ALMOST-ORTHOGONALITY IN THE SCHATTEN–VON NEUMANN CLASSES

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1. Introduction

In this note we consider almost-orthogonality in the $\mathcal{C}_p$ classes of Schatten and von Neumann, whose definition we now briefly recall. Let H be a Hilbert space. When $1 \leq p < \infty$, $\mathcal{C}_p$ is defined as the space of compact operators T on H such that if λ_j^2 is the sequence of eigenvalues of T^*T, then

$$\|T\|_p := \|\lambda_j\|_{l^p} < \infty.$$

When $p = \infty$, $\mathcal{C}_\infty$ is the space $\mathcal{L}(H)$ of all bounded linear operators on H with the usual operator norm.

The basic properties of the Schatten–von Neumann classes are set out for example in [2]. Amongst them are the following, which we shall use in what follows without further comment.

- $\mathcal{C}_p$ is a Banach space under this norm (and a Hilbert space when $p = 2$)
- $\|T\|_p = \|T^*\|_p = \|(T^*T)^{1/2}\|_p$
- If $S \geq 0$ and $pa \geq 1$, then $\|S^a\|_p = \|S\|_{pa}^a$
- Hölder's inequality: if $1 \leq p, q, r \leq \infty$, $1/r = 1/p + 1/q$, $S \in \mathcal{C}_p$ and $T \in \mathcal{C}_q$, then $ST \in \mathcal{C}_r$ and
$$\|ST\|_r \leq \|S\|_p \|T\|_q$$

Using merely the triangle inequality for $\mathcal{C}_p$, we see that if $T_j \in \mathcal{C}_p$ and $\sum_j \|T_j\|_p < \infty$, then $\sum_j T_j \in \mathcal{C}_p$ and

$$\|\sum_j T_j\|_p \leq \sum_j \|T_j\|_p.$$

In this note we shall be interested in obtaining the same conclusion $\sum_j T_j \in \mathcal{C}_p$, but we do *not* wish to make the rather strong assumption that $\sum_j \|T_j\|_p < \infty$.

A first indication of what we are aiming for is an instance of the Clarkson-McCarthy inequalities (see [2] again)

$$2(\|S\|_p^p + \|T\|_p^p) \leq \|S + T\|_p^p + \|S - T\|_p^p, \; 2 \leq p \leq \infty$$

and

$$2(\|S\|_p^p + \|T\|_p^p) \geq \|S + T\|_p^p + \|S - T\|_p^p, \; 1 \leq p \leq 2;$$

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equality holds for $p \neq 2$ if and only if $ST^* = S^*T = 0$. So if the operators S and T are *orthogonal* in the sense that $ST^* = S^*T = 0$, we have $(S+T)^*(S+T) = (S-T)^*(S-T) = S^*S + T^*T$ and so

$$\|S+T\|_p = \left(\|S\|_p^p + \|T\|_p^p\right)^{1/p},$$

which can also be seen in a variety of other ways. (Of course this is just the non-commutative analogue of the statement that if on a measure space f and g have disjoint supports, then $\|f+g\|_p = \left(\|f\|_p^p + \|g\|_p^p\right)^{1/p}$, where $\|\cdot\|_p$ denotes the L^p norm.)

Consequently, if we have a family of operators T_j such that for $j \neq k$ $T_jT_k^* = T_j^*T_k = 0$ (i.e. the operators are mutually orthogonal), then

$$\|\sum_j T_j\|_p = \left(\sum_j \|T_j\|_p^p\right)^{1/p}.$$

In this note we wish to examine what happens if we do not have *exact* mutual orthogonality of the operators T_j, but only some "almost-orthogonality". For example, what if we just know that the sizes of $T_jT_k^*$ and $T_j^*T_k$ decay at some reasonable rate as $|j-k| \to \infty$? In the commutative case, if $f_j \in L^p$ are not necessarily disjointly supported, but nevertheless "most of the mass" of f_j lives far from where "most of the mass" of the other f_k's lives, can we deduce that $\sum_j f_j$ belongs to L^p?

There are a few cases where we can give a satisfactory answer to this question more or less directly. The first is when $p = 2$. Suppose that $\mathcal{H}$ is a Hilbert space, $x_j \in \mathcal{H}$, and let β_{jk} be the cosine of the angle between x_j and x_k, i.e.

$$\langle x_j, x_k\rangle = \beta_{jk}\|x_j\|\|x_k\|.$$

Then

$$\|\sum_j x_j\|^2 = \sum_{j,k}\langle x_j, x_k\rangle = \sum_{j,k}\beta_{jk}\|x_j\|\|x_k\| \leq B\sum_j \|x_j\|^2$$

where B is the l^2-operator norm of the matrix (β_{jk}). Since when $p = 2$, $\mathcal{C}_2$ is a Hilbert space with inner product $\langle S, T\rangle = trace(T^*S)$, we conclude that $|trace\ (T_j^*T_k)| \leq \beta_{jk}\|T_j\|_2\|T_k\|_2$ implies

$$\|\sum_j T_j\|_2 \leq B^{1/2}\sum_j \left(\|T_j\|_2^2\right)^{1/2} \tag{1}$$

where B is the l^2-operator norm of the matrix (β_{jk}).

Secondly, when $p = 1$, the formulation of the problem does not admit any improvement on the trivial triangle bound $\|\sum_j T_j\|_1 \leq \sum_j \|T_j\|_1$.

Finally, when $p = \infty$, this problem has already been well studied. Let us first look at the commutative case. Suppose that f_j are functions defined on some measure space and are such that $\|f_jf_k\|_\infty \leq \gamma_{jk}^2 \sup_m \|f_m\|_\infty^2$ for some nonnegative γ_{jk}.

Then

$$\begin{aligned} (\sum_j |f_j(x)|)^2 &= \sum_{j,k} |f_j(x) f_k(x)| \\ &\leq \sum_{j,k} |f_j(x)|^{1/2} |f_k(x)|^{1/2} \gamma_{jk} \sup_m \|f_m\|_\infty \\ &\leq \Gamma \sum_j |f_j(x)| \sup_m \|f_m\|_\infty . \end{aligned}$$

where Γ is the l^2-operator norm of the matrix (γ_{jk}). Therefore

$$\| \sum_j f_j \|_\infty \leq \Gamma \sup_j \|f_j\|_\infty.$$

The fact that this result continues to hold in the non-commutative setting is the celebrated Cotlar–Stein Lemma whose elegant proof can for example be found in [3], together with a large collection of applications in harmonic analysis. Incidentally, in most applications, γ_{jk} can be taken to have exponential decay away from the diagonal.

Theorem 1. *(Cotlar–Stein Lemma). Suppose $T_j \in \mathcal{L}(H)$ satisfy*

$$\|T_j^* T_k\|_{\mathcal{L}(H)} \leq \gamma_{jk}^2 \sup_m \|T_m\|_{\mathcal{L}(H)}^2$$

and

$$\|T_j T_k^*\|_{\mathcal{L}(H)} \leq \gamma_{jk}^2 \sup_m \|T_m\|_{\mathcal{L}(H)}^2$$

for certain $\gamma_{jk} \geq 0$. If the l^2-operator norm of (γ_{jk}) (or indeed its spectral radius with respect to any sequence space) is denoted by Γ and is finite, then $\sum_j T_j$ converges in the strong operator topology and

$$\| \sum_j T_j \|_{\mathcal{L}(H)} \leq \Gamma \sup_j \|T_j\|_{\mathcal{L}(H)}.$$

Remark. This result can of course be used to give estimates for the operator norm of the operator $T \mapsto \sum_j A_j T B_j$ where $A_j, B_j \in \mathcal{L}(H)$. It is a matter of some interest to determine the exact operator norm of this operator in terms of the data A_j, B_j. See for example [4] and the references therein.

Turning now to other values of p, our main result is as follows:

Theorem 2. *Suppose for some $p \geq 2$ and some real symmetric matrix (α_{jk}) with nonnegative entries the operators T_j satisfy*

$$\|T_j^* T_k\|_{p/2} \leq \alpha_{jk}^2 \|T_j\|_p \|T_k\|_p$$

and

$$\|T_j T_k^*\|_{p/2} \leq \alpha_{jk}^2 \|T_j\|_p \|T_k\|_p.$$

Let $A = \sup_j \sum_k \alpha_{jk}$ be the Schur norm of the matrix (α_{jk}). If p is an even integer, then

$$\| \sum_j T_j \|_p \leq A^{1/p'} (\sum_j \|T_j\|_p^p)^{1/p}.$$

Note that by Hölder's inequality we may assume that each $\alpha_{jk} \leq 1$.

We make some remarks concerning the shortcomings of this theorem and other matters in the final section.

2. Proof of Theorem 2

The proof begins in a way reminiscent of that of the Cotlar–Stein Lemma. The additional ingredient that we employ is multilinear interpolation.

Let $p = 2k$ and $T = \sum_j T_j$. Then

$$\|T\|_p^p = \|T^*TT^*T \dots T^*T\|_1$$

(where there are k copies of T^*T)

$$\leq \sum_{j_1,\dots,j_{2k}} \|T_{j_1}^* T_{j_2} T_{j_3}^* \dots T_{j_{2k}}\|_1.$$

Each term in the sum can be estimated via Hölder's inequality by both

$$\|T_{j_1}^* T_{j_2}\|_{p/2} \dots \|T_{j_{2k-1}}^* T_{j_{2k}}\|_{p/2}$$

and

$$\|T_{j_1}^*\|_p \|T_{j_2} T_{j_3}^*\|_{p/2} \dots \|T_{j_{2k-2}} T_{j_{2k-1}}^*\|_{p/2} \|T_{j_{2k}}\|_p. \tag{2}$$

By hypothesis these are dominated respectively by

$$\alpha_{j_1 j_2}^2 \dots \alpha_{j_{2k-1} j_{2k}}^2 \|T_{j_1}\|_p \dots \|T_{j_{2k}}\|_p$$

and

$$\alpha_{j_2 j_3}^2 \dots \alpha_{j_{2k-2} j_{2k-1}}^2 \|T_{j_1}\|_p \dots \|T_{j_{2k}}\|_p$$

and hence by their geometric mean

$$\alpha_{j_1 j_2} \alpha_{j_2 j_3} \dots \alpha_{j_{2k-1} j_{2k}} \|T_{j_1}\|_p \dots \|T_{j_{2k}}\|_p.$$

Therefore

$$\|T\|_p^p \leq \sum \alpha_{j_1 j_2} \alpha_{j_2 j_3} \dots \alpha_{j_{2k-1} j_{2k}} \|T_{j_1}\|_p \dots \|T_{j_{2k}}\|_p. \tag{3}$$

Lemma 1. *Let Ω be any measure space and $K(x,y)$ a nonnegative symmetric integral kernel defined on $\Omega \times \Omega$. Suppose κ is the Schur norm of the operator with kernel K, i.e. $\kappa = \sup_x \int K(x,y)dy$. Let p be an integer greater than 1. Then*

$$\int_{\Omega^p} K(x_1,x_2)K(x_2,x_3)\dots K(x_{p-1},x_p) \prod_{s=1}^p F_s(x_s)dx_s$$

$$\leq \kappa^{p-1} \prod_{s=1}^p \|F_s\|_p.$$

Proof. By symmetry and multilinear interpolation, (see for example [1]), it is enough to show that the left hand side is dominated for each j by

$$\kappa^{p-1} \|F_j\|_1 \prod_{k \neq j} \|F_k\|_\infty.$$

To show this, we may assume that $F_k \equiv 1$ when $k \neq j$. The left hand side is now

$$\int K(x_1,x_2)K(x_2,x_3)\dots K(x_{p-1},x_p)F_j(x_j)dx_1 \dots dx_p.$$

Integrating in turn with repsect to $x_1, x_2, \ldots, x_{j-1}$; $x_p, x_{p-1} \ldots, x_{j+1}$ gives a factor κ each of $p-1$ times; finally integrating with respect to x_j gives the result.

□

We apply Lemma 1 with counting measure on $\mathbb{Z}$, $K(x_r, x_s) = \alpha_{j_r j_s}$ and $F_s(x) = \|T_j\|_p$ for all s. Then $\|F_s\|_p^p = \sum_j \|T_j\|_p^p$ and so (3) is dominated by $A^{p-1}\sum_j \|T_j\|_p^p$; upon taking p'th roots we obtain what we desire.

3. Concluding Remarks and Open Questions

1. The most striking deficiency of our theorem is that it is only proved for even integers p. In the commutative case, the proof given works for all integers p because we do not then need the preliminary step $\|T\|_p = \|T^*T\|_{p/2}$. But even in this case we do not know whether the theorem holds for other values of $p > 2$.

2. Is it possible to formulate a meaningful question or result in the case $1 < p < 2$?

3. The statement of the theorem does not formally recover the Cotlar–Stein Lemma in the limiting case $p = \infty$. At the other endpoint $p = 2$, estimate (2) is inefficient, and so neither does the statement of the theorem recover the discussion of the case $p = 2$ (see (1)) from the Introduction.

In fact, the Cotlar–Stein Lemma and the discussion of the case $p = 2$ (see (1)) actually suggest (via naïve "interploation") that an alternative result may be possible. Under the hypotheses of Theorem 2, let A_p be the l^2-operator norm of $(\alpha_{jk}^{p'})$. Do we have

$$\|\sum_j T_j\|_p \leq A_p^{1/p'}(\sum_j \|T_j\|_p^p)^{1/p}? \tag{4}$$

(Note that $A_p \leq A$ since the Schur norm is dominated by the l^2-operator norm and since we may assume via Hölder's inequality that each $\alpha_{jk} \leq 1$.) We do not know, even in the commutative case, whether (4) holds, even when p is an integer. As an easier question, what about (4) with the Schur norm of $(\alpha_{jk}^{p'})$ replacing the operator norm?

4. Applying the commutative version of Theorem 2 in the situation when there are only two summands we obtain an L^p inequality: if $f, g \in L^p$ and $p \in \mathbb{N}$, $p \geq 2$, then

$$\|f+g\|_p \leq \left[1 + \left\{\frac{\|fg\|_{p/2}}{\|f\|_p\|g\|_p}\right\}^{1/2}\right](\|f\|_p^p + \|g\|_p^p)^{1/p}. \tag{5}$$

(The Schur and the operator norms of the 2×2 symmetric matrix with 1's on the diagonal and α off-diagonal are both $1 + \alpha$.) However this is not in general best possible in terms of the exponents of the curly and square brackets. For example, if (4) held, we would be able to improve (5) to

$$\|f+g\|_p \leq \left[1 + \left\{\frac{\|fg\|_{p/2}}{\|f\|_p\|g\|_p}\right\}^{p'/2}\right]^{1/p'}(\|f\|_p^p + \|g\|_p^p)^{1/p}.$$

Even this is may not be best possible. When $p = \infty$ we have

$$\|f+g\|_\infty \leq [1 + \frac{\|fg\|_\infty}{\max\{\|f\|_\infty, \|g\|_\infty\}^2}]\max\{\|f\|_\infty, \|g\|_\infty\}, \tag{6}$$

(since $a \leq A,\ b \leq B \leq A,\ ab \leq \lambda$ implies $a+b \leq A+\lambda/A$). This suggests that the inequality

$$\|f+g\|_p \leq \left[1+\left\{\frac{\|fg\|_{p/2}}{\|f\|_p\|g\|_p}\right\}\right]^{1/p'} (\|f\|_p^p+\|g\|_p^p)^{1/p} \tag{7}$$

might hold for $2 \leq p \leq \infty$.

Proposition 1. *If f and g are characteristic functions of sets and $p \geq 2$, then* (7) *does indeed hold.*

Proof. We see quickly that it suffices to prove the inequality

$$a+b+(2^p-2)c \leq \left[1+\frac{c^{2/p}}{a^{1/p}b^{1/p}}\right]^{p-1}(a+b)$$

for $0 \leq c \leq a, b$. Fix λ and c and consider a, b with $a+b = 2\lambda$ and $\lambda \geq c$. The the worst case is when a and b are both equal to λ, and we are reduced to showing $1+(2^{p-1}-1)s \leq [1+s^{2/p}]^{p-1}$ for $0 \leq s \leq 1$. This in turn follows from concavity of $s \mapsto [1+s^{2/p}]^{p-1}$ on $0 \leq s \leq 1$ when $p \geq 2$. □

The power $1/p'$ occuring in (7) is optimal (consider $f = g$) but we do not know whether the power 1 of $\left\{\frac{\|fg\|_{p/2}}{\|f\|_p\|g\|_p}\right\}$ is optimal. Indeed, for the function $s \mapsto [1+s^\gamma]^{p-1}$ to be concave on $[0,1]$ we need that $\gamma \leq 2/p$, but in general $1+(2^{p-1}-1)s \leq [1+s^\gamma]^{p-1}$ for $0 \leq s \leq 1$ holds for some $\gamma > 2/p$. For example if $p = 3$ then it holds for all $\gamma \leq 3/4$.

5. The previous remark – in particular (6) – suggests that when $p = \infty$, it might be the case that

$$\|\sum_j f_j\|_\infty \leq \left(\sup_j \sum_k \alpha_{jk}^2\right) \sup_j \|f_j\|_\infty$$

where $\|f_j f_k\|_\infty \leq \alpha_{jk}^2 \|f_j\|_\infty \|f_k\|_\infty$. If so, naïve "interpolation" with

$$\|\sum_j f_j\|_2 \leq \left(\sup_j \sum_k \alpha_{jk}^2\right)^{1/2} \sum_j \|f_j\|_2^2$$

where $\|f_j f_k\|_1 \leq \alpha_{jk}^2 \|f_j\|_2 \|f_k\|_2$ (which follows from (1)) would suggest that for $p \geq 2$

$$\|\sum_j f_j\|_p \leq \left(\sup_j \sum_k \alpha_{jk}^2\right)^{1/p'} \sum_j \|f_j\|_p^p$$

where $\|f_j f_k\|_{p/2} \leq \alpha_{jk}^2 \|f_j\|_p \|f_k\|_p$. Again, this is open.

6. We state a combinatorial corollary which is obtained by applying Chebychev's inequality to the commutative version of Theorem 2 with $f_j = \chi_{E_j}$:

Corollary 1. *Suppose E_j are finite subsets of a set $\mathcal{A}$ and that*

$$\#(E_j \cap E_k) \leq \beta_{jk}\{\#E_j \#E_k\}^{1/2}.$$

Then if $p \in \mathbb{N}$,

$$\#\{a \in \mathcal{A} \mid a \text{ is in at least } M\ E_j\text{'s}\} \leq A^{p-1} \frac{\sum_j \#E_j}{M^p}$$

where $A = \sup_j \sum_k \beta_{jk}^{1/p}$.

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