$(DA + h\hat{A})\boldsymbol{U} = D\boldsymbol{f} + h\hat{\boldsymbol{f}}.$$

- Basic Galerkin formulation: find $u_h \in V_h$ s.t. $\forall q_h \in V_h$

$$\int_0^T q_h(t) \int_0^t K(\tau) \dot{u}_h(t-\tau) d\tau \, dt = \int_0^T q_h(t) \dot{f}(t) dt.$$

gives linear system

$$A\boldsymbol{U} = \boldsymbol{f}.$$

Galerkin for VIE $K * u = f$ with B_1 basis

The resulting linear system when B_1 basis functions used is

$$(DA + h\hat{A})\boldsymbol{U} = D\boldsymbol{f} + h\hat{\boldsymbol{f}}, \quad D = \mathrm{diag}(T-h, T-2h, \ldots, 2h, h, 0).$$

When $N_T = 4$: $\quad \boldsymbol{U} = (u_1, \ldots, u_4)^{\mathsf{T}}, \quad \boldsymbol{f}, \hat{\boldsymbol{f}} \in \mathbb{R}^4$

$$A = \begin{pmatrix} q_0 & \boldsymbol{q_{-1}} & 0 & 0 \\ q_1 & q_0 & \boldsymbol{q_{-1}} & 0 \\ q_2 & q_1 & q_0 & \boldsymbol{q_{-1}} \\ p_3 & p_2 & p_1 & p_0 \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} \hat{q}_0 & \boldsymbol{\hat{q}_{-1}} & 0 & 0 \\ \hat{q}_1 & \hat{q}_0 & \boldsymbol{\hat{q}_{-1}} & 0 \\ \hat{q}_2 & \hat{q}_1 & \hat{q}_0 & \boldsymbol{\hat{q}_{-1}} \\ \hat{p}_3 & \hat{p}_2 & \hat{p}_1 & \hat{p}_0 \end{pmatrix},$$

Structured, not lower triangular, nearly Toeplitz.

ASIDE: A nice property of B-splines

- Key term: $Y_{m,n}(\tau) = \int_\tau^T \boldsymbol{(T - t)}\phi_m(t)\dot{\phi}_n(t-\tau)dt$.
- Split $\boldsymbol{(T - t) = (T - mh) + (mh - t)}$ for each $m = 1, \ldots, N_T$.
- If $\phi_n(t) = B_\ell(t/h - n)$ (splines degree $\ell \geq 0$) then

$$\int_\tau^T \phi_m(t)\dot{\phi}_n(t-\tau)dt = \int_\tau^T B_\ell(t/h - m)\, \dot{B}_\ell(t/h - n - \tau/h)\, dt$$
$$= h\left(B_{2\ell}\left(\tfrac{\tau}{h} - \tfrac{1}{2} + n - m\right) - B_{2\ell}\left(\tfrac{\tau}{h} + \tfrac{1}{2} + n - m\right)\right)$$
$$= -h\dot{B}_{2\ell+1}\left(\tfrac{\tau}{h} + n - m\right)$$

- Away from 0 and T, B_1 spline Galerkin gives calculations involving (smoother) B_2 splines – good for TDBIE quadrature.
- Term $\int_\tau^T \boldsymbol{(mh - t)}\phi_m(t)\dot{\phi}_n(t-\tau)dt$ also reasonably nice.

Backwards-in-time approximation 1

- Volterra integral equation (VIE) with $u, f \equiv 0$ **for all** $t \leq 0$:

$$\int_0^t K(\tau)\, u(t-\tau)\, d\tau = f(t) = \int_0^\infty K(\tau)\, u(t-\tau)\, d\tau \quad t \in (0, T].$$

- Approximate VIE at $t = t_n = nh$ for $n = 1, 2, \ldots$. i.e. collocate.
- **Approximate solution with basis functions ϕ_m:**

$$\boldsymbol{u(t_n - \tau) \approx \sum_{m=0}^{n} u_{n-m}\, \phi_m(\tau)} \quad \textbf{NOT} \quad \boldsymbol{u(\tau) \approx \sum_k u_k\, \phi_k(\tau)}$$

- Plug into VIE at $t = t_n$:

$$\sum_{m=0}^{n} q_m u_{n-m} = f(t_n), \quad q_m = \int_0^\infty K(\tau)\phi_m(\tau) d\tau.$$

Backwards-in-time approximation 2

- Marching on in time for $n = 1, 2, \ldots$:

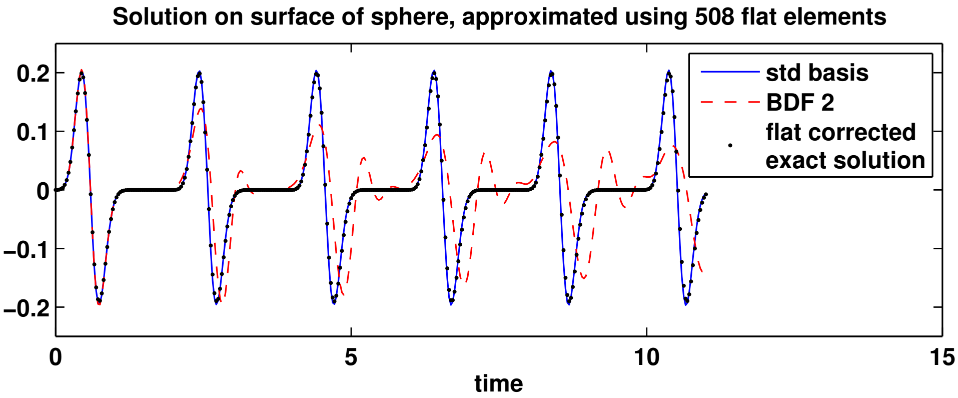
$$\sum_{m=0}^{n} q_m u_{n-m} = f(t_n), \quad \Leftrightarrow \quad u_n = \frac{1}{q_0}\left(f(t_n) - \sum_{m=1}^{n} q_m u_{n-m} \right)$$

  where

$$u(t_n - \tau) \approx \sum_{m=0}^{n} u_{n-m}\,\phi_m(\tau), \quad q_m = \int_0^\infty K(\tau)\phi_m(\tau)d\tau.$$

- Remarkably, convolution quadrature based on linear multistep methods has this format - but with globally supported time basis functions.
- We use mainly B-spline basis functions of degree ℓ since their local support with some global smoothness is an advantage in the full TDBIE. Need modification for $m = 0, \ldots, \ell - 1$, but have also used Gaussian basis functions.

Results for TDBIE – backward-in-time vs CQ



BDF2 is a 2nd order accurate CQ method.

The backward-in-time scheme is (formally) 2nd order witrh local Gaussian basis functions.

Backwards-in-time approximation 3

- Choose B_3 (cubic spline) basis functions with modifications to ϕ_0, ϕ_1:

$$\phi_0(t) = B_3(t/h) + 2B_3(t/h+1),\ \phi_1(t) = B_3(t/h-1) - B_3(t/h+1),$$

$$\phi_m(t) = B_3(t/h - m),\ m \geq 2$$

and approximate $K * \dot{u} = \dot{f}$ – time differentiated version of VIE.

$$n = 1, \ldots, N_T : \quad \sum_{m=0}^{n} g_m u_{n-m} = \dot{f}(t_n), \quad g_m = \int_0^\infty K(\tau)\, \dot{\phi}_m(\tau) d\tau.$$

- Closely related to simple Galerkin B_1 spline approximation from earlier:

$$n = 1, \ldots, N_T - 1 : \quad \boldsymbol{q_{-1} u_{n+1}} + \sum_{m=0}^{n} q_m u_{n-m} = f_n,$$

where, after scaling, $g_0 = q_0 + 2\boldsymbol{q_{-1}}$, $g_1 = q_1 - \boldsymbol{q_{-1}}$, $g_m = q_m$, $m \geq 2$.

Backwards-in-time approximation 4

- Choose B_3 (cubic spline) basis functions with modifications to ϕ_0, ϕ_1 and approximate $K * \dot{u} = \dot{f}$ – time differentiated version of VIE.

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where, after scaling, $g_0 = q_0 + 2\boldsymbol{q_{-1}}$, $g_1 = q_1 - \boldsymbol{q_{-1}}$, $g_m = q_m, m \geq 2$.

- Same as 2nd order extrapolation – replace $\boldsymbol{u_{n+1}}$ by $2u_n - u_{n-1}$.

Galerkin for VIE $K * u = f$ with B_1 basis revisited

The B_1 basis function full Galerkin approx is not lower triangular (and so is expensive to solve)

$$(DA + h\hat{A})\boldsymbol{U} = D\boldsymbol{f} + h\hat{\boldsymbol{f}}, \qquad D = \mathrm{diag}(T-h, T-2h, \ldots, 2h, h, 0).$$

Can write it as

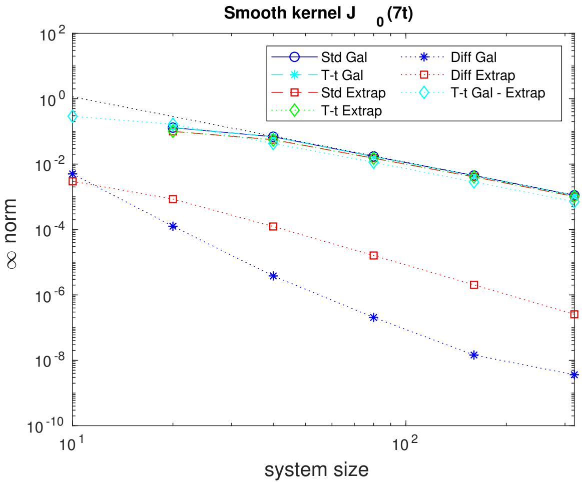
$$(T-nh)\left(\boldsymbol{q_{-1} u_{n+1}} + \sum_{m=0}^{n} q_m u_{n-m}\right) + \left(\boldsymbol{\hat{q}_{-1} u_{n+1}} + \sum_{m=0}^{n} \hat{q}_m u_{n-m}\right) = (T-nh)f_n + h\hat{f}_n$$

Extrapolate $\boldsymbol{u_{n+1}} = 2u_n - u_{n-1}$ gives a lower triangluar approximation:

$$(T-nh)\left(\sum_{m=0}^{n} g_m u_{n-m}\right) + \left(\sum_{m=0}^{n} \hat{g}_m u_{n-m}\right) = (T-nh)f_n + h\hat{f}_n$$

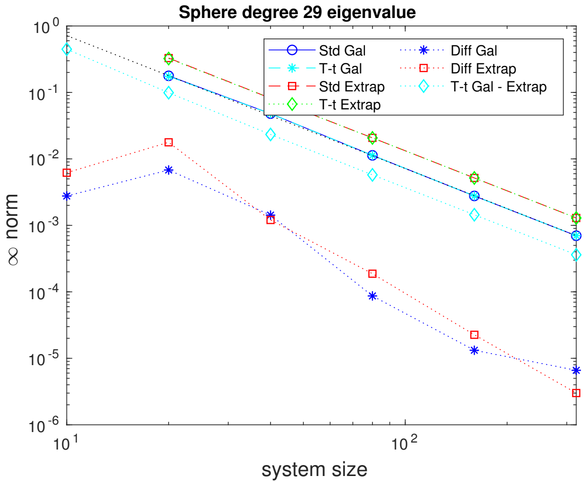
$$g_0 = q_0 + \boldsymbol{2q_{-1}}, \; g_1 = q_1 - \boldsymbol{q_{-1}}, \; g_k = q_k, \; k \geq 2. \qquad \hat{g} \text{ analagous}$$

Results for VIE



- VIE kernel $J_0(7t)$ – 1st kind Bessel function of order 0.
- VIE approximation by various methods: (a) standard Galerkin, (b) $(T - t)$-weighted Galerkin, (c,d) extrapolated versions of (a,b)
- all appear $\mathcal{O}(h^2)$

Results for VIE



- VIE kernel $P_{29}(1 - t^2/2)H(2 - t)$ – sphere scattering, harmonics of order 29.
- VIE approximation by various methods: (a) standard Galerkin, (b) $(T - t)$-weighted Galerkin, (c,d) extrapolated versions of (a,b)
- all appear $\mathcal{O}(h^2)$ – and the kernel is discontinuous at $t = 2$!

Results for TDBIE – full Ha Duong Galerkin

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Not a sausage so far.

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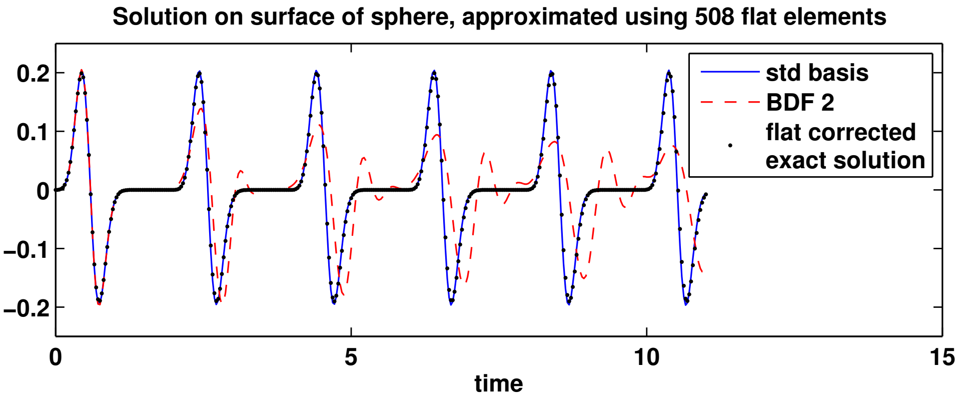
Not a sausage so far.

But plenty of results for other methods.
e.g. Simple Galerkin with extrapolation as preconditioner in Startk & Gimperlein – does a good job.

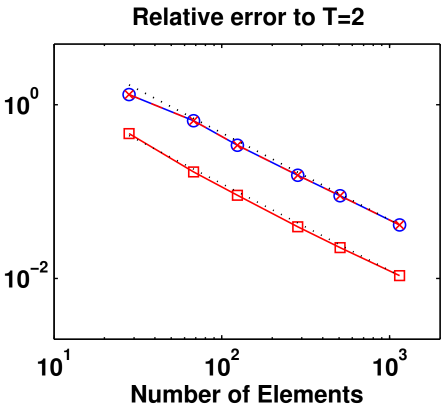
Results for TDBIE – backward-in-time vs CQ

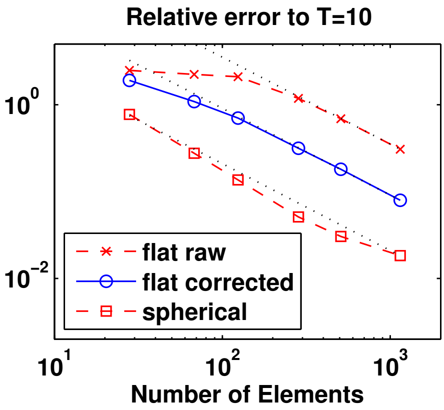


BDF2 is a 2nd order accurate CQ method.

The backward-in-time scheme is (formally) 2nd order witrh local Gaussian basis functions.

Results for TDBIE – backward in time, sphere surface

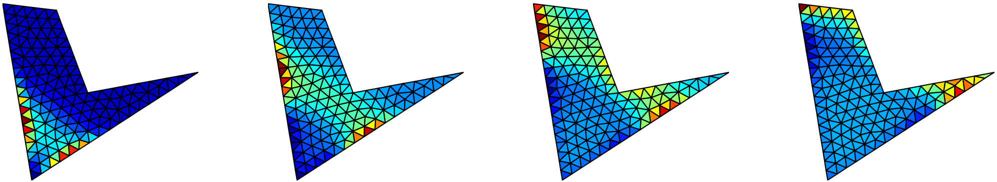




L_∞ Errors – appear 2nd order.

Results for TDBIE – backward in time, flat screen

Edge and corner singularities.



Gimperlien, Stark et al. get good results using mesh refinement at corners and edges.

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- **Outlook:**
  – get some full Ha Duong results
  – try to patch up theory, particularly of connections between schemes
  – move to B_2 spline Galerkin and B_4 backward-in-time collocation counterpart