Notes on Integration

1. **Null sets** A subset E of $\mathbb{R}$ is **null** if, given $\varepsilon > 0$, we can find a countable collection of open intervals $\{I_j\}$ such that $E \subseteq \bigcup_{j=1}^{\infty} I_j$ and $\sum_{j=1}^{\infty} |I_j| < \varepsilon$. (It's easy to see the word 'open' can be omitted.)

   Clearly singletons are null. So is any countable set. Subsets of null sets are null.

   Notation: For an interval I of the form $I = [a, b], (a, b), (a, b]$ or $[a, b)$, $|I| := b - a$ denotes its **length.** If E is a finite union of disjoint bounded intervals, $E = I_1 \cup .... \cup I_N$, then $|E| := \sum_{j=1}^{N} |I_j|$. It is tedious but routine to show that this is well-defined, and, as a consequence, if E_1 and E_2 are such sets with $E_1 \subseteq E_2$, then $|E_1| \leq |E_2|$.

   Theorem 1.1 If E_j $(j = 1, 2, ...\infty)$ is null, so is $\bigcup_{j=1}^{\infty} E_j$.

   Proof Let $\varepsilon > 0$; cover E_j by intervals of total length $< \varepsilon/2^{j+1}$. The union of all such intervals has length $\leq \varepsilon \sum_{j=1}^{\infty} 2^{-j-1} = \varepsilon/2 < \varepsilon$. $\square$

   However not every null set is countable:

   Example: **The Cantor middle third set**

   Let $E_0 = [0, 1]$; let $E_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, the set obtained by removing the middle third of E_0; let $E_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$, the set obtained by removing the middle third of each of the two intervals comprising E_1. Continuing in this way we obtain E_j, a union of 2^j intervals of length 3^{-j}, and E_{j+1} is obtained by removing the middle third of each of these intervals. Let $E = \bigcap_{j=0}^{\infty} E_j$. As the intervals comprising E_j have total length $(2/3)^j$ which tends to 0 as $j \to \infty$, E is null. E is uncountable by the usual Cantor argument.

   Example: Generalised Cantor sets

   Consider the unit interval $I = [0, 1]$. Let F_1 be the "middle" open subinterval of length $\frac{1}{5}$. Let F_2 be the union of the 2 "middle" open subintervals of $I \setminus F_1$, each of length $\frac{1}{5^2}$. Having defined $F_1, F_2, ..., F_j$ such that $I \setminus (F_1 \cup .... \cup F_j)$ consists of 2^j closed intervals, we define F_{j+1} as the union of the 2^j "middle" open subintervals of $I \setminus (F_1 \cup .... \cup F_j)$, each of length $\frac{1}{5^{j+1}}$. An example of a generalised Cantor set is given by $E = I \setminus \bigcup_{j=1}^{\infty} F_j$. Notice that E contains no nontrivial interval, and that $|F_j| = 2^{j-1}/5^j$, so that

$$\sum_{j=1}^{\infty} |F_j| = \frac{1}{2} \sum_{j=1}^{\infty} \left(\frac{2}{5}\right)^j = \frac{1}{2} \frac{2/5}{(1 - 2/5)} = \frac{1}{3}\,.$$

   We shall soon see that this shows that E is **not** null. (One can play games by varying the ratio 1/5.)

Theorem 1.2 If $[a, b]$ is covered by open intervals $\{I_j\}_{j=1}^{\infty}$, then $\sum_{j=1}^{\infty} |I_j| \geq b-a$. In particular, $[a, b]$ is not null, if $a < b$.

Proof Assume that $\sum_{j=1}^{\infty} |I_j| < b-a$. By the Heine-Borel Theorem, we may find a finite subcollection $\{I_1, ...I_N\}$ with $[a, b] \subseteq \bigcup_{j=1}^{N} I_j$. Then, by the remark preceding Theorem 1.1, $b - a \leq |\bigcup_{j=1}^{N} I_j| \leq \sum_{j=1}^{N} |I_j|$, which is a contradiction. (The final inequality while "obvious", can easily be proved by induction on N.) $\square$

Theorem 1.3 Let I be a bounded interval. Suppose that F_j $(j = 1, 2, ...\infty)$ is a finite union of disjoint intervals, $F_j \cap F_k = \emptyset$ when $j \neq k$, each $F_j \subseteq I$ and that $I \setminus \bigcup_{j=1}^{\infty} F_j$ is null. Then $\sum_{j=1}^{\infty} |F_j| = |I|$.

Proof Since $\bigcup_{j=1}^{N} F_j \subseteq I$ for all N we have $\sum_{j=1}^{N} |F_j| \leq |I|$ for all N (using disjointness of the $\{F_j\}$) and so $\sum_{j=1}^{\infty} |F_j| \leq |I|$. Thus we have to show $\sum_{j=1}^{\infty} |F_j| \geq |I|$. Without loss of generality we may assume that I is closed and that each F_j consists of *open* intervals as this affects neither $|F_j|, |I|$ nor the statement that $I \setminus \bigcup_{j=1}^{\infty} F_j$ is null. (We have modified matters only on a countable, hence null set.) Suppose for a contradiction that $\sum_{j=1}^{\infty} |F_j| < |I|$. Cover $I \setminus \bigcup_{j=1}^{\infty} F_j$ by open intervals $\{J_i\}_{i=1}^{\infty}$ with $\sum_{i=1}^{\infty} |J_i| < |I| - \sum_{j=1}^{\infty} |F_j|$. Then $\{F_j\}_{j=1}^{\infty} \cup \{J_i\}_{i=1}^{\infty}$ gives a cover of I by open intervals of total length less than $|I|$, in contradiction to Theorem 1.2. $\square$

Corollary The generalised Cantor set constructed above is **not** null – if it were we'd have to have $\sum_{j=1}^{\infty} |F_j| = 1$, and we showed that $\sum_{j=1}^{\infty} |F_j| = \frac{1}{3}$.

Null sets will be systematically regarded as 'negligible' in integration theory. A property of the real numbers holds **almost everywhere** or holds for **almost all x** if it holds for all real numbers except those in some null set. Thus, for example, $\chi_{\mathbb{Q}} = 0$ almost everywhere.
Notation: if $E \subseteq \mathbb{R}$, $\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E. \end{cases}$

2. **Integration – what are we aiming for?**

We wish to develop an integral with the following features:

(i) if $f : \mathbb{R} \to \mathbb{R}$ is "nice", then $\int f$ should represent the "area under the graph of f"

(ii) if $f \geq 0$, $\int f \geq 0$

(iii) the integral should be linear

(iv) if I_j $(j = 1....\infty)$ is a sequence of pairwise disjoint intervals with $\sum_{j=1}^{\infty} |I_j| < \infty$, then $\chi_{\bigcup_{j=1}^{\infty} I_j}$ should be integrable and $\int \chi_{\bigcup_{j=1}^{\infty} I_j}$ should equal $\sum_{j=1}^{\infty} |I_j|$.

Of course we also wish the integral to be calculable by the standard techniques of integral calculus (antiderivatves, parts, substitution ....) for sufficiently nice integrands.

If $\phi = \sum_{j=1}^{n} c_j \chi_{I_j}$ is a **step function** (i.e. a finite linear combination of characteristic functions of bounded intervals) then this wish list prescribes that we must have

$$\int \phi = \sum_{j=1}^{n} c_j |I_j|.$$

We will then use analysis to extend the definition of the integral to a wider class of functions.

Convention All functions f have domain $\mathbb{R}$, and usually have codomain $\mathbb{R}$, (but we will in occasion allow f to take the values $\pm\infty$). If $g : [a, b] \to \mathbb{R}$, we extend g to be zero outside $[a, b]$ to obtain a function with domain $\mathbb{R}$. (We make an exception to this convention in Sections 7-9.)

3. Integration of Step functions

Definition A **step function** $\phi : \mathbb{R} \to \mathbb{R}$ is a finite linear combination of characteristic functions of bounded intervals, i.e.

$$\phi = \sum_{j=1}^{n} c_j \chi_{I_j}$$

where $|I_j| < \infty$.

Evidently, ϕ is a step function if and only if $\exists x_0 < x_1 < ... < x_N$ such that

(i) $\phi(x) = 0$ for $x < x_0$ and $x > x_N$

(ii) ϕ is constant on (x_{j-1}, x_j) $1 \leq j \leq N$.

We say that such a ϕ is a **step function with respect to** $\{x_0, ..., x_N\}$. We define the **integral** of such a ϕ by

$$\int \phi := \sum_{j=1}^{N} \phi_j (x_j - x_{j-1})$$

where ϕ_j is the constant value of ϕ on (x_{j-1}, x_j). Note that if ϕ is a step function with respect to $\{x_0, ..., x_N\}$, and $x_{j-1} < c < x_j$, ϕ is also a step function with respect to $\{x_0, ..., x_{j-1}, c, x_j, ..., x_N\}$ and the two definitions of $\int \phi$ agree. Thus if ϕ is a step function with respect to $\{x_0, ..., x_N\}$ and also with respect to $\{y_0, ..., y_M\}$, upon ordering $\{x_0, ..., x_N\} \cup$

$\{y_0, ..., y_M\}$ as $z_0 < .... < z_K$ $(K \leq M + N)$ we see that the definitions of $\int \phi$ with respect to $\{x_0, ..., x_N\}, \{z_0, ..., z_K\}$ and $\{y_0, ..., y_M\}$ all agree.

Thus $\int \phi$ is well-defined for any step function ϕ.

It is clear that if ϕ and ψ are step functions and $\alpha, \beta \in \mathbb{R}$, then $\alpha\phi + \beta\psi$ is a step function; moreover

$$\int (\alpha\phi + \beta\psi) = \alpha \int \phi + \beta \int \psi \tag{*}$$

as if we list all the "jump points" of either ϕ or ψ together as $\{x_0 < .... < x_N\}$, the left hand side is $\sum_{j=1}^{N} (\alpha\phi_j + \beta\psi_j)(x_j - x_{j-1}) = \alpha \sum_{j=1}^{N} \phi_j(x_j - x_{j-1}) + \beta \sum_{j=1}^{N} \psi_j(x_j - x_{j-1}) = \alpha \int \phi + \beta \int \psi$.

Similarly if $\phi \geq \psi$, with ϕ, ψ step functions, then $\int \phi \geq \int \psi$.

Since $\int \chi_I = |I|$, linearity of the integral (*) implies that

$$\int \sum_{j=1}^{n} c_j \chi_{I_j} = \sum_{j=1}^{n} c_j |I_j|.$$

So points (i)-(iii) of our "wish list" are adequately resolved for step functions. How about point (iv)? As so far we've only defined the integral for step functions, (iv) is tantamount to asking:

If $\{I_j\}_{j=1}$ is a disjoint sequence of intervals with union $\bigcup_{j=1}^{\infty} I_j$ **which is also an interval** I, does $|I| = \sum_{j=1}^{\infty} |I_j|$? This is guaranteed by Theorem 1.3. In fact, Theorem 1.3 can be re-phrased as follows:

Theorem 1.3′ Let ϕ_n be the characteristic function of a finite union of bounded intervals. Suppose that $\phi_{n+1}(x) \leq \phi_n(x)$ for all $x \in \mathbb{R}$, and $\phi_n(x) \to 0$ a.e. then $\int \phi_n \to 0$.

Proof Let $\phi_n = \chi_{E_n}$, thus $E_{n+1} \subseteq E_n$. We may assume without loss of generality that E_1 is an interval. Set $F_n = E_n \setminus E_{n+1}$ and $I = E_1$. Then $\phi_n(x) \to 0$ *a.e.* is the same as saying $I \setminus \bigcup_{j=1}^{\infty} F_j$ is null. By Theorem 1.3, $\sum_{j=1}^{\infty} |F_j| = |I| = |E_1|$. But $\sum_{j=1}^{\infty} |F_j| = \sum_{j=1}^{\infty} |E_j \setminus E_{j+1}| = \sum_{j=1}^{\infty} (|E_j| - |E_{j+1}|) = |E_1| - \lim_{j\to\infty} |E_j|$. Thus $\lim_{j\to\infty} |E_j| = 0$ and so $\int \phi_j \to 0$ as $j \to \infty$. $\square$

A fundamental fact in the theory of Lebesgue integration is that Theorem 1.3′ extends to the case where each ϕ_n is an arbitrary non-negative step function:

Proposition 3.1 Suppose ϕ_n is a sequence of step functions with $\phi_n(x) \geq 0$ *a.e.*and $\phi_{n+1}(x) \leq \phi_n(x)$ *a.e.* Suppose that $\phi_n(x) \to 0$ *a.e.* as $n \to \infty$. Then $\int \phi_n \to 0$.

Remark 1 The assumptions $\phi_n \geq 0$ *a.e.*and $\phi_{n+1} \leq \phi_n$ *a.e.* are innocuous as in this context they mean $\phi_n \geq 0$ and $\phi_{n+1} \leq \phi_n$ except on a finite set, and any statement about $\int \phi_n$ is unchanged after modification of ϕ_n on such a finite set. On the other hand '$\phi_n(x) \to 0$ *a.e.*' does have content as the (usual) Cantor set construction shows.

[**Remark 2** There is also a "dominated" version of this Proposition which is formally stronger: if ϕ_n is a sequence of step functions with $0 \leq \phi_n \leq \phi_1$ *a.e.* and $\phi_n \to 0$ *a.e.*, then $\int \phi_n \to 0$. However this would require some machinery to prove.]

We first prove a lemma, which, roughly speaking, says that if ϕ_n is a sequence of step funcitons such that $\phi_n(x) \geq 0$ and $\phi_{n+1}(x) \leq \phi_n(x)$ for all x, and such that $\phi_n(x) \to 0$ *a.e.*, then the convergence is "almost uniform":

Lemma 3.2 Suppose ϕ_n is a sequence of step functions with $\phi_n(x) \geq 0$ and $\phi_{n+1}(x) \leq \phi_n(x)$ for all x, and such that $\phi_n(x) \to 0$ *a.e.* Let $\varepsilon > 0$. Then there is a finite union E of intervals such that $|E| < \varepsilon$ and an $N \in \mathbb{N}$ and such that $x \notin E,\ n \geq N \Rightarrow \phi_n(x) < \varepsilon$.

Proof Let $E_n = \{x \mid \phi_n(x) \geq \varepsilon\}$. Then E_n is a finite union of bounded intervals and, as $\phi_{n+1} \leq \phi_n$, $E_{n+1} \subseteq E_n$, and thus $\chi_{E_{n+1}} \leq \chi_{E_n}$. Moreover $\chi_{E_n} \to 0$ *a.e.* as

$$\begin{aligned}\{x \,|\chi_{E_n}(x) \quad \nrightarrow \quad 0\} &= \{x \mid x \text{ belongs to infinitely many } E_n\} \\ &= \{x \mid \phi_n(x) \geq \varepsilon \text{ for infinitely many } n\text{'s}\} \\ &\subseteq \{x \mid \phi_n(x) \nrightarrow 0\}\end{aligned}$$

which is null by hypothesis. By Theorem 1.3′, $\int \chi_{E_n} = |E_n| \to 0$. Choose N so that $|E_N| < \varepsilon$, and take $E = E_N$. So $x \notin E \Rightarrow x \notin E_n$ for all $n \geq N$, so $\phi_n(x) \leq \varepsilon$ for $x \notin E$ and $n \geq N$. $\square$

Proof of Proposition 3.1 According to Remark 1 after the statement of Proposition 3.1 we can assume that $\phi_n(x) \geq 0$ and $\phi_{n+1}(x) \leq \phi_n(x)$ for all $x \in \mathbb{R}$. Suppose ϕ_1 is zero outside $[a, b]$ and $\phi_1(x) \leq M$ for all x. Let $\varepsilon > 0$; take E and N as in the lemma. Then for $n \geq N$

$$\begin{aligned}\int \phi_n &= \int \phi_n \chi_{[a,b]\cap E} + \int \phi_n \chi_{[a,b]\cap E^C} \\ &\leq M\,|E| + \varepsilon(b-a) \\ &\leq (M + (b-a))\varepsilon.\end{aligned}$$

Thus $\int \phi_n \to 0$ as $n \to \infty$. $\square$

4. Extension of the integral

To capture the essence of the integral as the "area under the graph" it seems reasonable to define, for arbitrary $f : \mathbb{R} \to \mathbb{R}$, $\int f$ as $\sup \int \phi$ where the supremum is taken over all step functions ϕ with $\phi \leq f$ a.e. While this is a good definition for a wide class of f's, it's not good in general if we wish to maintain our wish list.

For example, let $f = \chi_C$ where C is a generalised Cantor set as constructed in Section 1, and where $\sum_{j=1}^{\infty} |F_j| = \alpha$ **is chosen to be less than 1.** Since C contains no nontrivial interval, the proposed definition for $\int \chi_C$ gives zero; while points (iii) and (iv) of our wish list require $\int \chi_C = 1 - \sum_{j=1}^{\infty} |F_j| = 1 - \alpha > 0$.

Clearly the problem with this f is that it is not well-accessed from below by step functions.

<u>Definition</u> $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is **accessible from below by step functions** if there exists a sequence of step functions ϕ_n, with $\phi_n \leq f$ a.e., such that $\lim_{n\to\infty} \phi_n(x) = f(x)$ a.e.

The class of such is denoted by $\mathcal{M}^-$ (the minus signifying "below"). Thus, while $\chi_{\bigcup_{j=1}^{\infty} I_j} \in \mathcal{M}^-$, $1 - \chi_{\bigcup_{j=1}^{\infty} I_j}$ may not be in $\mathcal{M}^-$.

<u>Definition</u> For $f \in \mathcal{M}^-$ we define

$$\int f = \sup \left\{ \int \phi \mid \phi \text{ a step function, } \phi \leq f \text{ a.e.} \right\}$$

(possibly with value $+\infty$) and we say that $f \in L^{inc}$ iff $f \in \mathcal{M}^-$ and $\int f < \infty$.

<u>Note</u>: For step functions, this definition agrees with the one previously given.

<u>Remark</u> The reason for the terminology "L^{inc}" will become clearer in a little while.

<u>Definition</u> We shall say that $f : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ is **Lebesgue-integrable,** or $f \in L^1$, if there exist $g, h \in L^{inc}$ such $f(x) = g(x) - h(x)$ for all x with $g(x), h(x) < \infty$. For such an f we shall define its **integral** to be

$$\int f = \int g - \int h.$$

<u>Remark</u> We shall show later that this is a good definition: If $g_j, h_j \in L^{inc}$ and $g_1 + h_2 = g_2 + h_1$, then $\int g_1 + \int h_2 = \int g_2 + \int h_1$ will follow from linearity of the integral on L^{inc} with positive scalars. This requires proof. (It is not even immediately obvious that $g, h \in L^{inc} \Rightarrow g + h \in L^{inc}$.)

<u>Remark</u> There is of course another way to deal with the question of accessibility from below by step functions, and simultaneously deal with the "above/below" symmetry, which is to declare that we should define $\int f$ as $\sup \left\{ \int \phi \mid \phi \leq f,\ \phi \text{ a step function} \right\}$ **only** for those f's for which we have

$$\sup_{\phi \leq f} \int \phi = \inf_{\psi \geq f} \int \psi$$

(where the sup and inf are taken over step functions). This of course is the Riemann integral; thus the Riemann integral is to "and" as the Lebesgue integral is to "or". If we want wish (iv) to be true we have to step outside the framework of Riemann integration, and it is wish (iv) that drives the powerful convergence theorems of the Lebesgue integral, the utility of the Lebesgue integral in PDE's and the relevance of the Lebesgue integral to probability theory.

5. **On L^{inc}**

We wish to address three issues concerning L^{inc}: Are "reasonable" functions automatically in L^{inc}? Is L^{inc} closed under addition, and the integral then linear with respect to positive scalars? How does one effectively **calculate** $\int f$ for $f \in L^{inc}$?

Lemma 5.1 Suppose $f : [a,b] \to \mathbb{R}$ is continuous. Let $\varepsilon > 0$. Then there is a step function ϕ such that $f(x) - \varepsilon \le \phi(x) \le f(x)$ for all $x \in [a,b]$.

Thus if $f : [a,b] \to \mathbb{R}$ is continuous, and is extended to be zero outside $[a,b]$, $f \in \mathcal{M}^-$. If M is an upper bound for $f(x)$ and ϕ is a step function with $\phi \le f$ a.e., then $\int \phi \le M(b-a)$. Hence $f \in L^{inc}$ and $\int f \le M(b-a)$. By "pieceing together" such f's we see that piecewise continuous functions of bounded support are always in L^{inc}.

Proof of Lemma

Given $\varepsilon > 0$, for each $x \in [a,b]$ there is an open interval I_x containing x such that $y \in I_x \Rightarrow |f(x) - f(y)| < \varepsilon/2$. Thus, $y, z \in I_x \Rightarrow |f(y) - f(z)| < \varepsilon$. Since $\{I_x \mid x \in [a,b]\}$ is an open cover for $[a,b]$ we may find a finite subcover $I_{x_1}, \dots. I_{x_n}$. Letting $J_1 = I_{x_1}$ and $J_j = J_{x_j} \setminus \bigcup_{\ell=1}^{j-1} J_\ell$ we obtain a disjoint cover $\{J_1, \dots J_n\}$ for $[a,b]$. Now define

$$\phi(x) = \begin{cases} \inf_{x \in J_j} f(x) & x \in J_j \\ 0 & x \notin \bigcup_{j=1}^{n} J_j \, . \end{cases}$$

Then ϕ is a step function satisfying

$$f(x) - \varepsilon \le \phi(x) \le f(x) \text{ for } x \in [a,b].$$

$\square$

Slightly more generally, it's not hard to see that if $f : [a,b] \to \mathbb{R}$ is bounded below and continuous at almost every point of $[a,b]$, then $f \in \mathcal{M}^-$ (when extended to be zero outside $[a,b]$). If such an f is also bounded above, then f will be in L^{inc}.

Theorem 5.2

(i) If ϕ_n is a sequence of step functions with $\phi_{n+1} \ge \phi_n$ a.e., $\phi_n \to f$ a.e. and $\int \phi_n$ convergent, then $f \in L^{inc}$ and $\int f = \lim \int \phi_n$.

(ii) If $f \in L^{inc}$ and ϕ_n is any sequence of step functions with $\phi_{n+1} \ge \phi_n$ a.e., $\phi_n \to f$ a.e., then $\lim_{n\to\infty} \int \phi_n$ exists and equals $\int f$.

(Hence the name "L^{inc}" for such f's realised as limits of increasing sequences of step functions.)

[**Remark**: There is also a formally stronger "dominated" version of this result.]

Proof

(i) If there is a sequence of step functions ϕ_n with $\phi_n \le \phi_{n+1} \le f$ a.e. and $\phi_n \to f$ a.e., then clearly $f \in \mathcal{M}^-$. If now ϕ is any step function with $\phi \le f$ a.e. then $\psi_n := (\phi - \phi_n)_+$ is a decreasing sequence of nonnegative step functions with $\psi_n \to 0$ a.e. By Proposition 3.1, $\int \psi_n \to 0$. Thus $f \in L^{inc}$ and $\int f \le \lim \int \phi_n$. But as $\phi_n \le f$ a.e. and as ϕ_n is a step function, it is a candidate in the definition of $\int f$, and so $\int \phi_n \le \int f$ for each n. Hence $\lim \int \phi_n = \int f$.

(ii) Let $f \in L^{inc}$. There **is** an increasing sequence of step functions ϕ_n with $\phi_n \to f$ a.e. Then as $\phi_n \le f$ a.e., $\int \phi_n \le \int f$ and the increasing sequence $\int \phi_n$ converges to some number less than or equal to $\int f$. By part (i), $\lim \int \phi_n = \int f$. □

This theorem is useful both theoretically – as we'll see below, we can deduce linearity of the integral with positive scalars from it – and practically as a method for calculating $\int f$ for $f \in L^{inc}$.

Theorem 5.3

(i) If $f, g \in L^{inc}$, so is $f + g$ and $\int (f+g) = \int f + \int g$.

(ii) If $f \in L^{inc}$ and $\alpha > 0$, then $\alpha f \in L^{inc}$ and $\int \alpha f = \alpha \int f$.

(iii) If $f, g \in L^{inc}$ and $f \ge g$ a.e., then $\int f \ge \int g$.

Proof

(i) If ϕ_n is a sequence of step functions increasing to f a.e., and ψ_n is a sequence of step functions increasing to g a.e., then $\phi_n + \psi_n$ is a sequence of step functions increasing to $f + g$ a.e. By Theorem 5.2, $\int \phi_n \to f$, $\int \psi_n \to \int g$, $f + g \in L^{inc}$ and $\int (f+g) = \lim \int (\phi_n + \psi_n) = \lim \int \phi_n + \lim \int \psi_n = \int f + \int g$.

(ii) is similar.

(iii) If ϕ_n and ψ_n are as in (i), let $\theta_n = \min(\phi_n, \psi_n)$. Then $\theta_n \le \phi_n$ and θ_n is a sequence of step functions increasing to g a.e. Thus $\int g = \lim \int \theta_n \le \lim \int \phi_n = \int f$.

Very useful in practical situations is the following.

Lemma 5.4 Let f_n be a piecewise continuous function, $f_n(x) \equiv 0$ outside a bounded interval I_n. Suppose $f_{n+1} \ge f_n$ a.e., $f_n \to f$ a.e. and $\int f_n$ is a convergent sequence. Then $f \in L^{inc}$ and $\int f = \lim \int f_n$.

Proof By Lemma 5.1, there exists a step function ϕ_n, vanishing outside I_n, such that $f_n - 2^{-n} \le \phi_n \le f_n$ a.e. As $f_n \to f$ a.e., $\phi_n \to f$ a.e. Let $\psi_n = \max\{\phi_1, ...\phi_n\}$. Then $\psi_{n+1} \ge \psi_n$, ψ_n is a step function and $\psi_n \to f$ a.e. Moreover as $\psi_n = \max\{\phi_1, ...\phi_n\} \le \max\{f_1, ...f_n\} = f_n$, $\int \psi_n \le \int f_n \le K < \infty$. Thus $f \in L^{inc}$ and $\int f = \lim \int \psi_n \le \lim \int f_n$. But as $f_n \le f$ we have $\int f_n \le \int f$ and thus $\int f = \lim \int f_n$. □

6. Practical Integration

Let $g : [a, b] \to \mathbb{R}$ be continuous. Then $\int_a^b g$ is defined as $\int g\chi_{[a,b]}$. Note that if g is continuous on $[a, b]$ and $a < c < b$, then $\int_a^b g = \int_a^c g + \int_c^b g$ (by linearity of the integral on L^{inc} with positive coefficients).

Theorem 6.1 Let $g : [a, b] \to \mathbb{R}$ be continuous. For $a \le x \le b$ let

$$G(x) = \int_a^x g.$$

Then G is differentiable on (a, b) and $G'(x) = g(x)$.

Proof For $x \in (a, b)$, let $h > 0$ be small and consider $\dfrac{G(x+h) - G(x)}{h} - g(x)$. (The argument for $h < 0$ is similar.) This quantity equals

$$\frac{1}{h}\int_x^{x+h} [g(t) - g(x)]dt \ .$$

Now as g is continuous at x, if $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x < t < x + h$ and $h < \delta$, then $-\varepsilon < g(t) - g(x) < \varepsilon$. So for such h,

$$-\varepsilon \le \frac{1}{h}\int_x^{x+h} [g(t) - g(x)]dt \le \varepsilon$$

by the properties of the integral developed above. Thus $h < \delta$ implies $\left|\dfrac{G(x+h) - G(x)}{h} - g(x)\right| < \varepsilon$ and so $G'(x)$ exists and equals $g(x)$.

Corollary 6.2 Suppose $f : [a, b] \to \mathbb{R}$ has continuous derivative f' on $[a, b]$. Then

$$\int_a^b f' = f(b) - f(a).$$

Proof Let $G(x) = \int_a^x f'$. Then by Theorem 6.1, $G'(x)$ exists for all x in (a, b) and $G'(x) = f'(x)$. Thus $G - f$, being continuous on $[a, b]$ and differentiable on (a, b), must be constant on $[a, b]$ by Rolle's theorem. So $\int_a^b f' = G(b) = G(a) + f(b) - f(a) = f(b) - f(a)$. $\square$

7. $\mathbf{L^1}$

The definition of L^{inc} is asymmetric in so far as it is concerned with accessibility from below rather than accessibility from above. We make a final extension of the integral to remedy this situation. The resulting space of integrable functions, L^1, will be a vector space, and the integral will be a linear function from L^1 to $\mathbb{R}$. First we need a preliminary result.

Proposition 7.1 If $f \in L^{inc}$, then $\{x \in \mathbb{R} \mid f(x) = \infty\}$ is a null set.

Proof In view of Theorem 5.2, it suffices to show that if ϕ_n is a sequence of step functions, $\phi_{n+1} \geq \phi_n$ a.e., $\int \phi_n \leq K$, then $\{x \mid \phi_n(x) \to \infty\}$ is null. Fix an M large. Then $E_1 = \{x \mid \phi_1(x) > M\}$ is the union of a finite number of intervals of total length $\leq K/M$. Let $E_2 = \{x \mid \phi_2(x) > M\}$; then $E_1 \subseteq E_2$ (as $\phi_2 \geq \phi_1$) and E_2 is also the union of a finite number of intervals of total length $\leq K/M$. With $F_2 = E_2 \setminus E_1$, we see that E_2 is the union of a finite number of intervals comprising E_1 and F_2 of total length $\leq K/M$. Continuing in this way, with $E_n = \{x \mid \phi_n(x) > M\}$, then $E_{n-1} \subseteq E_n$, and with $F_n = E_n \setminus E_{n-1}$ consisting of a finite number of intervals, we see that E_n is the union of a finite number of intervals comprising $E_1, F_2, F_3, ..., F_n$, of total length $\leq K/M$. Now $E_\infty := \{x \mid \lim\limits_{n\to\infty} \phi_n(x) > M\} = \bigcup\limits_{n=1}^{\infty} E_n =$

$E_1 \cup F_2 \cup F_3.......$. Thus E_∞ can be covered by a countable family of intervals of total length $\leq K/M$.

Finally, $\{x \mid \lim \phi_n(x) = \infty\} \subseteq \{x \mid \lim\limits_{n\to\infty} \phi_n(x) > M\}$ for each M, and so by choosing M sufficiently large we can cover $\{x \mid \lim\limits_{n\to\infty} \phi_n(x) = \infty\}$ by countably many intervals of arbitrarily small total length. $\square$

Definition 7.2 Let $f : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$. Then f is **Lebesgue-integrable**, or $f \in L^1$, if there exist $g, h \in L^{inc}$ with $f(x) = g(x) - h(x)$ for all x with $g(x), h(x) < \infty$. For such an f we define its **integral** to be

$$\int f := \int g - \int h.$$

Lemma 7.3 This is a good definition.

Proof If $\exists g_1, g_2, h_1, h_2 \in L^{inc}$ *s.t.*

$$f(x) = g_1(x) - h_1(x) = g_2(x) - h_2(x)$$

for all x with $g_i(x) < \infty,\ h_i(x) < \infty$, then $g_1(x) + h_2(x) = g_2(x) + h_1(x)$ for almost all x, by Proposition 7.1, and both $g_1 + h_2$ and $g_2 + h_1$ are in L^{inc} by Theorem 5.3. Thus

$$\int g_1 + \int h_2 = \int (g_1 + h_2) = \int (g_2 + h_1) = \int g_2 + \int h_1$$

by Theorem 5.3 once again, and since $g_1 + h_2 = g_2 + h_1$ *a.e.* Thus $\int g_1 - \int h_1 = \int g_2 - \int h_2$ and $\int f$ is well-defined.

Corollary 7.4 L^1 is a vector space and $\int : L^1 \to \mathbb{R}$ is a linear transformation. Moreover, if $f \in L^1$, then $|f| \in L^1$ and $\left|\int f\right| \leq \int |f|$. Finally if $f \in L^1$ and $g = f$ *a.e.*, then $g \in L^1$ and $\int g = \int f$.

8. The convergence theorems

Lemma 8.1 A nonnegative function f belongs to $\mathcal{M}^-$ if and only if there is a sequence ϕ_n of nonnegative step functions such that $f = \sum_{n=1}^{\infty} \phi_n$ a.e. In this case, $\int f = \sum_{n=1}^{\infty} \int \phi_n$.

Proof

'if part': Given such a sequence ϕ_n, let $\psi_n = \sum_{j=1}^{n} \phi_j$. Then ψ_n is an increasing sequence of step functions converging to f almost everywhere. If $\sum \int \phi_n = \lim \int \psi_n$ is finite then $f \in L^{inc}$ and $\int f = \lim \int \psi_n = \sum \int \psi_n$; if $\int \psi_n \to \infty$ then $\int f = \infty$.

'only if' part: If $f \in \mathcal{M}^-$, suppose θ_n is a sequence of step functions with $\theta_n \leq f$ a.e. and $\theta_n \to f$ a.e. Then $\psi_n := \max\{\theta_1, ...\theta_n\}$ gives an increasing sequence of step functions with $\psi_n \to f$ a.e. By replacing ψ_n by $\max(\psi_n, 0)$, we may assume that each $\psi_n \geq 0$. Let $\phi_1 = \psi_1,\ \phi_n = \psi_n - \psi_{n-1}$. Then $f = \sum_{n=1}^{\infty} \phi_n$ a.e. and, as in the 'if' part, $\int f = \sum_{n=1}^{\infty} \int \phi_n$. □

Lemma 8.2 Let f_n be a sequence of nonnegative functions in $\mathcal{M}^-$. Then $f = \sum_{n=1}^{\infty} f_n$ is in $\mathcal{M}^-$ and $\int f = \sum_{n=1}^{\infty} \int f_n$.

Proof For each n there is a sequence of nonnegative step functions $\{\phi_{n,v}\}_{v=1}^{\infty}$ such that $f_n = \sum_{v=1}^{\infty} \phi_{n,v}$ a.e. Then $f = \sum_{n,v} \phi_{n,v}$ a.e. and $f \in \mathcal{M}^-$ (as the set of pairs n, v is countable). Thus $\int f = \sum_{n,v} \int \phi_{n,v} = \sum_{n=1}^{\infty} \int f_n$ by Lemma 8.1. □

Corollary 8.3 Let f_n be a sequence of nonnegative functions in L^{inc}, with $\sum_{n=1}^{\infty} \int f_n < \infty$. Let $f = \sum_{n=1}^{\infty} f_n$. Then $f \in L^{inc}$ and $\int f = \sum_{n=1}^{\infty} \int f_n$.

Lemma 8.4 Let $f \in L^1$ and $\varepsilon > 0$. Then $\exists g, h \in L^{inc}$ with $f = g - h$ a.e., $h \geq 0$ a.e. and $\int h < \varepsilon$.

Proof As $f \in L^1$, $\exists G, H \in L^{inc}$ with $f = G - H$ a.e. By the definition of L^{inc}, $\exists$ step functions ϕ, ψ with $0 < \int(G - \phi) < \varepsilon$ and $0 < \int(H - \psi) < \varepsilon$. Then $h := H - \psi \geq 0$ a.e.,$\int h < \varepsilon$ and $f = g - h$ with $g = G - \psi \in L^{inc}$. □

Theorem 8.5 (Interchange of summation and integral theorem for nonnegative sequences in L^1)

Suppose $f_n \in L^1,\ f_n \geq 0$ a.e. and $\sum_{n=1}^{\infty} \int f_n$ converges. Then $\sum f_n \in L^1$ and $\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n$.

Proof By Lemma 8.4, we can write $f_n = g_n - h_n$ with $g_n, h_n \in L^{inc}$, $h_n \geq 0$ a.e. and $\int h_n < 1/2^n$. Corollary 8.3 applied to $\{h_n\}$ shows that $h = \sum h_n \in L^{inc}$ and $\int h \leq 1$. Now $g_n = f_n + h_n \geq 0$ a.e., $g_n \in L^{inc}$ and $\sum \int g_n = \sum \int f_n + \sum \int h_n < \infty$. So Corollary 8.3 applied to $\{g_n\}$ shows that $g = \sum g_n \in L^{inc}$ and $\int g = \sum \int g_n$. So $\sum f_n = \sum g_n - \sum h_n = g - h \in L^1$ and $\int \sum f_n = \int g - \int h = \sum \int g_n - \sum \int h_n = \sum \int f_n$. $\square$

Corollary 8.6 (Monotone convergence theorem for L^1)

Suppose $f_n \in L^1$, $f_{n+1} \geq f_n$ a.e. and $\lim\limits_{n\to\infty} \int f_n < \infty$. Then $f = \lim f_n \in L^1$ and $\int f = \lim\limits_{n\to\infty} \int f_n$.

Proof For $n \geq 2$, let $g_n = f_n - f_{n-1}$. Then $g_n \in L^1$, $g_n \geq 0$ a.e., $\sum\limits_{n=2}^{\infty} g_n = f - f_1$ and $\sum\limits_{n=2}^{\infty} \int g_n = \lim\limits_{n\to\infty} \int f_n - \int f_1$.

By Theorem 8.5, $\sum\limits_{n=2}^{\infty} g_n \in L^1$ and $\int \sum\limits_{n=2}^{\infty} g_n = \int (f - f_1) = \sum\limits_{n=2}^{\infty} \int g_n = \lim\limits_{n\to\infty} \int f_n - \int f_1$. Thus $f = \sum\limits_{n=2}^{\infty} g_n + f_1 \in L^1$ and $\int f = \lim \int f_n$. $\square$

Corollary 8.7 Let $f \in L^1$, $f \geq 0$ *a.e.* and $\int f = 0$. Then $f \equiv 0$ *a.e.*.

Proof Let $f_n = nf$; then $f_{n+1} \geq f_n$ *a.e.*, $f_n \in L^1$ and $\int f_n = 0 \forall n$. By Monotone Convergence, $\lim\limits_{n\to\infty} f_n \in L^1$, and by Proposition 7.1, $\lim\limits_{n\to\infty} f_n(x) < \infty$ *a.e.* The only way this can happen is if $f \equiv 0$ *a.e.*. $\square$

Remark There is also a version of the MCT when $f_{n+1} \leq f_n$, (obtained by applying the MCT to $\{-f_n\}$).

Observation 8.8 Suppose $f_n(x) \leq g(x)$ *a.e.*, $f_n, g \in L^1$. Let $U_k(x) = \max\{f_1(x), ..., f_k(x)\}$. Then $U_{k+1} \geq U_k$, and $U_k \leq g \Rightarrow \int U_k$ is bounded above by $\int g$. The MCT now implies that $\lim\limits_{k\to\infty} U_k(x) = U(x)$ is in L^1(and is finite a.e.). Of course, $U(x) = \sup\{f_1(x), f_2(x), ...\}$.

Theorem 8.9 (Dominated Convergence Theorem)

Suppose $f_n \in L^1$, $|f_n| \leq g$ *a.e.* and $g \in L^1$. Suppose $f_n \to f$ *a.e.* Then $f \in L^1$ and $\int f = \lim\limits_{n\to\infty} \int f_n$.

Remark Existence of $\lim \int f_n$ is part of the conclusion.

Proof For each fixed $n \geq 1$ let

$$V_n(x) = \sup\{f_n(x), f_{n+1}(x), ...\}.$$

Then $V_n \in L^1$ by Observation 8.8. Now $V_{n+1} \leq V_n$ and $f_n \leq V_n$ so that $-g \leq f_n \leq V_n$ and $-\int g \leq \int V_n$. Moreover $V_n \to f$ *a.e.* So by MCT $f \in L^1$ and $\int f = \lim \int V_n$. Similarly with $L_n(x) = \inf\{f_n(x), f_{n+1}(x), ...\}$, $\int f = \lim \int L_n$.

Now $L_n \leq f_n \leq V_n$, so $\int f_n$ also converges to $\int f$. $\square$

Exercise Suppose $f_n \in L^1$, $f_n \to f$ *a.e.*, $0 \leq f_n \leq f$ *a.e.*, and $\int f_n \leq C$. Then $f \in L^1$ and $\int f = \lim_{n\to\infty} \int f_n$.

Theorem 8.10 (Interchange of integral and summation for general sequences in L^1)
If $g_n \in L^1$, and **either**

(a) $\sum_{n=1}^{\infty} \int |g_n|$ converges to a finite number
or
(b) $\sum_{n=1}^{\infty} |g_n(x)|$ belongs to L^1,

then $\sum_{n=1}^{\infty} g_n(x)$ converges a.e. to a function in L^1 and $\int \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} \int g_n$.

Remark If $g_n \geq 0$, then part (a) is Theorem 8.5.

Proof

(a) Let $g_n^+ = \max(g_n, 0) \geq 0$. Then $g_n^+ \in L^1$, $\int g_n^+ \leq \int |g_n|$ so that $\sum \int g_n^+ < \infty$. By Theorem 8.5, $\sum g_n^+ \in L^1$ and $\int \sum g_n^+ = \sum \int g_n^+$. Similarly for $g_n^- = -\min(g_n, 0) \geq 0$. Now $g_n = g_n^+ - g_n^-$, so $\sum g_n$ converges to $\sum g_n^+ - \sum g_n^- \in L^1$, and $\int \sum g_n = \int \sum g_n^+ - \int \sum g_n^- = \sum \int g_n^+ - \sum \int g_n^- = \sum \int (g_n^+ - g_n^-) = \sum \int g_n$.

(b) Let $f_n = \sum_{k=1}^{n} g_k$, $g = \sum_{k=1}^{\infty} |g_k|$. Then $f_n \in L^1$, $|f_n| \leq g \in L^1$ and $f_n \to \sum_{k=1}^{\infty} g_k$ (which converges a.e.). By DCT, $\sum_{k=1}^{\infty} g_k \in L^1$ and $\int \sum_{k=1}^{\infty} g_k = \lim_{n\to\infty} \int f_n = \lim_{n\to\infty} \sum_{k=1}^{n} \int g_k = \sum_{k=1}^{\infty} \int g_k$. $\square$

9. Measurable functions

Definition 9.1 $f : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty)$ is **measurable** if there exists a sequence of step functions ϕ_n with $\phi_n \to f$ *a.e.*

Remarks So $f \in \mathcal{M}^-$ or $-f \in \mathcal{M}^- \Rightarrow f$ measurable; [in fact if f is measurable $\exists g, h \in \mathcal{M}^-$ *s.t.* $f = g - h$.]

Obviously $f \in L^1 \Rightarrow f$ measurable.

Proposition 9.2 If f is measurable and $\exists g \in L^1$ such that $|f(x)| \leq g(x)$ *a.e.*, then $f \in L^1$.

Proof Let ϕ_n be step functions with $\phi_n \to f$ *a.e.* Let $h_n = \text{mid}\{-g, \phi_n, g\}$, which (ex.) belongs to L^1. Now $|h_n| \leq g$, $h_n \to f$ *a.e.* By DCT, $f \in L^1$. $\square$

Corollary 9.3 f measurable, $|f| \in L^1 \Rightarrow f \in L^1$.

Proposition 9.4 Let f, g be measurable. Then $|f|$, $f \wedge g$, $f \vee g$ are measurable, and, if $f(x) \neq 0$ *a.e.*, $\dfrac{1}{f}$ is measurable.

Proof If ϕ_n, ψ_n are step functions with $\phi_n \to f$ *a.e.*, $\psi_n \to g$ *a.e.*, then $|\phi_n|$, $\phi_n \wedge \psi_n$, $\phi_n \vee \psi_n$ are step functions converging a.e. to $|f|$, $f \wedge g$, $f \vee g$ respectively, and θ_n defined by

$$\theta_n(x) = \begin{cases} 1/\phi_n(x) & \text{if } \phi_n(x) \neq 0 \text{ and } |x| \leq N \\ 0 & \text{otherwise} \end{cases}$$

is a step function converging a.e. to $f(x)$ if $f(x) \neq 0$ *a.e.*

Proposition 9.5 If f_n is measurable and $f_n \to f$ *a.e.*, then f is measurable.

Proof Let $q(x) = e^{-|x|}$; $q \in L^1$, $q(x) > 0$ all x. Let $g_n = \dfrac{f_n}{(1+|f_n|)} q$: g_n measurable, $|g_n| \leq q \in L^1$. So by Proposition 9.2, $g_n \in L^1$. Now the DCT implies $g = \lim_{n\to\infty} g_n = \dfrac{f}{(1+|f|)} q \in L^1$. Noting that $|g| < q$ *a.e.*, since $f = \dfrac{g}{q - |g|}$, f is measurable by Prop.9.4. □