On a class of nonlinear wave equations related to the Dirac-Klein-Gordon system with generalized Yukawa interaction

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1 Introduction

In this paper we study low regularity local solutions of nonlinear wave equations of the form

$$\Box u = u^k Du \tag{1}$$

where Du stands for any of the first order derivatives $\partial_t u$, $\partial_{x_j} u$, $j = 1, 2, 3$, with initial data

$$u(0, \cdot) = f \in H^s\left(\mathbb{R}^3\right) \ , \ \ u_t(0, \cdot) = g \in H^{s-1}\left(\mathbb{R}^3\right) \tag{2}$$

Our motivation is the following Dirac-Klein-Gordon system with generalized Yukawa interaction.

$$\mathcal{D}\psi = \phi \left(\overline{\psi}\psi\right)^{\alpha} \psi \tag{3a}$$

$$\Box\phi = \left(\overline{\psi}\psi\right)^{\beta} \tag{3b}$$

Applying the Dirac operator on both sides of (3a) we obtain a system of nonlinear wave equations with nonlinearities in which the main terms are of the form $u^k Du$.

Wave equations of the form

$$\Box u = u^k \left(Du\right)^l \ , \ u(0, \cdot) = f \in H^s\left(\mathbb{R}^3\right) \ , \ u_t(0, \cdot) = g \in H^{s-1}\left(\mathbb{R}^3\right) \tag{4}$$

where k and l are positive integers, $l \geq 2$ and $s > \max\left\{2, \frac{5l-7}{2l-2}\right\}$ have been studied by Ponce and Sideris in [15]. The most usefull case for the Dirac-Klein-Gordon equations is $l = 1$, but unfortunately the proof in [15] works only for $l \geq 2$. In this paper we show how to deal with the case $l = 1$ using an observation of Klainerman and a Strichartz-type estimate due to Escobedo and Vega.

Applying the generalized energy estimate to (4) leads to a term of the form $\int_0^T \|Du(t,x)\|_{L^\infty_x}^{l-1} dt$. If we use the Sobolev inequality to estimate that L^∞ norm by $\|u(t,\cdot)\|_{H^s}$ then we must have $s > \frac{5}{2}$. This is the content of the classical local existence theorem for (4). Klainerman has observed in [8] that for $l = 1$ it is possible to avoid the L^∞ norm of the derivative and manage with estimating only a term of the form $\int_0^T \|u(t,x)\|^r_{L^\infty_x}\,dt$. Classical methods require $s > \frac{3}{2}$, but a suitable Strichartz estimate will gain $1/2$ derivatives, hence the restriction $s > 1$.

The Strichartz estimate we need arises from the following question. Let ϕ be a solution of

$$\Box\phi = 0\ ,\ \ \phi(0,\cdot) = f \in H^s\left(\mathbb{R}^3\right)\ ,\ \ \phi_t(t,\cdot) = g \in H^{s-1}\left(\mathbb{R}^3\right) \tag{5}$$

in three space dimensions. How large should s be so that mixed norms of the form $\left(\int \|u(t,x)\|^r_{L^\infty_x}\,dt\right)^{1/r}$ can be controlled by the quantity $\|f\|_{H^s} + \|g\|_{H^{s-1}}$? Klainerman and Machedon in [9] have shown that if $s = 1$ the estimate

$$\left(\int_0^\infty \|u(t,x)\|^2_{L^\infty_x}\,dt\right)^{1/2} \le C\left[\|f\|_{H^1} + \|g\|_{L^2}\right] \tag{6}$$

fails (it is true for spherically symmetric data)(see also [14]). However, when $s > 1$ we gain integrability in time and we can estimate a mixed norm $\left(\int \|u(t,x)\|^r_{L^\infty_x}\,dt\right)^{1/r}$ where r is allowed to take values strictly larger than 2. See Lemma 2.2 for a precise formulation.

The Dirac-Klein-Gordon system (3) with generalized Yukawa interaction has been studied by Chadam in [2], and Reed in [16], and more recently by Machihara in [12] and Machihara, Nakamura and Ozawa in [13] where global solutions are constructed under the assumption that the initial data are small. In this paper we study local low regularity solutions without any smallness assumption on the data.

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2 Strichartz Estimates

In this section we collect the Strichartz-type estimates we are going to need in our proofs.

Lemma 2.1.

1. *Let $T > 0$ and let u be the solution of:*

$$\Box u = F\ ,\ \ u(0,\cdot) = f\ ,\ \ u_t(0,\cdot) = g. \tag{7}$$

Then for any $p \in (6,\infty)$,

$$\left(\int_0^T \|u\|^{\frac{2p}{p-6}}_{L^p}\,dt\right)^{\frac{p-6}{2p}} \le C\left[\|f\|_{H^1} + \|g\|_{L^2} + \int_0^T \|F(t,\cdot)\|_{L^2}\,dt\right] \tag{8}$$

where C is independent of T.

2. *If u solves (7) then,*

   (a) *for any $p \in (6,\infty)$, $T>0$, $s \geq 1$,*

$$\left(\int_0^T \|u\|_{H_p^{s-1}}^{\frac{2p}{p-6}}\,dt\right)^{\frac{p-6}{2p}} \leq C\left[D_s + \int_0^T \|F\|_{H^{s-1}(\mathbb{R}^3)}\,dt\right] \tag{9}$$

   where $D_s = \|f\|_{H^s(\mathbb{R}^3)} + \|g\|_{H^{s-1}(\mathbb{R}^3)}$ and C is independent of T.

   (b) *for any $p \in (6,\infty)$, $s \geq 1$,*

$$\left(\int_0^\infty \|u\|_{H_p^{s-1}}^{\frac{2p}{p-6}}\,dt\right)^{\frac{p-6}{2p}} \leq C\left[D_s + \int_0^\infty \|F\|_{H^{s-1}(\mathbb{R}^3)}\,dt\right] \tag{10}$$

   with D_s as above.

Proof. Estimates (9) and (10) follow easily from (8). Estimate (8) is well known, see for example [10, 11]. □

The next estimate is due to Escobedo and Vega [5]. A special case appeared earlier as Lemma 9 of [1].

Lemma 2.2. *Let u be the solution of:*

$$\Box u = F\ ,\ u(0,\cdot) = f\ ,\ u_t(0,\cdot) = g$$

1. *For any $s \in \left(1,\frac{3}{2}\right)$, $\lambda \in \left(2,\frac{1}{\frac{3}{2}-s}\right)$, $T>0$, there exist $\theta>0$ and $C = C(s,\lambda) > 0$ such that*

$$\left(\int_0^T \|u\|_{L_x^\infty}^{\lambda}\,dt\right)^{1/\lambda} \leq CT^\theta\left[D_s + \int_0^T \|F\|_{H^{s-1}(\mathbb{R}^3)}\,dt\right] \tag{11}$$

   where $D_s = \|f\|_{H^s(\mathbb{R}^3)} + \|g\|_{H^{s-1}(\mathbb{R}^3)}$.

2. *(endpoint estimate) Let $s \in \left(1,\frac{3}{2}\right)$ and set $\lambda = \frac{1}{\frac{3}{2}-s}$. Let I denote either $(0,\infty)$ or an interval of the form $(0,T)$ with $T \in (0,\infty)$. Then there is a positive constant C, independent of T, such that*

$$\left(\int_I \|u(t,\cdot)\|_{L^\infty(\mathbb{R}^3)}^{\lambda}\,dt\right)^{1/\lambda} \leq C\left[D_s + \int_I \|F\|_{H^{s-1}(\mathbb{R}^3)}\,dt\right] \tag{12}$$

   where D_s is as above.

Proof. See [5]. □

In addition to these Strichartz estimates we shall use the following 'fractional Leibniz rules'.

Lemma 2.3. *Suppose $s \geq 0$, $b, c \in [1, \infty]$ $a, d, p \in (1, \infty)$ with $\frac{1}{p} = \frac{1}{a} + \frac{1}{b} = \frac{1}{c} + \frac{1}{d}$. If f, g are elements of the spaces indicated in the right-hand side of* (13) *then*

$$\|fg\|_{H^s_p} \leq C \left[\|f\|_{H^s_a} \|g\|_{L^b} + \|f\|_{L^c} \|g\|_{H^s_d} \right] \tag{13a}$$

If k is a positive integer, then

$$\left\| f^k \right\|_{H^s_p} \leq C \|f\|^{k-1}_{L^\infty} \|f\|_{H^s_p} \tag{13b}$$

Proof. For (13a) see [6] and [7]. Estimate (13b) follows easily from (13a). □

3 The nonlinear wave equation $\Box u = u^k Du$

3.1 The subcritical case

In this Section we study the subcritical case $s > s_{cr} = \frac{3}{2} - \frac{1}{k}$. In order to be able to apply Lemma 2.2 we assume $k \geq 2$ so that $s > 1$.

Theorem 3.1. *Let $f \in H^s(\mathbb{R}^3)$, $g \in H^{s-1}(\mathbb{R}^3)$, where $s_{cr} = \frac{3}{2} - \frac{1}{k} < s < \frac{3}{2}$ and $k \geq 2$ is an integer. Fix $\lambda \in \left(k, \frac{1}{\frac{3}{2}-s}\right)$ and let $p = \frac{3}{s-1}$. Let Du denote any one of the first order derivatives $\partial_t u$, $\partial_{x_j} u, j = 1, 2, 3$. Then there is a $T > 0$, depending only on the quantity $D_s = \|f\|_{H^s(\mathbb{R}^3)} + \|g\|_{H^{s-1}(\mathbb{R}^3)}$, such that the Cauchy problem*

$$\Box u = u^k Du \ , \ u(0, \cdot) = f \ , \ u_t(0, \cdot) = g$$

has a unique solution u with

$$u \in C^0\left([0, T], H^s\left(\mathbb{R}^3\right)\right) \cap C^1\left([0, T], H^{s-1}\left(\mathbb{R}^3\right)\right)$$

and

$$\left(\int_0^T \|u\|^\lambda_{L^\infty_x} dt\right)^{\frac{1}{\lambda}} + \left(\int_0^T \|u\|^{\frac{2p}{p-6}}_{H^{s-1}_p(\mathbb{R}^3)} dt\right)^{\frac{p-6}{2p}} < \infty.$$

Proof. Fix two constants $T \in (0, 1)$ and $M > 0$ to be determined in the course of the proof. Define

$$X = \left\{ u \in C^0\left([0, T]; H^s\left(\mathbb{R}^3\right)\right) \cap C^1\left([0, T]; H^{s-1}\left(\mathbb{R}^3\right)\right) \ : \ |||u||| \leq M \right\}$$

where

$$|||u||| = E_s(u) + \left(\int_0^T \|u\|^\lambda_{L^\infty_x} dt\right)^{1/\lambda} + \left(\int_0^T \|u\|^{\frac{2p}{p-6}}_{H^{s-1}_p(\mathbb{R}^3)} dt\right)^{\frac{p-6}{2p}},$$

$$E_s(u) = \sup_{0 \leq t \leq T} \left[\|u(t, \cdot)\|_{H^s(\mathbb{R}^3)} + \|u_t(t, \cdot)\|_{H^{s-1}(\mathbb{R}^3)} \right],$$

λ is a fixed number in $\left(k, \frac{1}{\frac{3}{2}-s}\right)$ and $p = \frac{3}{s-1}$. Equiped with this norm X is complete. Consider the map $\mathcal{F} : X \to X$ defined as follows: Given $u \in X$, $v = \mathcal{F}(u)$ is the unique solution of the Cauchy problem,

$$\Box v = u^k Du \ , \ v(0, \cdot) = f \ , \ v_t(0, \cdot) = g \ .$$

We first need to prove that $\mathcal{F}$ maps X into X. So let $u \in X$ and $v = \mathcal{F}(u)$. The energy estimate and the Strichartz estimates (9) and (11) give

$$|||v||| \leq C \left[D_s + \int_0^T \left\| u^k Du \right\|_{H^{s-1}(\mathbb{R}^3)} dt \right] . \tag{14}$$

Using the Leibniz rules of Lemma 2.3 we get

$$\begin{aligned} \int_0^T \left\| u^k Du \right\|_{H^{s-1}} dt &\leq C \int_0^T \left\| u^k \right\|_{H_p^{s-1}} \|Du\|_{L^q} dt \\ &\quad + C \int_0^T \|u\|_{L_x^\infty}^k \|Du\|_{H^{s-1}} dt \\ &=: I + II \end{aligned} \tag{15}$$

where $p = \frac{3}{s-1}$, $q = \frac{6}{5-2s}$. By Sobolev embedding,

$$\|Du(t, \cdot)\|_{L^q} \leq C \|Du(t, \cdot)\|_{H^{s-1}} \leq C E_s(u) \leq C|||u|||$$

and using (13b),

$$\left\| u^k(t, \cdot) \right\|_{H_p^{s-1}} \leq C \|u(t, \cdot)\|_{L^\infty}^{k-1} \|u(t, \cdot)\|_{H_p^{s-1}}$$

therefore

$$I \leq C|||u||| \int_0^T \|u\|_{L_x^\infty}^{k-1} \|u\|_{H_p^{s-1}} dt$$

Observe that $\frac{k-1}{\lambda} + \frac{p-6}{2p} < \frac{k-1}{k} + \frac{p-6}{2p} = 1 - (s - s_{cr}) < 1$. Therefore there is an r such that $\frac{k-1}{\lambda} + \frac{p-6}{2p} + \frac{1}{r} = 1$. Then,

$$\begin{aligned} I &\leq C|||u|||T^{\frac{1}{r}} \left(\int_0^T \|u\|_{L_x^\infty}^\lambda dt \right)^{\frac{k-1}{\lambda}} \left(\int_0^T \|u\|_{H_p^{s-1}}^{\frac{2p}{p-6}} dt \right)^{\frac{p-6}{2p}} \\ &\leq C|||u|||T^{\frac{1}{r}} |||u|||^{k-1} |||u||| \\ &\leq C|||u|||^{k+1} T^{\frac{1}{r}} \end{aligned}$$

To estimate II use once more, $\|Du(t,\cdot)\|_{H^{s-1}} \le CE_s(u) \le C|||u|||$, to obtain

$$\begin{aligned} II &\le CE_s(u) \int_0^T \|u\|_{L_x^\infty}^k \, dt \\ &\le C|||u||| \left(\int_0^T \|u\|_{L_x^\infty}^\lambda \, dt \right)^{k/\lambda} T^{1-\frac{k}{\lambda}} \\ &\le C|||u|||^{k+1} T^{1-\frac{k}{\lambda}} \end{aligned}$$

We conclude that

$$\int_0^T \left\| u^k Du \right\|_{H^{s-1}} dt \le C|||u|||^{k+1} T^\sigma \tag{16}$$

where $\sigma = \min\left\{\frac{1}{r}, 1 - \frac{k}{\lambda}\right\} > 0$. Using (16) into (14) we obtain

$$|||v||| \le CD_s + C|||u|||^{k+1} T^\sigma \tag{17}$$

Now define $M = 2CD_s$ and choose T small enough so that $CM^kT^\sigma \le \frac{1}{2}$. Since $u \in X$ we have $|||u||| \le M$, therefore,

$$|||v||| \le CD_s + CM^{k+1}T^\sigma \le \frac{M}{2} + \frac{M}{2} = M \ .$$

Since $\Box v = u^k Du \in L^1([0,T]; H^{s-1})$ (see (16)) it follows from linear theory that $v \in C^0\left([0,T], H^s\left(\mathbb{R}^3\right)\right) \cap C^1\left([0,T], H^{s-1}\left(\mathbb{R}^3\right)\right)$. This completes the proof that $v \in X$ whenever $u \in X$.

Next, we need to show that $\mathcal{F}$ defines a contraction on X. Let $u_1, u_2 \in X$ and set $v_1 = \mathcal{F}(u_1)$, $v_2 = \mathcal{F}(u_2)$. An argument similar to the one we have just presented shows that

$$|||v_1 - v_2||| \le C(M)T^\theta |||u_1 - u_2|||$$

where $C(M)$ is a constant depending only on $M = 2CD_s$ and θ is some positive number. Thus, if $T < \left(\frac{1}{C(M)}\right)^{1/\theta} = \left(\frac{1}{C'(D_s)}\right)^{1/\theta}$ then $L = C(M)T^\theta < 1$ as required. Uniqueness follows along the same lines. Observe that the T we have used depends only on the quantity D_s.

□

3.2 The critical case

We now consider the critical case $s = s_{cr} = \frac{3}{2} - \frac{1}{k} > 1$. We use the endpoint estimate (12) in Lemma 2.2.

Theorem 3.2. *Let $k > 2$ be an integer and let $s = s_{cr} = \frac{3}{2} - \frac{1}{k}$. Let $f \in H^s\left(\mathbb{R}^3\right)$, $g \in H^{s-1}\left(\mathbb{R}^3\right)$. Let $p = \frac{3}{s-1}$.*

1. *There is a $T > 0$, depending only on f and g, and a unique solution $u \in C^0\left([0,T], H^s\left(\mathbb{R}^3\right)\right) \cap C^1\left([0,T], H^{s-1}\left(\mathbb{R}^3\right)\right)$ of*
$$\Box u = u^k Du \quad , \quad u(0,\cdot) = f \quad , \quad u_t(0,\cdot) = g$$
with $\left(\int_0^T \|u(t,\cdot)\|_{L^\infty}^k \, dt\right)^{1/k} + \left(\int_0^T \|u(t,\cdot)\|_{H_p^{s-1}}^{\frac{2p}{p-6}} \, dt\right)^{(p-6)/2p} < \infty.$

2. *If, moreover, $D_s = \|f\|_{H^s(\mathbb{R}^3)} + \|g\|_{H^{s-1}(\mathbb{R}^3)}$ is small enough, the solution exists globally in time.*

Proof. We prove part 2 first. Suppose D_s is small. The precise smallness condition on D_s will be determined in the course of the proof. Fix a positive number M, to be determined later, and define

$$X = \left\{u \in C^0\left([0,\infty), \dot{H}^s\left(\mathbb{R}^3\right)\right) \cap C^1\left([0,\infty), \dot{H}^{s-1}\left(\mathbb{R}^3\right)\right) : |||u||| \leq M\right\}$$

where

$$|||u||| = \dot{E}_1(u) + \dot{E}_s(u) + \left(\int_0^\infty \|u(t,\cdot)\|_{L^\infty}^k \, dt\right)^{\frac{1}{k}} + \left(\int_0^\infty \|u(t,\cdot)\|_{H_p^{s-1}}^{\frac{2p}{p-6}} \, dt\right)^{\frac{p-6}{2p}},$$

and $\dot{E}_s$ is the homogeneous s-energy,

$$\dot{E}_s(u) = \sup_{t\in[0,\infty)} \left[\|u(t,\cdot)\|_{\dot{H}^s(\mathbb{R}^3)} + \|u_t(t,\cdot)\|_{\dot{H}^{s-1}(\mathbb{R}^3)}\right] .$$

Note that in the critical case we have $\frac{2p}{p-6} = \frac{1}{\frac{3}{2}-s} = k$. We consider the same map $\mathcal{F} : X \to X$ as before and show first that it maps X into X. Let $u \in X$, $v = \mathcal{F}(u)$. By energy estimates, the Strichartz estimate (10) and the second part of Lemma 2.2 we have

$$|||v||| \leq C\left[D_s + \int_0^\infty \left\|u^k Du\right\|_{H^{s-1}} dt\right] \tag{18}$$

where the constant C is independent of time (we have to use the homogeneous version of the energy here because the nonhomogeneous energy estimate gives a constant which depends on T). We estimate the last term in the right-hand side of (18) as follows.

$$\begin{aligned} \int_0^\infty \left\|u^k Du\right\|_{H^{s-1}} dt &\leq C \int_0^\infty \left\|u^k\right\|_{H_p^{s-1}} \|Du\|_{L^q} \, dt \\ &\quad + C \int_0^\infty \left\|u^k\right\|_{L^\infty} \|Du\|_{H^{s-1}} \, dt \\ &=: A + B \end{aligned} \tag{19}$$

where $p = \frac{3}{s-1}$ and $q = \frac{6}{5-2s}$. To estimate the integral A observe that $\|Du\|_{L^q} \le C\,\|Du\|_{\dot{H}^{s-1}} \le C\dot{E}_s(u)$. Using this, the Leibniz rule (13b) and the fact that $\frac{k-1}{k} + \frac{p-6}{2p} = 1 - (s - s_{cr}) = 1$ we have,

$$
\begin{aligned}
A &\le C \int_0^\infty \|u\|_{L^\infty_x}^{k-1} \|u\|_{H_p^{s-1}} \|Du\|_{L^q}\, dt \\
&\le C\dot{E}_s(u) \left(\int_0^\infty \|u\|_{L^\infty_x}^k \, dt \right)^{\frac{k-1}{k}} \left(\int_0^\infty \|u\|_{H_p^{s-1}}^{\frac{2p}{p-6}}\, dt \right)^{\frac{p-6}{2p}} \\
&\le C|||u|||^{k+1}
\end{aligned}
$$

We estimate B as follows:

$$
\begin{aligned}
B &\le C \left(\int_0^\infty \|u\|_{L^\infty_x}^k \, dt \right) \sup_t \|Du(t,\cdot)\|_{H^{s-1}} \\
&\le C|||u|||^k \left[\dot{E}_1(u) + \dot{E}_s(u) \right] \\
&\le C|||u|||^{k+1}
\end{aligned}
$$

Therefore

$$
\int_0^\infty \left\| u^k Du \right\|_{H^{s-1}} dt \le C|||u|||^{k+1} \tag{20}
$$

and using this into (18) we obtain:

$$
|||v||| \le C_1 D_s + C_2 |||u|||^{k+1} \tag{21}
$$

where C_1 and C_2 are absolute constants. Since $u \in X$ we have $|||u||| \le M$, therefore $|||v||| \le C_1 D_s + C_2 M^{k+1}$. Choose $M = 2C_1 D_s$ and make the smallness assumtion $C_2 M^{k+1} \le \frac{M}{2}$ (equivalently: $D_s \le \frac{1}{2C_1 (2C_2)^{1/k}}$) to get $|||v||| \le \frac{M}{2} + \frac{M}{2} = M$. Thus $v \in X$ whenever $u \in X$. Similarly one can show that $\mathcal{F}$ is a contraction and thus construct a solution u in the space X. Since $\Box u = u^k Du \in L^1([0,\infty), H^{s-1})$, linear theory guarantees that actually $u \in C^0([0,\infty), H^s) \cap C^1([0,\infty), H^{s-1})$.

We turn now to part 1. In this case we can't use the approach of the proof of Theorem 3.1, because it gives an estimate similar to (17) but with $\sigma = 0$ (this is due to the fact that $s = s_{cr}$). Neither can we use the approach of part 2 that we have just discussed because D_s is no longer assumed to be small. To come up with a small quantity we introduce the modified s-energy E_s^m. Given any $\phi \in C^0\left([0,T]; H^s\left(\mathbb{R}^3\right)\right) \cap C^1\left([0,T]; H^{s-1}\left(\mathbb{R}^3\right)\right)$ we define

$$
E_s^m(\phi) = \sup_{0 \le t \le T} \left[\|\phi(t,\cdot) - f\|_{H^s(\mathbb{R}^3)} + \|\partial_t \phi(t,\cdot) - g\|_{H^{s-1}(\mathbb{R}^3)} \right] \tag{22}
$$

Observe that $E_s^m(\phi) \le E_s(\phi - \psi) + E_s^m(\psi)$, $E_s \le E_s^m + D_s$ and $E_s^m \le E_s + D_s$, where E_s is the standard s-energy,

$$
E_s(u) = \sup_{0 \le t \le T} \left[\|u(t,\cdot)\|_{H^s(\mathbb{R}^3)} + \|u_t(t,\cdot)\|_{H^{s-1}(\mathbb{R}^3)} \right]
$$

and $D_s = \|f\|_{H^s} + \|g\|_{H^{s-1}}$. We also introduce the solution w of the homogeneous Cauchy problem

$$\Box w = 0 \ , \ w(0,x) = f(x) \ , \ w_t(x,0) = g(x)$$

By linear theory,

$$E_s^m(w) \to 0 \quad , \quad \text{as} \ \ T \to 0 \ . \tag{23}$$

By Strichartz, $\int_0^\infty \|w\|_{H_p^{s-1}}^{\frac{2p}{p-6}} \, dt < \infty$, therefore,

$$\left(\int_0^T \|w\|_{H_p^{s-1}}^{\frac{2p}{p-6}} \, dt \right)^{(p-6)/2p} \to 0 \ , \quad \text{as} \ \ T \to 0. \tag{24a}$$

By (12), $\int_0^\infty \|w(t,\cdot)\|_{L_x^\infty}^k \, dt < \infty$, therefore,

$$\left(\int_0^T \|w(t,\cdot)\|_{L_x^\infty}^k \, dt \right)^{1/k} \to 0 \ , \quad \text{as} \ \ T \to 0. \tag{24b}$$

Now fix two positive constants $T \in (0,1)$ and M, to be determined later, and define

$$X = \left\{ u \in C^0 \left([0,T]; H^s \left(\mathbb{R}^3 \right) \right) \cap C^1 \left([0,T]; H^{s-1} \left(\mathbb{R}^3 \right) \right) : |u| \leq M \right\}$$

where

$$|u| = E_s^m(u) + \left(\int_0^T \|u(t,\cdot)\|_{L^\infty}^k \, dt \right)^{\frac{1}{k}} + \left(\int_0^T \|u(t,\cdot)\|_{H_p^{s-1}}^{\frac{2p}{p-6}} \, dt \right)^{\frac{p-6}{2p}} ,$$

and $p = \frac{3}{s-1}$ (hence $\frac{2p}{p-6} = \frac{1}{\frac{3}{2}-s} = k$). Note that $|\cdot|$ is not a norm. We equip X as usual with

$$|||u||| = E_s(u) + \left(\int_0^T \|u(t,\cdot)\|_{L^\infty}^k \, dt \right)^{1/k} + \left(\int_0^T \|u(t,\cdot)\|_{H_p^{s-1}}^{\frac{2p}{p-6}} \, dt \right)^{\frac{p-6}{2p}} .$$

Then $(X, |||\cdot|||)$ is complete. Let $\mathcal{F}$ be defined as usual. We show first that $\mathcal{F}$ maps X into X.

Let $u \in X$ and let $v = \mathcal{F}(u)$. We wish to show that $|v| \leq M$. Using

$$E_s^m(v) \leq E_s^m(w) + E_s(v-w)$$

we obtain

$$|v| \leq |w| + |||v-w||| \tag{25}$$

Observe that (23) and (24) give

$$|w| \to 0 \ \ \text{as} \ \ T \to 0. \tag{26}$$

As before,

$$|||v-w||| \le C\left[\|(v-w)(0)\|_{H^s} + \|(v-w)_t(0)\|_{H^{s-1}} + \int_0^T \|\Box(v-w)\|_{H^{s-1}}\,dt\right]$$

However $(v-w)(0) = (v-w)_t(0) = 0$ and $\Box w = 0$ therefore

$$|||v-w||| \le \int_0^T \|\Box v\|_{H^{s-1}}\,dt = \int_0^T \left\|u^k Du\right\|_{H^{s-1}}\,dt$$

hence

$$|v| \le |w| + \int_0^T \left\|u^k Du\right\|_{H^{s-1}}\,dt \tag{27}$$

Estimates similar to the ones we have employed earlier show that

$$\int_0^T \left\|u^k Du\right\|_{H^{s-1}}\,dt \le C\left(|u| + D_s\right)|u|^k \tag{28}$$

hence

$$|v| \le |w| + C\left(|u| + D_s\right)|u|^k \le |w| + C\left(M + D_s\right)M^k$$

We can choose T small enough so that $C\left(2|w| + D_s\right)\left(2|w|\right)^{k-1} \le \frac{1}{2}$ (thanks to (26)). Set $M = 2|w|$. Then $C(M+D_s)M^k \le \frac{M}{2}$. With these choices of T and M we obtain $|v| \le \frac{M}{2} + \frac{M}{2} = M$. The rest of the proof proceeds as usual.

□

Remarks: It is clear from our proofs that both in Theorem 3.1 and in Theorem 3.2 we can replace the right hand side by a linear combination of terms of the form $u^k Du$. Moreover, both theorems are true for systems of the form $\Box u_i = c^{i\alpha\mu\nu} u_\alpha^k \partial_\mu u_\nu$, where the $c^{i\alpha\mu\nu}$ are constants.

We have assumed k to be an integer for the sake of simplicity. With suitable modifications our proofs can be made to work for non-integral values of k too.

The restriction $s < \frac{3}{2}$ is not essential and can easily be removed.

3.3 The Dirac-Klein-Gordon equations with generalized Yukawa interaction

The Dirac-Klein-Gordon equations with generalized Yukawa interaction are:

$$\mathcal{D}\psi = \phi\left(\overline{\psi}\psi\right)^\alpha \psi \tag{29a}$$

$$\Box\phi = \left(\overline{\psi}\psi\right)^\beta \tag{29b}$$

where ϕ is a scalar field, ψ is a 4-spinor field $\mathcal{D} = i\gamma^\mu\partial_\mu$ is the Dirac operator, γ^μ are the Dirac matrices, $\overline{\psi}\psi = |\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2$, and α and β are positive integers. As a consequence of $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I$, where $(g^{\mu\nu})$ is the Minkowski metric, we have $\mathcal{D}^2 = \Box$. We have taken the mass to be zero for simplicity. The mass terms can easily be incorporated in local existence theorems. They are vital only for global existence results. We use the standard summation notation with Greek indices summed from 0 to 3 and Latin indices summed from 1 to 3.

If we apply the Dirac operator to both sides of (29a) we obtain a wave equation of the form:

$$\Box\psi = \partial_\mu\phi\gamma^\mu F(\psi,\overline{\psi}) + F^\mu(\phi,\psi,\overline{\psi})\partial_\mu\psi \tag{30a}$$

where $F(\psi,\overline{\psi}) = i\left(\overline{\psi}\psi\right)^\alpha\psi$ and $F^\mu(\phi,\psi,\overline{\psi})$ are homogeneous polynomials of degree $2\alpha+1$. The method we used earlier in the paper can be used to study this equation coupled with the following generalization of (29b):

$$\Box\phi = G(\psi,\overline{\psi}) \tag{30b}$$

where G is a smooth real valued map which 'behaves like' $|\psi|^{2\beta}$. We prescribe initial data

$$\phi(0,\cdot) = \phi_0\ ,\ \phi_t(0,\cdot) = \phi_1\ ,\ \psi(0,\cdot) = \psi_0 \tag{30c}$$

Equations (30) form a system of non-linear wave equations with nonlinearities which are linear combinations of terms of the form u^kDu and powers u^q, schematically $\Box u = u^kDu + u^q$. The nonlinearities of the form u^kDu were studied above in part 3. Now we have to make sure that the method we used there can accommodate the extra terms of the form u^q.

Theorem 3.3. *Let k and q be positive integers with $q \le 1+2k$. Let G be a smooth function with $|G(u)| \le C|u|^q$ and $|G'(u)| \le C|u|^{q-1}$. Let D stand for any of the first order derivatives ∂_t , ∂_{x_j} , $j = 1,2,3$. Consider initial data $f \in H^s(\mathbb{R}^3)$, $g \in H^{s-1}(\mathbb{R}^3)$.*

1. *(subcritical case) Suppose $k \ge 2$ and $s > \frac{3}{2} - \frac{1}{k}$. Fix $\lambda \in \left(k, \frac{1}{\frac{3}{2}-s}\right)$ and let $p = \frac{3}{s-1}$. Then there is a $T > 0$, depending only on the quantity $D_s = \|f\|_{H^s} + \|g\|_{H^{s-1}}$, and a unique solution u of the Cauchy problem*

$$\Box u = u^kDu + G(u)\ ,\ u(0,\cdot) = f\ ,\ u_t(0,\cdot) = g \tag{31}$$

with

$$u \in C^0\left([0,T]; H^s(\mathbb{R}^3)\right) \cap C^1\left([0,T]; H^{s-1}(\mathbb{R}^3)\right)$$

$$\left(\int_0^T \|u\|_{L_x^\infty}^\lambda\, dt\right)^{\frac{1}{\lambda}} + \left(\int_0^T \|u\|_{H_p^{s-1}}^{\frac{2p}{p-6}}\, dt\right)^{\frac{p-6}{2p}} < \infty$$

2. *(critical case) Suppose $k > 2$ and $s = \frac{3}{2} - \frac{1}{k}$. Let $p = \frac{3}{s-1}$. Then there is a $T > 0$, depending only on the initial data f and g, and a unique solution u of the Cauchy problem*

$$\Box u = u^kDu + G(u)\ ,\ u(0,\cdot) = f\ ,\ u_t(0,\cdot) = g \tag{32}$$

with

$$u \in C^0\left([0,T]; H^s\left(\mathbb{R}^3\right)\right) \cap C^1\left([0,T]; H^{s-1}\left(\mathbb{R}^3\right)\right)$$

$$\left(\int_0^T \|u\|_{L_x^\infty}^k \, dt\right)^{\frac{1}{k}} + \left(\int_0^T \|u\|_{H_p^{s-1}}^{\frac{2p}{p-6}} \, dt\right)^{\frac{p-6}{2p}} < \infty$$

If, moreover, $q = 1 + 2k$ and $D_s = \|f\|_{H^s} + \|g\|_{H^{s-1}}$ is small, then the solution exists globally in time.

Proof.
1. We only deal with the case $q = 1 + 2k$. Smaller values of q are easier to handle. We define $(X, |||\cdot|||)$ and $\mathcal{F}$ as in the proof of Theorem 3.1. Then (14) becomes

$$|||v||| \leq C\left[D_s + \int_0^T \left\|u^k Du\right\|_{H^{s-1}} dt + \int_0^T \|G(u)\|_{H^{s-1}} dt\right] \tag{33}$$

We already know from (16) that

$$\int_0^T \left\|u^k Du\right\|_{H^{s-1}} dt \leq C|||u|||^{k+1} T^\sigma \tag{34}$$

for some positive σ. The new term to be estimated is $\int_0^T \|G(u)\|_{H^{s-1}} dt$. We have

$$\int_0^T \|G(u)\|_{H^{s-1}} dt \leq \int_0^T \|G(u)\|_{L^2} dt + \int_0^T \|G(u)\|_{\dot{H}^{s-1}} dt := I + II$$

To estimate II choose μ with $k < \mu < \min\{2k, \lambda\}$ and proceed as follows:

$$\begin{aligned}
II &\leq C \int_0^T \|G'(u)\|_{L^3} \|u\|_{\dot{H}_6^{s-1}} dt \\
&\leq C \int_0^T \left\||u|^{q-1}\right\|_{L^3} \|u\|_{H^s} dt \\
&\leq C E_s(u) \int_0^T \||u|^\mu\|_{L^\infty} \left\||u|^{q-1-\mu}\right\|_{L^3} dt \\
&\leq C|||u||| \sup_{0 \leq t \leq T} \|u\|_{L^{3(q-1-\mu)}}^{q-1-\mu} \int_0^T \|u\|_{L^\infty}^\mu dt
\end{aligned}$$

where we have used the 'fractional chain rule' of Proposition 25, p. 48 of [3] in the first line and the Sobolev embedding theorem $H^s\left(\mathbb{R}^3\right) \hookrightarrow H_6^{s-1}\left(\mathbb{R}^3\right)$ in the second line. Observe now that $3(q-1-\mu) = 3(2k-\mu) < 3k < \frac{6}{3-2s}$ therefore

$\|u\|_{L^{3(q-1-\mu)}} \leq C\|u\|_{H^s} \leq C|||u|||$. Thus,

$$
\begin{aligned}
II &\leq C|||u|||^{q-\mu} \int_0^T \|u\|_{L^\infty}^{\mu}\, dt \\
&\leq C|||u|||^{q-\mu} \left(\int_0^T \|u\|_{L^\infty}^{\lambda}\, dt \right)^{\frac{\mu}{\lambda}} T^{1-\frac{\mu}{\lambda}} \\
&\leq C|||u|||^{q} T^{1-\frac{\mu}{\lambda}}
\end{aligned}
$$

To estimate I choose ν such that $\nu < \lambda$, $\nu < q$ and $2(q-\nu) \leq \frac{6}{3-2s}$. Then

$$
\begin{aligned}
I &\leq C \int_0^T \|u\|_{L^\infty}^{\nu} \|u\|_{L^{2(q-\nu)}}^{q-\nu}\, dt \\
&\leq C \sup_{0\leq t\leq T} \|u(t,\cdot)\|_{L^{2(q-\nu)}}^{q-\nu} \int_0^T \|u\|_{L^\infty}^{\nu}\, dt \\
&\leq C E_s(u)^{q-\nu} \left(\int_0^T \|u\|_{L^\infty}^{\lambda}\, dt \right)^{\frac{\nu}{\lambda}} T^{1-\frac{\nu}{\lambda}} \\
&\leq C|||u|||^{q} T^{1-\frac{\nu}{\lambda}}
\end{aligned}
$$

Therefore

$$
\int_0^T \|G(u)\|_{H^{s-1}}\, dt \leq C|||u|||^{q} T^{\delta} \tag{35}
$$

for some positive δ. From (33), (34) and (35),

$$
|||v||| \leq C\left[D_s + |||u|||^{k+1} T^{\sigma} + |||u|||^{q} T^{\delta}\right]
$$

The rest of the proof is similar to that of Theorem 3.1.

2. Again we consider only the largest value of q, $q = 1+2k$. We define X, $\mathcal{F}$, $|\cdot|$ and $|||\cdot|||$ as in the proof of part 1 of Theorem 3.2. Let $u \in X$ and set $v = \mathcal{F}(u)$. Then, as in (27),

$$
|v| \leq |w| + \int_0^T \left\|u^k Du\right\|_{H^{s-1}} dt + \int_0^T \|G(u)\|_{H^{s-1}}\, dt
$$

where w is the solution of the Cauchy problem

$$
\Box w = 0\ ,\ w(0,\cdot) = f\ ,\ w_t(0,\cdot) = g\ .
$$

The new term to be estimated here is $\int_0^T \|G(u)\|_{H^{s-1}}\, dt$. This is done as follows. Recall that $|G'(u)| \leq C|u|^{q-1} = C|u|^{2k}$. Also observe that $3k = \frac{6}{3-2s}$, therefore,

$\left\|u(t,\cdot)^k\right\|_{L^3} = \|u(t,\cdot)\|_{L^{3k}}^k \leq C\,\|u(t,\cdot)\|_{H^s}^k \leq E_s(u)^k$, hence,

$$\begin{aligned}
\int_0^T \|G(u)\|_{H^{s-1}}\,dt &\leq C\int_0^T \|G'(u)\|_{L^3}\,\|u\|_{H^{s-1}_6}\,dt \\
&\leq CE_s(u)\int_0^T \left\|u^k\right\|_{L^\infty}\left\|u^k\right\|_{L^3}dt \\
&\leq CE_s(u)^{k+1}\int_0^T \|u\|_{L^\infty}^k\,dt \\
&\leq C\left(E_s^m(u)+D_s\right)^{k+1}|u|^k \\
&\leq C\left(|u|+D_s\right)^{k+1}|u|^k \qquad (36)
\end{aligned}$$

and we continue as in the proof of part 1 of Theorem 3.2.

If D_s is small, we follow the proof of part 2 of Theorem 3.2. The extra term now is $\int_0^\infty \|G(u)\|_{H^{s-1}}\,dt$ and it can be treated using estimates similar to the ones leading to (36). □

As a consequence of Theorem 3.3 and the discussion preceding it we obtain the following result for local low regularity solutions of the Dirac-Klein-Gordon equations with generalized Yukawa interaction.

Theorem 3.4. *Let α and β be positive integers with $2\beta \leq 4\alpha+3$. Consider initial data $\phi_0 \in H^s\left(\mathbb{R}^3 \to \mathbb{R}\right)$, $\phi_1 \in H^{s-1}\left(\mathbb{R}^3 \to \mathbb{R}\right)$, $\psi_0 \in H^s\left(\mathbb{R}^3 \to \mathbb{C}^4\right)$.*

1. *If $s > \frac{3}{2} - \frac{1}{2\alpha+1}$ then there is a $T > 0$, depending only on the quantity $D_s = \|\phi_0\|_{H^s} + \|\phi_1\|_{H^{s-1}} + \|\psi_0\|_{H^s}$, and a unique solution (ϕ,ψ) of the Cauchy problem (30) with*

$$\phi \in C^0\left([0,T];H^s\right)\cap C^1\left([0,T];H^{s-1}\right)\ ,\ \psi \in C^0\left([0,T];H^s\right)$$

2. *If $s = \frac{3}{2} - \frac{1}{2\alpha+1}$ then there is a $T > 0$, depending only on the initial data ϕ_0 , ϕ_1 and ψ_0, and a unique solution (ϕ,ψ) of the Cauchy problem (30) with*

$$\phi \in C^0\left([0,T];H^s\right)\cap C^1\left([0,T];H^{s-1}\right)\ ,\ \psi \in C^0\left([0,T];H^s\right)$$

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