FINITE BOUNDS FOR HÖLDER-BRASCAMP-LIEB MULTILINEAR INEQUALITIES

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Abstract. A criterion is established for the validity of multilinear inequalities of a class considered by Brascamp and Lieb, generalizing well-known inequalities of Hölder, Young, and Loomis-Whitney. The proof relies on interpolation, linear algebra, and Hölder's inequality.

1. Formulation

Consider multilinear functionals

$$\Lambda(f_1, f_2, \cdots, f_m) = \int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j(\ell_j(y))\, dy \tag{1.1}$$

where each $\ell_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ is a surjective linear transformation, and $f_j : \mathbb{R}^{n_j} \to [0, +\infty]$. Let $p_1, \cdots, p_m \in [1, \infty]$. For which m-tuples of exponents and linear transformations is

$$\sup_{f_1,\cdots,f_m} \frac{\Lambda(f_1, f_2, \cdots, f_m)}{\prod_j \|f_j\|_{L^{p_j}}} < \infty? \tag{1.2}$$

The supremum is taken over all m-tuples of nonnegative Lebesgue measurable functions f_j having positive, finite norms. If $n_j = n$ for every index j then (1.2) is essentially a restatement of Hölder's inequality. Other well-known particular cases include Young's inequality for convolutions and the Loomis-Whitney inequality [13].

In this paper we characterize finiteness of the supremum (1.2) in linear algebraic terms, and discuss certain variants and a generalization. In this level of generality, the question was to our knowledge first posed by Brascamp and Lieb [3]. A primitive version of the problem involving Cartesian product rather than linear algebraic structure was posed and solved by Finner [10]; see §7 below. In the case when the dimension n_j of each target space equals one, Barthe [1] characterized (1.2). Carlen, Lieb and Loss [6] gave an alternative proof for that case. They developed an inductive analysis closely related to that of Finner, and emphasized the pivotal concept of a critical subspace. Our analysis is based on those ideas.

An alternative line of analysis exists. Although rearrangement inequalities such as that of Brascamp, Lieb, and Luttinger [4] do not apply when the target spaces have dimensions greater than one, Lieb [12] nonetheless showed that the supremum in (1.2) equals the supremum over all m-tuples of Gaussian functions,[1] meaning

Date: May 23, 2005.
The third author was supported in part by NSF grant DMS-040126.

[1] This situation should be contrasted with that of multilinear *operators* of the same general form, mapping $\otimes_j L^{p_j}$ to L^q. When $q \geq 1$, such multilinear operators are equivalent by duality to multilinear forms Λ. This is not so for $q < 1$, and Gaussians are then quite far from being extremal [7].

those of the form $f_j = \exp(-Q_j(y,y))$ for some positive definite quadratic form Q_j. See [6] and references cited there for more on this approach. In a companion paper [2] we have given other proofs of our characterization of (1.2), by using heat flow to continuously deform arbitrary functions f_j to Gaussians while increasing the ratio in (1.2). Once again, this extends work of Carlen, Lieb, and Loss [6] by a method they introduced.

2. Results

Denote by $\dim(V)$ the dimension of a vector space V. It is convenient to reformulate the problem in a more invariant fashion. Let $H, H_1, \ldots, H_m$ be Hilbert spaces of finite, positive dimensions. Each is equipped with a canonical Lebesgue measure, by choosing orthonormal bases, thus obtaining identifications with $\mathbb{R}^{\dim(H)}$, $\mathbb{R}^{\dim(H_j)}$. Let $\ell_j : H \to H_j$ be surjective linear mappings. Let $f_j : H_j \to \mathbb{R}$ be nonnegative. Then $\Lambda(f_1, \cdots, f_m)$ equals $\int_H \prod_{j=1}^m f_j \circ \ell_j(y)\, dy$.

Theorem 2.1. *For $1 \le j \le m$ let H, H_j be Hilbert spaces of finite, positive dimensions. For each index j let $\ell_j : H \to H_j$ be surjective linear transformations, and let $p_j \in [1,\infty]$. Then* (1.2) *holds if and only if*

$$\dim(H) = \sum_j p_j^{-1} \dim(H_j) \tag{2.1}$$

and

$$\dim(V) \le \sum_j p_j^{-1} \dim(\ell_j(V)) \quad \textit{for every subspace } V \subset H. \tag{2.2}$$

This equivalence is established by other methods in [2], Theorem 1.15.

The necessity of (2.1) follows from scaling: if $f_j^\lambda(x_j) = g_j(\lambda x_j)$ for each $\lambda \in \mathbb{R}^+$ then $\Lambda(\{f_j^\lambda\})$ is proportional to $\lambda^{-\dim(H)}$, while $\prod_j \|f_j^\lambda\|_{p_j}$ is proportional to $\prod_j \lambda^{-\dim(H_j)/p_j}$. That (2.2) is also necessary will be shown in §5 in the course of the proof of the more general Theorem 2.3.

Throughout the paper, $\operatorname{codim}_W(V)$ will denote the codimension of a subspace $V \subset W$ in W. Given that (2.1) holds, the hypothesis (2.2) can be equivalently restated as (2.6): $\operatorname{codim}_H(V) \ge \sum_j p_j^{-1} \operatorname{codim}_{H_j}(\ell_j(V))$; any two of these three conditions (2.1), (2.2), (2.6) imply the third. As will be seen through the discussion of variants below, (2.2) expresses a necessary condition governing large-scale geometry (compare Theorem 2.5), while (2.6) expresses a necessary condition governing small-scale geometry (compare Theorem 2.2). See also the discussion of necessary conditions for Theorem 2.3.

Remark 2.1. Λ can be alternatively expressed as a constant multiple of $\int_\Sigma \prod_j f_j \, d\sigma$, where Σ is a linear subspace of $\oplus_j H_j$ and σ is Lebesgue measure on Σ. More exactly, Σ is the range of the map $H \ni x \mapsto \oplus_j \ell_j(x)$. Denote by π_j the restriction to Σ of the natural projection $\pi_j : \oplus_i H_i \to H_j$. Then condition (2.2) can be restated as

$$\dim(\tilde\Sigma) \le \sum_j p_j^{-1} \dim(\pi_j(\tilde\Sigma)) \text{ for every linear subspace } \tilde\Sigma \subset \Sigma. \tag{2.3}$$

A local variant is also natural. Consider

$$\Lambda_{\mathrm{loc}}(f_1, \cdots, f_m) = \int_{\{y \in H : |y| \le 1\}} \prod_j f_j \circ \ell_j(y)\, dy. \tag{2.4}$$

Theorem 2.2. *Let H, H_j, ℓ_j, and $f_j : H_j \to [0,\infty)$ be as in Theorem 2.1. A necessary and sufficient condition for there to exist $C < \infty$ such that*

$$\Lambda_{\mathrm{loc}}(f_1, \cdots, f_m) \le C \prod_j \|f\|_{L^{p_j}} \tag{2.5}$$

for all nonnegative measurable functions f_j is that for every subspace V of H,

$$\operatorname{codim}_H(V) \ge \sum_j p_j^{-1} \operatorname{codim}_{H_j}(\ell_j(V)). \tag{2.6}$$

This is equivalent to Theorem 8.17 of [2], proved there by a very different method.

The next theorem, in which some but not necessarily all coordinates of y are constrained to a bounded set, unifies Theorems 2.1 and 2.2.

Theorem 2.3. *Let $H, H_0, \cdots, H_m$ be finite-dimensional Hilbert spaces and assume that $\dim(H_j) > 0$ for all $j \ge 1$. Let $\ell_j : H \to H_j$ be linear transformations for $0 \le j \le m$, which are surjective for all $j \ge 1$. Let $p_j \in [1,\infty]$ for $1 \le j \le m$. Then there exists $C < \infty$ such that*

$$\int_{\{y \in H : |\ell_0(y)| \le 1\}} \prod_{j=1}^m f_j \circ \ell_j(y)\, dy \le C \prod_{j=1}^m \|f_j\|_{L^{p_j}} \tag{2.7}$$

for all nonnegative Lebesgue measurable functions f_j if and only if

$$\dim(V) \le \sum_{j=1}^m p_j^{-1} \dim(\ell_j(V)) \qquad \text{for all subspaces } V \subset \operatorname{kernel}(\ell_0) \tag{2.8}$$

and

$$\operatorname{codim}_H(V) \ge \sum_{j=1}^m p_j^{-1} \operatorname{codim}_{H_j}(\ell_j(V)) \quad \text{for all subspaces } V \subset H. \tag{2.9}$$

This subsumes Theorem 2.2, by taking $H_0 = H$ and $\ell_0 : H \to H$ to be the identity; (2.8) then only applies to $\{0\}$, for which it holds automatically, so that the only hypothesis is then (2.9). On the other hand, Theorem 2.1 is the special case $\ell_0 \equiv 0$ of Theorem 2.3. In that case $\operatorname{kernel}(\ell_0) = H$, so (2.8) becomes (2.2). In addition, the case $V = \{0\}$ of (2.9) yields the reverse inequality $\dim(H) \ge \sum_j p_j^{-1} \dim(H_j)$. Thus the hypotheses of Theorem 2.3 imply those of Theorem 2.1 when $\ell_0 \equiv 0$. The converse implication also holds, as was pointed out in the discussion of Theorem 2.2.

Our next result is one of several possible discrete analogues.

Theorem 2.4. *Let G and $\{G_j : 1 \le j \le N\}$ be finitely generated Abelian groups. Let $\varphi_j : G \to G_j$ be homomorphisms whose ranges are subgroups of finite indices. Let $p_j \in [1,\infty]$. Then there exists $C < \infty$ such that*

$$\sum_{y \in G} \prod_{j=1}^N f_j \circ \varphi_j(y) \le C \prod_j \|f_j\|_{\ell^{p_j}(G_j)} \quad \text{for all nonnegative functions } f_j \tag{2.10}$$

if and only if

$$\operatorname{rank}(H) \le \sum_j p_j^{-1} \operatorname{rank}(\varphi_j(H)) \quad \text{for every subgroup } H \text{ of } G. \tag{2.11}$$

Here the ℓ^{p_j} norms are of course defined with respect to counting measure. The constant C depends of course on the torsion subgroups of the groups G_j.

In $\mathbb{R}^d$, for each $n \in \mathbb{Z}^d$ define $Q_n = \{x \in \mathbb{R}^d : |x - n| \le \sqrt{d}\}$. The space $\ell^p(L^\infty)(\mathbb{R}^d)$ is the space of all $f \in L^\infty(\mathbb{R}^d)$ for which the norm $(\sum_{n\in\mathbb{Z}^d} \|f\|^p_{L^\infty(Q_n)})^{1/p}$ is finite.

Theorem 2.5. *Let $\ell_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$ be surjective linear transformations. Let $p_j \in [1,\infty]$. Then there exists $C < \infty$ such that*

(2.12)
$$\int_{\mathbb{R}^d} \prod_{j=1}^N f_j \circ \ell_j(y)\, dy \le C \prod_j \|f_j\|_{\ell^{p_j}(L^\infty)(\mathbb{R}^{d_j})} \text{ for all nonnegative functions } f_j$$

if and only if for every subspace $V \subset \mathbb{R}^d$,

$$\dim(V) \le \sum_j p_j^{-1} \dim(\ell_j(V)). \tag{2.13}$$

A related result is Corollary 8.11 of [2].

We have assumed in all these theorems that all exponents satisfy $p_j \ge 1$. In Theorems 2.1, 2.2, and 2.3, the inequalities in question are false if some $p_j < 1$. To see this, fix one index j. Take f_i to be the characteristic function of a fixed ball centered at the origin for each $i \ne j$, take f_j to be the characteristic function of a ball of measure δ centered at the origin, and let $\delta \to 0$. Then $\tilde{\Lambda}(f_1, \cdots, f_m)$ has order of magnitude δ, while $\prod_i \|f_i\|_{L^{p_i}}$ has order of magnitude $\delta^{1/p_j} \ll \delta$.

Valid inequalities can hold in Theorems 2.4 and 2.5 with some exponents strictly less than one, but they are always implied by stronger inequalities already contained in those theorems. More precisely, if the inequality holds for some m-tuple $(p_1, \cdots, p_m)$, then it also holds with each p_i replaced by $\max(p_i, 1)$. In the case of Theorem 2.4, that p_j can be replaced by 1 if $p_j < 1$ can be shown by considering the case when the support of f_i is a single point, then exploiting linearity and symmetry.

Two quite distinct investigations motivated our interest in these problems. One derives from multilinear versions of the Kakeya-Nikodym maximal functions, as will be explored in a forthcoming paper of the first, second, and fourth authors. A second motivator was work [9] on multilinear operators with additional oscillatory factors; see Proposition 3.1 and Corollary 3.2 below. Further applications of Theorem 2.1 to oscillatory integrals will appear in a forthcoming paper [8].

3. An application to oscillatory integrals

Proposition 3.1. *Let $m > 1$. For $1 \le j \le m$ let $\ell_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ be surjective linear mappings. Let $P : \mathbb{R}^n \to \mathbb{R}$ be a polynomial. Let $\varphi \in C^1_0(\mathbb{R}^n)$ be a compactly supported, continuously differentiable cutoff function. For $\lambda \in \mathbb{R}$ and $f_j \in L^{p_j}(\mathbb{R}^{n_j})$ define $\Lambda_\lambda(f_1, \cdots, f_m) = \int_{\mathbb{R}^n} e^{i\lambda P(x)} \prod_{j=1}^m f_j(\ell_j(x))\, \varphi(x)\, dx$. Suppose that there exist $\delta > 0$ and $C < \infty$ such that for all functions $f_j \in L^\infty$ and all $\lambda \in \mathbb{R}$*

$$|\Lambda_\lambda(f_1, \cdots, f_m)| \le C|\lambda|^{-\delta} \prod_{j=1}^m \|f_j\|_{L^\infty}. \tag{3.1}$$

Let $(p_1, \cdots, p_m) \in [1, \infty]^m$, and suppose that for every proper subspace $V \subset \mathbb{R}^n$,

$$\operatorname{codim}_{\mathbb{R}^n}(V) > \sum_j p_j^{-1} \operatorname{codim}_{\mathbb{R}^{n_j}}(\ell_j(V)). \tag{3.2}$$

Then there exist $\delta > 0$ and $C < \infty$, depending on $(p_1, \cdots, p_m)$, such that

$$|\Lambda_\lambda(f_1, \cdots, f_m)| \le C|\lambda|^{-\delta} \prod_{j=1}^m \|f_j\|_{L^{p_j}} \tag{3.3}$$

for all parameters $\lambda \in \mathbb{R}$ and functions $f_j \in L^{p_j}(\mathbb{R}^{n_j})$.

In the formulation of the hypothesis it is implicitly assumed that the integral defining $\Lambda_\lambda(f_1, \cdots, f_m)$ converges absolutely for all functions $f_j \in L^{p_j}$; thus by Theorem 2.2 it is necessary that $\operatorname{codim}_{\mathbb{R}^n}(V) \le \sum_j p_j^{-1} \operatorname{codim}_{\mathbb{R}^{n_j}}(\ell_j(V))$ for every subspace $V \subset \mathbb{R}^n$. The conclusion of Proposition 3.1 then follows directly from Theorem 2.2 by complex interpolation.

A polynomial P is said [9] to be nondegenerate, relative to the collection $\{\ell_j\}$ of mappings, if P cannot be expressed as $P = \sum_j P_j \circ \ell_j$ for any collection of polynomials $P_j : \mathbb{R}^{n_j} \to \mathbb{R}$.

Corollary 3.2. *Let $\{\ell_j\}, P, \varphi$ be as in Proposition 3.1. Suppose that P is nondegenerate relative to $\{\ell_j\}$. Suppose that either (i) $n_j = 1$ for all j, $m < 2n$, and the family $\{\ell_j\}$ of mappings is in general position, or (ii) $n_j = n - 1$ for all j. Let $(p_1, \cdots, p_m) \in [1, \infty]^m$ and suppose that for every proper subspace $V \subset \mathbb{R}^n$,* $\operatorname{codim}_{\mathbb{R}^n}(V) > \sum_j p_j^{-1} \operatorname{codim}_{\mathbb{R}^{n_j}}(\ell_j(V))$. *Then there exists $\delta > 0$ such that for any $\varphi \in C_0^1$ there exists $C < \infty$ such that for all functions $f_j \in L^{p_j}(\mathbb{R}^{n_j})$,*

$$|\Lambda_\lambda(f_1, \cdots, f_m)| \le C|\lambda|^{-\delta} \prod_{j=1}^m \|f_j\|_{L^{p_j}}.$$

Here general position means that for any subset $S \subset \{1, 2, \cdots, m\}$ of cardinality $|S| \le n$, $\cap_{j \in S} \operatorname{kernel}(\ell_j)$ has dimension $n - |S|$.

By Theorems 2.1 and 2.2 of [9], the hypotheses imply (3.1). Proposition 3.1 then implies the Corollary.

4. Proof of sufficiency in Theorem 2.1

We begin with the proof of sufficiency of the hypotheses (2.1), (2.2) for the finiteness of the supremum in (1.2). Necessity will be established in the next section.

The next definition is made for the purposes of the discussion of Theorem 2.1; alternative notions of criticality are appropriate for the other theorems.

Definition 4.1. A subspace $V \subset H$ is said to be critical if

$$\dim(V) = \sum_j p_j^{-1} \dim(\ell_j(V)), \tag{4.1}$$

to be supercritical if the right-hand side is less than $\dim(V)$, and to be subcritical if the right-hand side is greater than $\dim(V)$.

In this language, the hypothesis (2.1) states that $V = H$ is critical, while (2.2) states that no subspace of H is supercritical.

Proof of sufficiency in Theorem 2.1. The proof proceeds by induction on $\dim(H)$. When H has dimension one, necessarily $\dim(H_j) = 1$ for all j. The hypothesis of the theorem in this case is that $\sum_j p_j^{-1} = 1$, and the conclusion is simply a restatement of Hölder's inequality for functions in $L^{p_j}(\mathbb{R}^1)$.

Suppose now that $\dim(H) > 1$. There are two cases. Case 1 arises when there exists some proper nonzero critical subspace $W \subset H$. The analysis then follows the pattern of [10] and [6]. Express $H = W^\perp \oplus W$ where $W^\perp$ is the orthocomplement of W, with coordinates $y = (y', y'') \in W^\perp \oplus W$; we will identify $(y', 0)$ with y' and $(0, y'')$ with y''. Define $U_j \subset H_j$ to be

$$U_j = \ell_j(W). \tag{4.2}$$

Define $\tilde{\ell}_j = \ell_j|_W : W \to U_j$, which is surjective. For $y' \in W^\perp$ and $x_j \in U_j$ define

$$g_{j,y'}(x_j) = f_j(x_j + \ell_j(y')). \tag{4.3}$$

Then

$$f_j(\ell_j(y', y'')) = f_j(\ell_j(y') + \tilde{\ell}_j(y'')) = g_{j,y'}(\tilde{\ell}_j(y'')). \tag{4.4}$$

Now

$$\Lambda(f_1, \cdots, f_m) = \int_{W^\perp} \int_W \prod_j f_j(\ell_j(y', y''))\, dy''\, dy' = \int_{W^\perp} \int_W \prod_j g_{j,y'}(\tilde{\ell}_j(y''))\, dy''\, dy',$$

so

$$\Lambda(f_1, \cdots, f_m) = \int_{W^\perp} \tilde{\Lambda}(g_{1,y'}, \cdots, g_{m,y'})\, dy' \tag{4.5}$$

where

$$\tilde{\Lambda}(g_1, \cdots, g_m) = \int_W \prod_j g_j(\tilde{\ell}_j(y''))\, dy''. \tag{4.6}$$

We claim that

$$\tilde{\Lambda}(g_1, \cdots, g_m) \le C \prod_j \|g_j\|_{p_j}. \tag{4.7}$$

Since W has dimension strictly less than $\dim(H)$, this follows from the induction hypothesis provided that W is critical and no subspace $V \subset W$ is supercritical, relative to the mappings $\tilde{\ell}_j$ and exponents p_j. But since $\tilde{\ell}_j$ is the restriction of ℓ_j to W, this condition is simply the specialization of the original hypothesis from arbitrary subspaces of H to those subspaces contained in W, together with the criticality of W hypothesized in Case 1. Thus

$$\Lambda(f_1, \cdots, f_m) = \int_{W^\perp} \tilde{\Lambda}(g_{1,y'}, \cdots, g_{m,y'})\, dy' \le C \int_{W^\perp} \prod_j \|g_{j,y'}\|_{L^{p_j}(U_j)}\, dy'. \tag{4.8}$$

We will next show how this last integral is another instance of the original problem, with H replaced by the lower-dimensional vector space $W^\perp$. For $z_j \in U_j^\perp$ define

$$F_j(z_j) = \Big(\int_{U_j} f_j(x_j + z_j)^{p_j}\, dx_j \Big)^{1/p_j}, \tag{4.9}$$

recalling that $f_j \ge 0$, with $F_j(z_j) = \text{ess sup}_{\,x_j\in U_j} f_j(x_j+z_j)$ if $p_j=\infty$. Thus[2]

$$\|F_j\|_{L^{p_j}(U_j^\perp)} = \|f_j\|_{L^{p_j}(H_j)}. \tag{4.10}$$

Denote by $\pi_{U_j^\perp} : H_j \to U_j^\perp$ and $\pi_{U_j} : H_j \to U_j$ the orthogonal projections. Define $L_j : W^\perp \to U_j^\perp$ by

$$L_j = \pi_{U_j^\perp} \circ \ell_j. \tag{4.11}$$

Decomposing $\ell_j(y') = L_j(y') + u_j$ where $u_j = \pi_{U_j}(\ell_j(y'))$, and making the change of variables $\tilde{x}_j = x_j + u_j$ in U_j, gives (if $p_j < \infty$)

$$\|g_{j,y'}\|^{p_j}_{L^{p_j}(U_j)} = \int_{U_j} |g_{j,y'}(x_j)|^{p_j}\,dx_j = \int_{U_j} |f_j(x_j+\ell_j(y'))|^{p_j}\,dx_j \tag{4.12}$$
$$= \int_{U_j} |f_j(x_j+u_j+L_j(y'))|^{p_j}\,dx_j = \int_{U_j} |f_j(\tilde{x}_j+L_j(y'))|^{p_j}\,d\tilde{x}_j = F_j(L_j(y'))^{p_j}.$$

Consequently we have shown thus far that

$$\Lambda(f_1,\cdots,f_m) \le C\int_{W^\perp}\prod_j F_j\circ L_j \tag{4.13}$$

where $\|F_j\|_{L^{p_j}(U_j^\perp)} = \|f_j\|_{L^{p_j}(H_j)}$. Since $\ell_j : H \to H_j$ is surjective, H_j is spanned by $\ell_j(W) = U_j$ together with $\ell_j(W^\perp)$; thus the orthogonal projection of $\ell_j(W^\perp)$ onto $U_j^\perp$ is all of $U_j^\perp$; thus each $L_j : W^\perp \to U_j^\perp$ is surjective.

To complete the argument for Case 1 we need only show that

$$\int_{W^\perp}\prod_j F_j\circ L_j \le C\prod_j \|F_j\|_{L^{p_j}(U_j^\perp)}. \tag{4.14}$$

By induction on the ambient dimension, this follows from the next lemma.

Lemma 4.1. *Suppose that H is critical, and has no supercritical subspaces. Suppose that $W \subset H$ is a nonzero proper critical subspace. Define surjective linear transformations $L_j = \pi_{\ell_j(W)^\perp}\circ\ell_j : W^\perp \to \ell_j(W)^\perp$. Then for any subspace $V \subset W^\perp$, $\dim(V) \le \sum_j p_j^{-1}\dim(L_j(V))$.*

Proof. Associate to V the subspace $V+W \subset H$. Since $V \subset W^\perp$, $\dim(V+W) = \dim(V)+\dim(W)$. Moreover, for any j,

$$\dim(\ell_j(V+W)) = \dim(L_j(V)) + \dim(\ell_j(W)), \tag{4.15}$$

since $L_j = \pi_{\ell_j(W)^\perp}\circ\ell_j$.

Therefore

$$\begin{aligned}\sum_j p_j^{-1}\dim(L_j(V)) &= \sum_j p_j^{-1}\dim(\ell_j(V+W)) - \sum_j p_j^{-1}\dim(\ell_j(W))\\ &= \sum_j p_j^{-1}\dim(\ell_j(V+W)) - \dim(W)\\ &\ge \dim(V+W)-\dim(W) = \dim(V),\end{aligned}$$

by the criticality of W and subcriticality of $V+W$. Thus V is not supercritical.

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[2]If $U_j = \{0\}$ then the domain of F_j is H_j, and $F_j \equiv f_j$. If $U_j = H_j$ then the domain of F_j is $\{0\}$, and $\|F_j\|_{p_j}$ is by definition $F_j(0) = \|f_j\|_{p_j}$.

When $V = W^\perp$, one has $V + W = H$, whence $\sum_j p_j^{-1} \dim(\ell_j(V + W)) = \dim(V + W)$ since H is assumed to be critical. With this information the final inequality of the preceding display becomes an equality, demonstrating that $W^\perp$ is critical. □

The proof of Case 1 of Theorem 2.1 is complete. Turn next to Case 2, in which every nonzero proper subspace of H is subcritical. ∞^{-1} is to be interpreted as zero throughout the discussion.

Consider the set K of all m-tuples $t = (t_1, \cdots, t_m) \in [0, 1]^m$ such that relative to the exponents $p_j = t_j^{-1}$, H is critical and has no supercritical subspace. Then K equals the intersection of $[0, 1]^m$ with a hyperplane and with various closed half-spaces. Thus K is convex and compact, whence it equals the closed convex hull of its extreme points.

For any $t = (t_1, \cdots, t_m) \in [0, \infty)^m$, if (2.6) holds, that is if $\mathrm{codim}_H(V) \geq \sum_j t_j \,\mathrm{codim}_{H_j}(V)$ for all subspaces $V \subset H$, then necessarily $t \in [0, 1]^m$. Indeed, consider any index i and let V be the nullspace of ℓ_i. Then

(4.16)
$$\dim(H_i) = \mathrm{codim}_H(V) \geq \sum_j t_j \,\mathrm{codim}_{H_j}(\ell_j(V)) \geq t_i \,\mathrm{codim}_{H_i}\{0\} = t_i \dim(H_i).$$

(2.6) holds whenever $(t_j) = (p_j^{-1})$ satisfies the hypotheses (2.1) and (2.2) of Theorem 2.1. Consequently if t is an extreme point of K, then some nonzero proper subspace of H is critical relative to t, or at least one coordinate t_i equals 0, or $m = 1$ and $p_1 = 1$. In the first subcase we are in Case 1, not Case 2. For the third subcase, see below.

In the second subcase, we may proceed by induction on the number m of indices j, for an inequality $\Lambda(f_1, \cdots, f_m) \leq C \|f_i\|_{L^\infty} \prod_{j \neq i} \|f_j\|_{L^{p_j}}$ is equivalent to

$$\Lambda(f_1, \cdots, f_{i-1}, 1, f_{i+1}, \cdots, f_m) \leq C \prod_{j \neq i} \|f_j\|_{L^{p_j}}. \tag{4.17}$$

The hypotheses of Theorem 2.1 are inherited by this multilinear operator of one lower degree, acting on $\{f_j : j \neq i\}$, whence the desired inequality follows by induction.

This induction is founded by the subcase where $m = 1$, so that $\Lambda(f_1) = \int_H f_1 \circ \ell_1$; moreover $p_1 = 1$. Then $\ell_1 : H \to H_1$ is surjective, so $\dim(H) \geq \dim(H_1)$. The hypothesis $\dim(H) = p_1^{-1} \dim(H_1) \leq \dim(H_1)$ thus forces $\ell_1 : H \to H_1$ to be invertible, and p_1^{-1} to equal 1. Then $\Lambda(f_1) = c \int f_1$ for some finite constant c, which is the desired result. □

Remark 4.1. When $\dim(H_j) = 1$ for all j, every extreme point $(p_1^{-1}, \cdots, p_m^{-1})$ of K has each $p_j^{-1} \in \{0, 1\}$ [1],[6]. This is not the case in general; in the Loomis-Whitney inequality for $\mathbb{R}^n$, K consists of a single point, with $p_j = n - 1$ for all j.

5. Proof of Theorem 2.3

Consider $\int_{\{y \in H : |\ell_0(y)| \leq 1\}} \prod_{j=1}^m f_j \circ \ell_j \, dy$ where the linear transformation ℓ_0 has domain H and range H_0 with $\dim(H_0)$ possibly equal to zero. Thus some components of y are constrained to a bounded set, while the rest are free. Set

$$\mathcal{V} = \mathrm{kernel}(\ell_0); \tag{5.1}$$

the component of y lying in $\mathcal{V}$ is completely unconstrained, while the component in $\mathcal{V}^\perp$ is constrained to a bounded set.

Proof of necessity of (2.8) *and* (2.9). For any subspace $V \subset H$ define $V_{\rm big} = V \cap \mathcal{V}$ and $V_{\rm small} = V \ominus V_{\rm big}$, so that $V = V_{\rm small} \oplus V_{\rm big}$. Let $r \le 1 \le R$ be arbitrary. Define $f_j = f_j(x_j)$ to be the characteristic function of the region S_j where $|x_j| \le R$ if $x_j \in \ell_j(V_{\rm big})$, $|x_j| \le 1$ if $x_j \in \ell_j(V) \cap (\ell_j(V_{\rm big}))^\perp$, and $|x_j| \le r$ if $x_j \in (\ell_j(V))^\perp$.

Let $c_0 > 0$ be a small constant, independent of r, R, and define $S \subset H$ to be the set of all y such that $|y| \le c_0 r$ if $y \in V^\perp$, $|y| \le c_0$ if $y \in V_{\rm small}$, and $|y| \le c_0 R$ if $y \in V_{\rm big}$. Then provided c_0 is chosen sufficiently small, $y \in S \Rightarrow f_j(\ell_j(y)) = 1$ for all indices j. Indeed, if $y \in V^\perp$ then $|\ell_j(y)| \le C|y| \le Cc_0 r$, so $\ell_j(y) \in S_j$. If $y \in V_{\rm small}$ then $|\ell_j(y)| \le C|y| \le Cc_0$, so since $\ell_j(y) \in \ell_j(V)$, $\ell_j(y) \in S_j$. Finally if $y \in V_{\rm big}$ then $|\ell_j(y)| \le Cc_0 R$, which implies that $\ell_j(y) \in S_j$ since $\ell_j(y) \in \ell_j(V_{\rm big})$.

Moreover $y \in S \Rightarrow |\ell_0(y)| \le 1$. Therefore

$$\tilde\Lambda_{\rm loc}(\{f_j\}) \ge |S| \sim R^{\dim(V_{\rm big})} \cdot r^{\operatorname{codim}_H(V)} \tag{5.2}$$

while

$$\|f_j\|_{p_j} \sim R^{p_j^{-1}\dim(\ell_j(V_{\rm big}))} r^{p_j^{-1}\operatorname{codim}_{H_j}(\ell_j(V))}. \tag{5.3}$$

Suppose that the ratio $\tilde\Lambda_{\rm loc}(\{f_j\})/\prod_j \|f_j\|_{p_j}$ is bounded uniformly as a function of r, R. By letting $r \to 0$, we conclude that $\dim(V_{\rm big}) \le \sum_j p_j^{-1}\dim(\ell_j(V_{\rm big}))$. Letting $R \to \infty$ gives $\operatorname{codim}_H(V) \ge \sum_j p_j^{-1}\operatorname{codim}_{H_j}(\ell_j(V))$. □

The following lemma will be used in the proof of Theorem 2.3.

Lemma 5.1. *Suppose that* $\operatorname{codim}_H(V) \ge \sum_j p_j^{-1}\operatorname{codim}_{H_j}(\ell_j(V))$ *for every subspace* $V \subset H$*, and that* $W \subset H$ *is a subspace satisfying* $\operatorname{codim}_H(W) = \sum_j p_j^{-1}\operatorname{codim}_{H_j}(\ell_j(W))$*. Then for any subspace* $V \subset W$*,* $\operatorname{codim}_W(V) \ge \sum_j p_j^{-1}\operatorname{codim}_{\ell_j(W)}(\ell_j(V))$*. Likewise for any subspace* $V \subset W^\perp$*,* $\operatorname{codim}_{W^\perp}(V) \ge \sum_j p_j^{-1}\operatorname{codim}_{\ell_j(W)^\perp}(L_j(V))$*.*

Proof. For the first conclusion,

$$\begin{aligned}\operatorname{codim}_W(V) &= \dim(W) - \dim(V) = \operatorname{codim}_H(V) - \operatorname{codim}_H(W)\\ &\ge \sum_j p_j^{-1}\operatorname{codim}_{H_j}(\ell_j(V)) - \sum_j p_j^{-1}\operatorname{codim}_{H_j}(\ell_j(W))\\ &= \sum_j p_j^{-1}(\dim(\ell_j(W)) - \dim(\ell_j(V))) = \sum_j p_j^{-1}\operatorname{codim}_{\ell_j(W)}(\ell_j(V)).\end{aligned} \tag{5.4}$$

For the second conclusion,

$$\begin{aligned}
\operatorname{codim}_{W^\perp}(V) &= \dim(H) - \dim(W) - \dim(V) \\
&= \operatorname{codim}_H(V+W) \\
&\geq \sum_j p_j^{-1} \operatorname{codim}_{H_j}(V+W) \\
&= \sum_j p_j^{-1} \big(\dim(H_j) - \dim(\ell_j(W)) - \dim(L_j(V)) \big) \qquad (5.5) \\
&= \sum_j p_j^{-1} \big(\dim(L_j(W^\perp)) - \dim(L_j(V)) \big). \\
&= \sum_j p_j^{-1} \operatorname{codim}_{L_j(W^\perp)}(L_j(V)).
\end{aligned}$$

The identity $\dim(H_j) = \dim(\ell_j(W)) + \dim(L_j(W^\perp))$ used to obtain the final line is (4.15) specialized to $V = W^\perp$. □

Proof of sufficiency in Theorem 2.3. The proof follows the inductive scheme of the proof of Theorem 2.1. To simplify notation set $t_j = p_j^{-1} \in [0,1]$. Case 1 now breaks down into two subcases. Case 1A arises when there exists a nonzero proper subspace W of H that is contained in $\mathcal{V}$ and is critical in the sense of (2.8), that is,[3] $\sum_j t_j \dim(\ell_j(W)) = \dim(W)$.

With coordinates (y', y'') for $W^\perp \oplus W$, ℓ_0 is independent of y'', and for every subspace $V \subset W$, $\sum_j t_j \dim(\ell_j(V)) \geq \dim(V)$ by (2.8). Thus the collection of mappings $\{\ell_j|_W\}$ satisfies the hypothesis of Theorem 2.1, whence $\int_W \prod_j f_j \circ \ell_j(y', y'')\, dy'' \leq C \prod_j F_j(y')$ where $\|F_j\|_{L^{p_j}(W^\perp)} \leq C\|f_j\|_{L^{p_j}(H_j)}$.

It remains to bound $\int_{W^\perp} \chi_B \circ \ell_0(y', 0) \prod_j F_j \circ L_j(y')\, dy'$, where B denotes the characteristic function of a ball of finite radius. Theorem 2.3 can be invoked by induction on the ambient dimension, provided that (2.8) and (2.9) hold for the data $W^\perp, \mathcal{V} \cap W^\perp, \{U_j^\perp, L_j, p_j\}$. We will write $(2.8)_H$, $(2.8)_W$, and $(2.8)_{W^\perp}$ to distinguish between this hypothesis for the three different data that arise in the discussion; likewise for (2.9).

$(2.9)_W$ is the condition that $\operatorname{codim}_{W^\perp}(V) \geq \sum_j t_j \operatorname{codim}_{L_j(W^\perp)}(L_j(V))$ for every subspace $V \subset W^\perp$, which is the second conclusion of Lemma 5.1. $(2.8)_W$ is the condition

$$\dim(V) \leq \sum_j t_j \dim(L_j(V)) \quad \text{for all subspaces } V \subset \mathcal{V} \cap W^\perp. \qquad (5.6)$$

Since V, W are both contained in $\mathcal{V}$ so is $V + W$, so $\sum_j t_j \dim(\ell_j(V+W)) \geq \dim(V+W) = \dim(V) + \dim(W)$ by $(2.8)_H$. This together with the previously established identity $\dim(\ell_j(V+W)) = \dim(\ell_j(W)) + \dim(L_j(V))$ and the criticality condition $\sum_j t_j \dim(\ell_j(W)) = \dim(W)$ yields (5.6). Thus Case 1A is treated by applying Theorem 2.1 for W and the induction hypothesis for $W^\perp$.

Case 1B arises when there exists a nonzero proper subspace $W \subset H$ that is critical in the sense of (2.9), that is, $\operatorname{codim}_H(W) = \sum_j t_j \operatorname{codim}_{H_j}(\ell_j(W))$. The analysis follows the same inductive scheme. Lemma 5.1 guarantees that $(2.9)_W$

[3]All summations with respect to j are taken over $1 \leq j \leq m$.

holds, while $(2.8)_W$ is simply the specialization of $(2.8)_H$ to subspaces $V \subset W \cap \mathcal{V}$. Thus Theorem 2.3 may be applied by induction to $W, \{\ell_j(W), \ell_j|_W, p_j\}$.

This reduces matters to $\int_{W^\perp \cap \{|L_0(y')| \le 1\}} \prod_j F_j \circ L_j \, dy'$, where the nullspace $\tilde{V}$ of L_0 is the set of all $y' \in W^\perp$ for which there exists $y'' \in W$ such that $\ell_0(y', y'') = 0$; thus the subspace $\mathcal{V} \subset H$ is now replaced by $\pi_{W^\perp} \mathcal{V} \subset W^\perp$.

Now it is natural to expect to use $(2.8)_H$ to establish $(2.8)_{W^\perp}$, but the latter pertains to certain subspaces not contained in $\mathcal{V}$, about which the former says nothing. Luckily the inequality in (5.6) holds for arbitrary subspaces $V \subset W^\perp$, not merely those contained in $\pi_{W^\perp}\mathcal{V}$. Indeed,

$$\begin{aligned}
\sum_j t_j \dim(L_j(V)) &= \sum_j t_j \dim(\ell_j(V+W)) - \sum_j t_j \dim(\ell_j(W)) \\
&= \sum_j t_j \operatorname{codim}_{H_j}(\ell_j(W)) - \sum_j t_j \operatorname{codim}_{H_j}(\ell_j(V+W)) \\
&= \operatorname{codim}_H(W) - \sum_j t_j \operatorname{codim}_{H_j}(\ell_j(V+W)) \\
&\ge \operatorname{codim}_H(W) - \operatorname{codim}_H(V+W) \\
&= \dim(V).
\end{aligned}$$

The assumption that W is critical in the sense that equality holds in $(2.9)_H$ implies $(2.9)_W^\perp$, by the second conclusion of Lemma 5.1. Thus by induction on the dimension, Theorem 2.3 may be applied to the integral over $W^\perp$, concluding the proof for Case 1B.

Case 2 arises when no subspace W is critical in either sense. Consider the set $K \subset [0,1]^m$ of all $(t_1, \cdots, t_m)$ such that $\sum_j t_j \dim(\ell_j(V)) \ge \dim(V)$ for all subspaces $V \subset \mathcal{V} = \operatorname{kernel}(\ell_0)$, and $\operatorname{codim}_H(V) \ge \sum_j t_j \operatorname{codim}_{H_j}(\ell_j(V))$ for all subspaces $V \subset H$. It suffices to prove that $\int_H \chi_B \circ \ell_0 \prod_{j\ge1} f_j \circ \ell_j \le C \prod_j \|f_j\|_{q_j}$ for every extreme point $(t_1, \cdots, t_m)$ of K, where $q_j = t_j^{-1}$. Consider such an extreme point. If there exists a nonzero proper subspace $V \subset \mathcal{V}$ that is critical in the sense that $\sum_j t_j \dim(\ell_j(V)) = \dim(V)$, or a nonzero proper subspace $V \subset H$ that is critical in the sense that $\operatorname{codim}_H(V) = \sum_j t_j \operatorname{codim}_{H_j}(\ell_j(V))$, then Case 1A or Case 1B apply.

There are other cases in which equality might hold in (2.8) or (2.9), besides those subsumed under Case 1. If equality holds for $V = \{0\}$ in (2.9) with $p_j^{-1} = t_j$, then $\dim(H) = \sum_j t_j \dim(H_j)$, which is the first hypothesis of Theorem 2.1. In conjunction with (2.9) this implies that (2.8) holds for every subspace $V \subset H$, which is the second hypothesis of Theorem 2.1. Therefore the conclusion (2.7) of Theorem 2.3 holds without the restriction $|\ell_0(y)| \le 1$ in the integral, by Theorem 2.1.

If on the other hand $H = \mathcal{V} = \operatorname{kernel}(\ell_0)$ and equality holds for $V = H$ in (2.8) with $p_j^{-1} = t_j$, then $\dim(H) = \sum_j t_j \dim(H_j)$, so Theorem 2.1 applies once more.

Therefore matters reduce to the case where equality holds in (2.8) for no subspace of $\mathcal{V}$ except $V = \{0\}$, and where furthermore equality holds in (2.9) for no subspace of H except for $V = H$ itself. Equality always holds in both of those cases, so they play no part in defining K.

t satisfies $\operatorname{codim}_H(V) \ge \sum_j t_j \operatorname{codim}_{H_j}(V))$ for every subspace $V \subset H$. Therefore as in Case 2 of the proof of Theorem 2.1, every remaining extreme point $(t_1, \cdots, t_m)$ of K must have $t_i = 0$ for at least one index i.

By induction on m, it therefore suffices to treat the case $m = 1$, with $p_1 = \infty$. By (2.8) applied to $V = \text{kernel}\,(\ell_0)$, $\dim(\text{kernel}\,(\ell_0)) \le 0 \dim(H_1) = 0$, so ℓ_1 has no kernel. Therefore the restriction $|\ell_0(y)| \le 1$ constrains y to a bounded region, whence $\int_{|\ell_0(y)|\le 1} f_1 \circ \ell_1(y)\,dy \le C\|f_1\|_{L^\infty}$ for some finite constant C. $\square$

6. Proof of Theorem 2.4

This proof contains no new elements, so will merely be outlined. We denote the identity element of a group by 0. Recall that if H_1, H_2 are subgroups of a finitely generated discrete Abelian group G, and if $H_1 \cap H_2 = \{0\}$, then $\text{rank}\,(H_1 + H_2) = \text{rank}\,(H_1) + \text{rank}\,(H_2)$. Likewise if H' is a subgroup of the quotient group G/H then $\text{rank}\,(H) + \text{rank}\,(H')$ equals $\text{rank}\,(\pi^{-1}(H'))$ where $\pi : G \to G/H$ is the natural projection. A finitely generated Abelian group is finite if and only if its rank is zero.

Let groups G, G_j, homomorphisms φ_j, and exponents p_j satisfy the hypotheses of Theorem 2.4. Consider first the case where there exists a subgroup $G' \subset G$, satisfying $0 < \text{rank}\,(G') < \text{rank}\,(G)$, that is critical in the sense that $\sum_j p_j^{-1}\,\text{rank}\,(\varphi_j(G')) = \text{rank}\,(G')$. Define $G'_j = \varphi_j(G') \subset G_j$. Since every subgroup of G inherits the hypothesis of the theorem, we may conclude by induction on the rank that

$$\sum_{y\in G'} \prod_j f_j \circ \varphi_j(y) \le C \prod_j \|f_j\|_{\ell^{p_j}(G'_j)}. \tag{6.1}$$

Define $F_j \in \ell^{p_j}(G_j/G'_j)$ by

$$F_j(x + G'_j) = \Big(\sum_{z\in G'_j} |f_j(x+z)|^{p_j}\Big)^{1/p_j}.$$

Then $\|F_j\|_{\ell^{p_j}(G_j/G'_j)} \le C\|f_j\|_{\ell^{p_j}(G_j)}$. Define homomorphisms $\psi_j : G/G' \to G_j/G'_j$ by composing φ_j with the quotient map from G_j to G'_j. Then

$$\sum_{y\in G} \prod_j f_j \circ \varphi_j(y) = \sum_{x\in G/G'} \sum_{z\in G'} f_j \circ \varphi_j(x+z) \le \sum_{x\in G/G'} F_j \circ \psi_j(x). \tag{6.2}$$

It suffices to show that the homomorphisms ψ_j inherit the hypothesis of Theorem 2.4, which may then be applied by induction on the rank to yield the desired bound $O(\prod_j \|F_j\|_{\ell^{p_j}})$. This hypothesis is verified using the criticality of G' and the additivity of ranks, just as in the proof of Theorem 2.1.

There remains the case in which no critical subgroup G' of strictly smaller but strictly positive rank exists. Once again we consider the compact convex set K of all $(q_1^{-1}, \cdots, q_N^{-1}) \in [0,1]^m$ for which $\text{rank}\,(H) \le \sum_j q_j^{-1}\,\text{rank}\,(\varphi_j(H))$ for all subgroups $H \subset G$, and it suffices to prove that $\sum_{y\in G}\prod_j f_j \circ \varphi_j(y) \le C \prod_j \|f_j\|_{q_j}$ for all extreme points $(q_1^{-1}, \cdots, q_N^{-1})$ of K.

If $(q_1^{-1}, \cdots, q_N^{-1})$ is an extreme point then either $\sum_j q_j^{-1}\,\text{rank}\,(\varphi_j(G')) = \text{rank}\,(G')$ for some subgroup G' satisfying $0 < \text{rank}\,(G') < \text{rank}\,(G)$, or $q_j^{-1} \in \{0,1\}$ for all indices j, or $\text{rank}\,(G) = \sum_j q_j^{-1}\,\text{rank}\,(\varphi_j(G))$ and $q_j^{-1} \in \{0,1\}$ for all but at most one index j. In the first case we are in the critical case treated above.

Suppose that $(q_1^{-1}, \cdots, q_N^{-1}) \in K$ and $q_j \in \{0,1\}$ for all j. Let $S = \{j : q_j^{-1} = 1\}$, and consider the subgroup $G' = \cap_{j:q_j=1}\,\text{kernel}\,(\varphi_j)$. The hypothesis (2.11) states that $0 = \sum_{j\in S} \text{rank}\,(\varphi_j(G')) \ge \text{rank}\,(G')$, so G' is finite. If f_j is the characteristic function of a single point z_j for each $j \in S$, then $\sum_{y\in G}\prod_{j\in S} f_j \circ \phi_j(y)$ equals the cardinality of $\{y : \phi_j(y) = z_j \ \forall j \in S\}$, which is $\le |G'|$. The inequality then follows for arbitrary functions by multilinearity.

Suppose finally that $q_i^{-1} \in (0,1)$, $q_j^{-1} = 1$ if and only if $j \in S$, and $q_j^{-1} = 0$ if neither $j \in S$ nor $j = i$. Let $S = \{j : q_j^{-1} = 1\}$ and consider $G' = \cap_{j\in S} \operatorname{kernel}(\varphi_j)$. The hypothesis (2.11) states that $\operatorname{rank}(G') \le q_i^{-1} \operatorname{rank}(\varphi_i(G')) + \sum_{j\in S} \operatorname{rank}(\varphi_j(G')) = q_i^{-1} \operatorname{rank}(\varphi_i(G'))$; the right-hand side is necessarily $\le q_i^{-1}\operatorname{rank}(G')$, which is strictly less than $\operatorname{rank}(G')$ unless $\operatorname{rank}(G') = 0$; hence $\operatorname{rank}(G')$ must vanish. Therefore for any nonnegative functions,

$$\sum_{y\in G} \prod_j f_j \circ \varphi_j(y) \le C \prod_{j\in S} \|f_j\|_{\ell^1} \prod_{j\notin S} \|f_j\|_{\ell^\infty},$$

as in the preceding paragraph. Since $\|f_i\|_\infty \le \|f_i\|_{q_i}$, this completes the proof. □

The proof of the variant Theorem 2.5 is nearly identical to that of Theorem 2.4 and is left to the reader. Likewise the proofs of the necessity of the hypotheses in both theorems, which are simplifications of the reasoning shown above for their continuum analogues, are omitted.

7. Variants based on product structure

A variant of our results, based on combinatorial rather than linear algebraic or group theoretic structure, has been obtained earlier by Finner [10]; see also [11] for a discussion of some special cases from another point of view. Let $\{(X_i, \mu_i)_{i\in I}\}$ be a finite collection of measure spaces, and let $(X, \mu) = \prod_{i\in I}(X_i, \mu_i)$ be their product. Let J be another finite index set. For each $j \in J$, let S_j be some nonempty subset of I. Let $Y_j = \prod_{i\in S_j} X_i$, equipped with the associated product measure, and let $\pi_j : X \to Y_j$ be the natural projection map. Let $f_j : Y_j \to [0, \infty]$ be measurable. To avoid trivialities, we assume throughout the discussion that I, J are nonempty and that $\mu(X)$ is strictly positive. Define

$$\Lambda(f_j)_{j\in J} = \int_X \prod_{j\in J} f_j \circ \pi_j \, d\mu. \tag{7.1}$$

Denote by $|\cdot|$ the cardinality of a finite set.

Let $p_j \in [1, \infty]$ for each $j \in J$. Finner's theorem then asserts that if

$$1 = \sum_{j : i\in S_j} p_j^{-1} \text{ for all } i \in I \tag{7.2}$$

then

$$\Lambda(f_j)_{j\in J} \le \prod_{j\in J} \|f_j\|_{L^{p_j}(Y_j)}. \tag{7.3}$$

The hypothesis (7.2) can be equivalently restated as

$$|K| = \sum_{j\in J} p_j^{-1} |S_j \cap K| \text{ for every subset } K \subset I, \tag{7.4}$$

or again as the conjunction of $|I| = \sum_{j\in J} p_j^{-1}|S_j|$ and $|K| \le \sum_{j\in J} p_j^{-1}|S_j \cap K|$ for every $K \subset I$. The analogue of a subspace is now a subset $K \subset I$, and the analogue of criticality is (7.4); the inequality need not hold, in general, unless every subset is critical. This contrasts with the situation treated by Carlen, Lieb, and

Loss [6] and in Theorem 2.1, where generic subspaces will be subcritical even if critical subspaces exist.

When each space X_i is some Euclidean space equipped with Lebesgue measure, the hypotheses in this last form are precisely those of Theorem 2.1, specialized to this limited class of linear mappings. A special case is the Loomis-Whitney inequality

$$\int_{\mathbb{R}^n} \prod_{j=1}^{n} f_j \circ \pi_j(x)\, dx \le \prod_{j=1}^{n} \|f_j\|_{L^{n-1}}\,,$$

where $\pi_j : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the mapping that forgets the j-th coordinate.

Our next result is analogous to a unification of Theorems 2.3 and 2.5. We say that a measure space (X, μ) is atomic if there exists $\delta > 0$ such that $\mu(E) \ge \delta$ for every measurable set E having strictly positive measure.

Proposition 7.1. *Suppose that the index set I is a disjoint union $I = I_0 \cup I_\infty \cup I_\star$, where X_i is a finite measure space for each $i \in I_0$, is atomic for each $i \in I_\infty$, and is an arbitrary measure space for each $i \in I_\star$. Then a sufficient condition for the inequality* (7.3) *is that*

$$1 \ge \sum_{j:i\in S_j} p_j^{-1} \text{ for all } i \in I_0 \tag{7.5}$$

$$1 \le \sum_{j:i\in S_j} p_j^{-1} \text{ for all } i \in I_\infty \tag{7.6}$$

$$1 = \sum_{j:i\in S_j} p_j^{-1} \text{ for all } i \in I_\star. \tag{7.7}$$

That these sufficient conditions are also necessary, in general, is a consequence of the necessity of the hypotheses of Theorem 2.3.

Consider the case where each X_i is a finite measure space. If $(p_j)_{j\in J}$ satisfies the hypothesis (7.2), and if $q_j \ge p_j$ for all $j \in J$, then $\Lambda(f_j)_{j\in J} \le C\prod_j \|f_j\|_{p_j} \le C\prod_j \|f_j\|_{q_j}$ by Finner's theorem and Hölder's inequality. However, there are situations[4] in which $(q_j)_{j\in J}$ satisfies (7.5) yet there exists no $(p_j)_{j\in J}$ satisfying (7.2) with $q_j \ge p_j$ for all $j \in J$.

To construct an example, begin with any situation where there is an extreme point $(q_j^{-1})_{j\in J}$ of $K = \{(t_j)_{j\in J} \in [0,1]^J : 1 = \sum_{j:i\in S_j} t_j \text{ for all } i \in I\}$, such that $q_j^{-1} < 1$ for all j; for instance, the Loomis-Whitney example. Augment I by adding a single new index i', choose one index j' already in J, and replace $S_{j'}$ by $S_j \cup \{i'\}$, while keeping S_j unchanged for all $j \ne j'$. Thus $\sum_{j:i'\in S_j} q_j^{-1} = q_{j'}^{-1} < 1$. Clearly no $(p_j)_{j\in J}$ can then be found with the required properties.

Proposition 7.1 can be proved by repeating Case 1 of the proofs of Theorems 2.1 and 2.3, arguing by induction on $|I|$, and integrating with respect to the m-th coordinate in $\prod_{i\in I} X_i$ while all other coordinates are held constant. The basis case $m = 1$ is Hölder's inequality. Indeed, this is the argument given in [10] for the special case when $I = I_\star$.

[4]The special case of Proposition 7.1 in which all X_i are finite measure spaces is stated in [10], p. 1898, but no proof is given.

Alternatively, when I_0 is empty,[5] Proposition 7.1 can be reduced to the case where each X_i is $\mathbb{R}^1$ equipped with Lebesgue measure, by approximating general functions by finite linear combinations of characteristic functions of product sets, and then embedding any particular situation measure-theoretically into a (product of copies of) $\mathbb{R}^1$. The inequality (7.3) then follows from an application of Theorem 2.1.

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[5] To treat the general case in this way would require a unification of Theorems 2.3 and 2.5 analogous to Proposition 7.1. We see no obstruction to such a result.