RESTRICTION FOR FLAT SURFACES OF REVOLUTION IN $\mathbf{R}^3$

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Abstract. We investigate restriction theorems for hypersurfaces of revolution in $\mathbf{R}^3$, with affine curvature introduced as a mitigating factor. Abi-Khuzam and Shayya recently showed that a Stein-Tomas restriction theorem can be obtained for a class of convex hypersurfaces that includes the surfaces $\Gamma(x) = (x, e^{-1/|x|^m}), m \geq 1$. We enlarge their class of hypersurfaces and give a much simplified proof of their result.

1. Introduction and Statement of Results

If S is a smooth $(n-1)$-dimensional submanifold in $\mathbf{R}^n$ $(n \geq 3)$, S_0 is a compact subset with non-vanishing Gaussian curvature and $d\sigma$ is the induced Lebesgue measure, then the $L(p,q)$ Stein-Tomas restriction theorem ([13],[14]) says that, for all $f \in L^p(\mathbf{R}^n)$, $\left(\int_{S_0} |\hat{f}(\xi)|^q d\sigma(\xi)\right)^{1/q} \leq C\|f\|_p$, for $1 \leq p \leq \frac{2n+2}{n+3},\ q \leq \left(\frac{n-1}{n+1}\right) p'$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

The key result is when $q = 2, p = \frac{2n+2}{n+3}$, i.e.

$$\|\hat{f}\|_{L^2(d\sigma)} \leq C\|f\|_{L^{\frac{2n+2}{n+3}}}. \tag{1}$$

The full range then follows by interpolation with the case $p = 1$.

Iosevich and Lu [7] proved that restriction is equivalent to non-vanishing curvature. More precisely, if (1) holds, then the surface must have non-vanishing Gaussian curvature.

There are various related results for hypersurfaces whose Gaussian curvature may vanish but which nevertheless satisfy some other conditions such as being finite-type or having non-vanishing principal curvatures. See, for example, [13], [5], [12], [10].

Our interest lies with analogues of the Stein-Tomas restriction theorem for surfaces that may be flat, possibly to infinite order. Other authors to consider this case include Brandolini/Iosevich/Travaglini, [3], Bak, [2], Oberlin, [9], and most recently Abi-Khuzam/Shayya, [1]. The work of Oberlin and that of Abi-Khuzam/Shayya plays a major role in our result. This will be discussed further in a moment.

In this paper we consider surfaces $\Gamma(\xi,\mu) = (\xi, \mu, \gamma(\xi,\mu))$, in $\mathbf{R}^3$, where $\xi, \mu \in \mathbf{R}$ and $\gamma : \mathbf{R}^2 \longrightarrow \mathbf{R}$. To compensate for the possible flatness we replace the induced Lebesgue measure with affine surface area. So we insert a mitigating factor

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$|K_\Gamma(\xi,\mu)|^{1/4} = |\det \mathrm{Hess}\gamma(\xi,\mu)]|^{1/4}$ into the left-hand-side of (1) and look for inequalities of the form

$$\left(\int |\hat{f}(\Gamma(\xi,\mu))|^2 |K_\Gamma(\xi,\mu)|^{1/4} d\xi d\mu\right)^{1/2} \le C\|f\|_{4/3}. \tag{2}$$

With this choice of mitigating factor the affine invariance of the restriction inequality is preserved. Moreover (2) is invariant under reparametrisation of the hypersurface. Because of this, we consider $|K_\Gamma|^{1/4}$ to be the optimal choice of mitigating factor. We note that in general, for surfaces in $\mathbf{R}^n$, the corresponding mitigating factor is $|K_\Gamma|^{\frac{1}{n+1}}$.

The analogous inequality for $n = 2$ holds for all convex curves γ, with a constant independent of γ. This was shown by Sjőlin in [11]. (In fact, Sjőlin proved the optimal result, namely $L(p,q)$ restriction for $p < \frac{4}{3},\ q \le \frac{p'}{3}$.) Thus, there is a *universal* restriction theorem for convex curves in $\mathbf{R}^2$. We would like to know whether there is a universal restriction theorem for an analogous class of surfaces in $\mathbf{R}^3$. For radial surfaces $(\Gamma(\xi,\mu) = (\xi,\mu,\gamma(|(\xi,\mu)|)))$ some progress has been made on this question, most recently in [9] and [1], and our Theorem 1.2 below is a further step. We note that for radial surfaces in $\mathbf{R}^3$, $K_\Gamma(r) = \frac{\gamma''(r)\gamma'(r)}{r}$.

The standard approach to prove L^2 restriction theorems is via decay estimates for the Fourier transform of the measure supported on the surface. More precisely, in [8], it was shown that, for γ defined on $[0,b)$, (2) follows from a decay estimate of the form $\left|\int_0^b e^{it\gamma(r)} J_0(r|(x,y)|)|K_\Gamma(r)|^{1/2+i\alpha} r dr\right| \le C\frac{(1+|\alpha|)^N}{|t|}$, for all (x,y,t) with C independent of γ.

This approach was pursued in [4]. The result there showed restriction for a class of convex surfaces satisfying some additional curvature and normalization conditions. This class included the examples $\gamma(r) = r^m,\ m \ge 2$, and $\gamma(r) = r^m \log\frac{1}{r},\ m > 2$, but did not include the exponentially flat surfaces $\gamma(r) = e^{-\frac{1}{r^m}}$, $m > 0$. In fact, as was shown in [4] the decay for these surfaces is of the order $\frac{(\log|t|)^{1/2}}{|t|}$ and no better.

The $\mathbf{R}^2$ result of Sjőlin mentioned above was proved without decay estimates. Instead the proof exploited the relationship $\left(\frac{4}{3}\right)' = 4 = 2^2$. Recently, in [9], Oberlin developed an approach for surfaces in $\mathbf{R}^3$ via this relationship, thus avoiding decay estimates. Oberlin was able to prove a uniform restriction theorem for the class of γ that are $C^3[0,b),\ \gamma(0) = \gamma'(0) = 0, \gamma^i(r) > 0$, for $r > 0, i = 1,2,3$. However the mitigating factor used was not $|K_\Gamma|^{1/4}$, but the smaller factor $\left(\frac{\gamma'(r)}{r}\right)^{1/2}$. Oberlin's class includes the asymptotically flat examples.

Most recently Abi-Khuzam and Shayya, [1], used the ideas of [9], as well as some very intricate calculations to prove restriction with the optimal mitigating factor, $|K_\Gamma|^{1/4}$, for a class of radial hypersurfaces that includes the examples $\gamma(r) = e^{-\frac{1}{r^m}}$. This result includes that of [4]; however the restriction theorem they give is not universal as stated, since the constant depends on γ. Nevertheless a universal restriction theorem may be deduced from their result; see Remark 2 below.

In the following we use the notation $a \approx b$ if there are absolute constants c and C such that $cb \le a \le Cb$.

Theorem 1.1. *(Abi-Khuzam and Shayya 2004)*[1]

Suppose that $\gamma : [0,b) \longrightarrow \mathbf{R}$, $\gamma \in C^3[0,b)$, $\gamma''(r) > 0$, *for* $0 < r < b$, $\gamma''' \geq 0$ *for* $0 \leq r < b$, $\gamma(0) = \gamma'(0) = 0$. *Then* L^2 *restriction* (2) *holds for the surface* $\Gamma(\xi,\mu) = (\xi,\mu,\gamma(|(\xi,\mu)|)$, *with constant* $C \approx \sup_{r\in(0,b)} \left[\frac{\gamma(r)\gamma''(r)}{\gamma'(r)^2}\right]^{1/8}$.

Our result improves on that of [1], however the most notable aspect of our result may be the simplicity of our proof.

Theorem 1.2. *Let* $\gamma \in C^1[0,b)$ *and* $C^2(0,b)$, $\gamma(0) = 0, \gamma'(r) \geq 0$, *for* $r \in [0,b)$, $\gamma''(r) > 0$, *for* $r \in (0,b)$. *Suppose, for* $k \geq 0$,

$$\sup_{r\in[2^{-k-1},2^{-k+1}]} \frac{\gamma(r)\gamma''(r)}{\gamma'(r)^2} \leq C_1 \inf_{r\in[2^{-k-1},2^{-k+1}]} \frac{\gamma(r)\gamma''(r)}{\gamma'(r)^2} \tag{3}$$

$$\begin{aligned} &\tfrac{\gamma(r_1)\gamma''(r_1)}{\gamma'(r_1)^2}\gamma(r_2)\gamma'(r_2) + \tfrac{\gamma(r_2)\gamma''(r_2)}{\gamma'(r_2)^2}\gamma(r_1)\gamma'(r_1) \geq \\ &\tfrac{1}{2}(\gamma(r_1)-\gamma(r_2))(\gamma'(r_1)-\gamma'(r_2)), \end{aligned} \tag{4}$$

$\forall\ 2^{-k-1} \leq r_2 \leq r_1 \leq 2^{-k+1}$. *Then* L^2 *restriction* (2) *holds with constant* $C \approx C_1^{1/8}$.

Remark 1: a) We first point out that conditions (3) and (4) are implied by the conditions in [1], and thus our theorem contains Theorem 1.1. To see this we begin by noting that the conditions of Theorem 1.1 in fact give $\frac{1}{2} \leq \frac{\gamma(r)\gamma''(r)}{\gamma'(r)^2} \leq C$, for all $r \in (0,b)$. The left-hand inequality is a consequence of $\gamma''' \geq 0$, and the normalization conditions $\gamma(0) = \gamma'(0) = 0$. This is easily seen:

$$\begin{aligned} \frac{1}{2}\gamma'(r)^2 &= \int_0^r \gamma'(u)\gamma''(u)du = \gamma(r)\gamma''(r) - \int_0^r \gamma(u)\gamma'''(u)du \\ &\leq \gamma(r)\gamma''(r). \end{aligned}$$

It is now trivial to see that (3) follows from the conditions of [1]. For (4) we have

$$\begin{aligned} &\tfrac{\gamma(r_1)\gamma''(r_1)}{\gamma'(r_1)^2}\gamma(r_2)\gamma'(r_2) + \tfrac{\gamma(r_2)\gamma''(r_2)}{\gamma'(r_2)^2}\gamma(r_1)\gamma'(r_1) \\ &\geq \tfrac{1}{2}(\gamma(r_2)\gamma'(r_2) + \gamma(r_1)\gamma'(r_1)) \\ &\geq \tfrac{1}{2}(\gamma(r_1)-\gamma(r_2))(\gamma'(r_1)-\gamma'(r_2)). \end{aligned}$$

b) Our condition (3) implies $\sup_{0\leq r\leq b} \frac{\gamma(r)\gamma''(r)}{\gamma'(r)^2} \leq \frac{4}{3}C_1$. This is because, for all k,

$$\begin{aligned} \inf_{2^{-k-1}\leq r\leq 2^{-k+1}} \frac{\gamma(r)\gamma''(r)}{\gamma'(r)^2} &\leq \frac{2^{k+1}}{3}\int_{2^{-k-1}}^{2^{-k+1}} \frac{\gamma(u)\gamma''(u)}{\gamma'(u)^2}du \\ &\leq \frac{2^{k+1}}{3}\int_0^{2^{-k+1}} \frac{\gamma(u)\gamma''(u)}{\gamma'(u)^2}du \\ &= \frac{2^{k+1}}{3}\int_0^{2^{-k+1}} \left(1-\frac{\gamma}{\gamma'}\right)'(u)du \\ &= \frac{2^{k+1}}{3}\left(2^{-k+1} - \frac{\gamma(2^{-k+1})}{\gamma'(2^{-k+1})}\right) \\ &\leq \frac{4}{3}. \end{aligned}$$

Remark 2: Universal Theorems

If $\gamma(0)=0$, $\gamma\in C^2$ is convex and increasing and $\frac{\gamma\gamma''}{\gamma'^2}$ is monotone (either increasing or decreasing) then $\frac{\gamma(r)\gamma''(r)}{\gamma"(r)^2}\le 2$, for all r. (For details of this, see Corollary 3.1 below.) Consequently Theorem 1.1 gives a universal restriction theorem for the class of $\gamma\in C^3[0,b)$, with $\gamma(0)=\gamma'(0)=0, \gamma''>0$ on $(0,b)$, $\gamma'''\ge 0$ on $[0,b)$, and $\frac{\gamma\gamma''}{\gamma'^2}$ monotone.

It is shown below in §3 that if $\gamma\in C^1[0,b), \gamma\in C^2(0,b),\ \gamma(0)=0, \gamma'\ge 0$ on $[0,b)$, $\gamma''>0$ on $(0,b)$, $\frac{\gamma\gamma''}{\gamma'^2}$ monotone, and $\lim_{r\longrightarrow 0}\dfrac{\gamma\gamma''}{\gamma'^2}\ne 0$ then (3) and (4) hold automatically, and therefore Theorem 1.2 gives a universal restriction theorem for this class of curves.

If $\frac{\gamma\gamma''}{\gamma'^2}$ is monotone-increasing and $\lim_{r\longrightarrow 0}\dfrac{\gamma\gamma''}{\gamma'^2}=0$ then (4) (with 2 replaced by $\frac{5}{4}$) holds automatically, but (3) need not. In this case we obtain a universal restriction theorem if we also assume, for example, that $\frac{\gamma\gamma''}{\gamma'^2}$ is concave, since then (3) holds.

Further details pertaining to this remark can be found in Section 3 below.

Remark 3: Although conditions (3) and (4) are perhaps unwieldy, they do allow for examples not previously covered. For example, $\gamma(r)=r^m$ and $\gamma(r)=r^m log\frac{1}{r}$ can now be dealt with for $m>1$. Other examples are $\gamma(r)=\frac{r}{(log\frac{1}{r})^m},\ \ m>0$ and $\gamma(r)=\frac{r}{(m+1-r^m)^{1/m}},\ m>0$. These examples are of particular interest since, in both cases, we have $\lim_{r\longrightarrow 0}\dfrac{\gamma(r)\gamma''(r)}{\gamma'(r)^2}=0$.

2. Proof of Theorem 1.2

Proof. We assume that $b=\frac{3}{2}$ and supp $\hat{u}_0\subset\{(\xi,\mu):|(\xi,\mu)|\le\frac{3}{2}\}$. We let

$$U(t)u_0(x,y)=\int\int e^{it\gamma(\xi,\mu)}e^{i(x\xi+y\mu)}\hat{u}_o(\xi,\mu)|K_\Gamma(\xi,\mu)|^{1/8}d\xi d\mu.$$

Then the desired restriction theorem is equivalent to

$$(5)\qquad \|U(t)u_0\|_{L^4_{xyt}}\le C\|u_0\|_{L^2_{xy}},\ \text{or equivalently,}\quad \|[U(t)u_0]^2\|_{L^2_{xyt}}\le C\|u_0\|^2_{L^2_{xy}}.$$

We now define the radial function $\phi(\xi,\mu)=\begin{cases}1 & 1\le|(\xi,\mu)|\le\frac{3}{2}\\ 0 & |(\xi,\mu)|\le\frac{1}{2}\text{ or }|(\xi,\mu)|\ge 2.\end{cases}$ We then define $\phi_k(\xi,\mu)=\phi(2^k\xi,2^k\mu)$ and $\widehat{P_kf}(\xi,\mu)=\phi_k(\xi,\mu)\hat{f}(\xi,\mu)$. We may choose ϕ such that $\sum_{k\ge 0}|\phi_k(\xi,\mu)|^4\ge c$, for $|(\xi,\mu)|\le\frac{3}{2}$, and then $\{P_k\}$ and $\{P_kP_k\}$ are both Littlewood-Paley families of operators, i.e.,

$$(6)\qquad \left\|\left(\sum|P_kf|^2\right)^{1/2}\right\|_{L^p_{xy}}\approx\|f\|_{L^p_{xy}},\quad 1<p<\infty.$$

and

$$(7)\qquad \left\|\left(\sum|P_kP_kf|^2\right)^{1/2}\right\|_{L^p_{xy}}\approx\|f\|_{L^p_{xy}},\quad 1<p<\infty.$$

We now claim that (5) follows once we have

(8)

$$\|U(t)P_ku_0\|_{L^4_{xyt}}\le C\|u_0\|_{L^2_{xy}},\ \text{or equivalently,}\ \|[U(t)P_ku_0]^2]\|_{L^2_{xyt}}\le C\|u_0\|^2_{L^2_{xy}}.$$

To see this, we first use a vector-valued version of (7), then (8), followed by (6) to obtain

$$\begin{aligned}\|U(t)u_0\|_{L^4_{xyt}} &\le C \left\| \left(\sum |P_k^2(U(t)u_0)|^2\right)^{1/2} \right\|_{L^4_{xyt}} \\ &\le C\left(\sum \|U(t)P_k^2 u_0\|^2_{L^4_{xyt}}\right)^{1/2} \\ &\le C\left(\sum \|P_k u_0\|^2_{L^2_{xy}}\right)^{1/2} = \left\| \left(\sum |P_k u_0|^2\right)^{1/2}\right\|_{L^2_{xy}} \\ &\le C\|u_0\|_{L^2_{xy}}.\end{aligned}$$

This proves the claim and so we are left with showing (8), i.e.

$$\begin{aligned}(9)\qquad &\Big\|\int\int\int\int e^{it[\gamma(\xi_1,\mu_1)+\gamma(\xi_2,\mu_2)]+i[x(\xi_1+\xi_2)+y(\mu_1+\mu_2)]}\hat{u}_0(\xi_1,\mu_1)\cdot\hat{u}_0(\xi_2,\mu_2) \\ &|K_\Gamma(\xi_1,\mu_1)|^{\frac{1}{8}}|K_\Gamma(\xi_2,\mu_2)|^{\frac{1}{8}}\phi_k(\xi_1,\mu_1)\phi_k(\xi_2,\mu_2)d\xi_1 d\mu_1 d\xi_2 d\mu_2\Big\|_{L^2_{xyt}} \\ &\le C\|u_0\|^2_{L^2_{xy}}.\end{aligned}$$

Next we change variables.

$$\begin{aligned} u &= \xi_1+\xi_2 \\ v &= \mu_1+\mu_2 \\ w &= \gamma(\xi_1,\mu_1)+\gamma(\xi_2,\mu_2) \\ z &= \text{to be chosen later}\end{aligned}$$

Then, with $J = \frac{\partial(u,v,w,z)}{\partial(\xi_1,\xi_2,\mu_1,\mu_2)}$, (9) becomes

$$\left\|\int\int\int\int e^{i[tw+xu+yv]}F(u,v,w,z)\frac{dudvdwdz}{J}\right\|_{L^2_{xyt}} \le C\|u_0\|^2_{L^2_{xy}}.$$

By Plancherel in u, v, w, this is $\left\|\int \frac{F(u,v,w,z)}{J}dz\right\|_{L^2_{u,v,w}} \le C\|u\|^2_{L^2_{xy}}$.

Now, by Cauchy-Schwarz we can bound the left-hand side by

$$\begin{aligned}&\Big\|\left(\int |\hat{u}_0(\xi_1,\mu_1)|^2|\hat{u}_0(\xi_2,\mu_2)|^2\tfrac{dz}{|J|}\right)^{1/2}\cdot \\ &\left(\int |K_\Gamma(\xi_1,\mu_1)|^{\frac{1}{4}}|K_\Gamma(\xi_2,\mu_2)|^{\frac{1}{4}}\phi_k(\xi_1,\mu_1)^2\phi_k(\xi_2,\mu_2)^2\tfrac{dz}{|J|}\right)^{\frac{1}{2}}\Big\|_{L^2_{uvw}} \\ &\le C\sup_{u,v,w}\left(\int_{R_k}|K_\Gamma(\xi_1,\mu_1)|^{\frac{1}{4}}|K(\xi_2,\mu_2)|^{\frac{1}{4}}\frac{dz}{|J|}\right)^{\frac{1}{2}}\|u_0\|^2_{L^2_{xy}},\end{aligned}$$

where $R_k = R_k(u,v,w) = \{z \in \mathbf{R} | 2^{-k-1} \le |(\xi_1,\mu_1)|, |(\xi_2,\mu_2)| \le 2^{-k+1}\}$ and ξ_1, μ_1, ξ_2, μ_2 are understood to be functions of u, v, w, z.

Thus to prove (9) and hence the restriction theorem (2), it suffices to prove that

$$(10)\qquad \sup_{u,v,w}\left(\int_{R_k}|K_\Gamma(\xi_1,\mu_1)|^{1/4}|K_\Gamma(\xi_2,\mu_2)|^{1/4}\frac{dz}{|J|}\right) \le C.$$

The choice of z is at our disposal.

If we carry out the same process using polar coordinates

$$\begin{aligned} u &= r_1\cos\theta_1 + r_2\cos\theta_2 \\ v &= r_1\sin\theta_1 + r_2\sin\theta_2 \\ w &= \gamma(r_1)+\gamma(r_2) \\ z &= \text{to be chosen later,} \end{aligned}$$

then $J = \frac{\partial(u,v,w,z)}{\partial(r_1,r_2,\theta_1,\theta_2)}$ and (10) becomes

$$\sup_{u,v,w}\left(\int_{R_k}\left(\frac{\gamma''(r_1)\gamma'(r_1)}{r_1}\right)^{1/4}\left(\frac{\gamma''(r_2)\gamma'(r_2)}{r_2}\right)^{1/4} r_1r_2\frac{dz}{|J|}\right)\le C. \tag{11}$$

Following [9] (and [1]) we choose $z = \arctan\sqrt{\frac{\gamma(r_1)}{\gamma(r_2)}}$. Then $\gamma(r_1) = w\sin^2 z$ and $\gamma(r_2) = w\cos^2 z$. Also $|J| = \frac{r_1r_2}{2}\frac{\gamma'(r_1)\gamma'(r_2)}{\sqrt{\gamma(r_1)\gamma(r_2)}}|\sin(\theta_1-\theta_2)|$. Then (11) becomes

$$\sup_{u,v,w}\left(\int_{R_k}\frac{[\frac{\gamma''\gamma}{\gamma'^2}(r_1)\frac{\gamma''\gamma}{\gamma'^2}(r_2)]^{1/4}}{|\sin(\theta_1-\theta_2)|}\left(\frac{\gamma(r_1)\gamma(r_2)}{r_1r_2\gamma'(r_1)\gamma'(r_2)}\right)^{1/4}dz\right)\le C.$$

If we let $\inf_{r\in[2^{-k-1},2^{-k+1}]}\frac{\gamma(r)\gamma''(r)}{\gamma'(r)^2} = \epsilon_k$, and use symmetry to reduce to the case $r_1\ge r_2$, then, by (3), it suffices to show

$$\sup_{u,v,w}\int_{\pi/4}^{\pi/2}\frac{\sqrt{\epsilon_k}}{|\sin(\theta_1-\theta_2)|}\left(\frac{\gamma(r_1)\gamma(r_2)}{r_1r_2\gamma'(r_1)\gamma'(r_2)}\right)^{1/4}dz\le C. \tag{12}$$

We now note that $u^2+v^2 = r_1^2+r_2^2+2r_1r_2\cos(\theta_1-\theta_2)$ and so, for $|\theta_1-\theta_2|\le\frac{1}{100}$, (which we may assume, since it is enough to prove the theorem with $\hat{u}_0$ having support in a a narrow angle) we have

$$\begin{aligned} \sin(\theta_1-\theta_2) &= \sqrt{1-\cos(\theta_1-\theta_2)}\sqrt{1+\cos(\theta_1-\theta_2)} \\ &\approx \sqrt{\frac{(r_1+r_2)^2-(u^2+v^2)}{2r_1r_2}} \\ &\approx 2^{k/2}\sqrt{r_1+r_2-\sqrt{u^2+v^2}} \\ &\approx \sqrt{\frac{r_1+r_2}{\sqrt{u^2+v^2}}-1}. \end{aligned}$$

We note that for $|\theta_1-\theta_2|\le\frac{1}{100}$, the quantity $\frac{r_1+r_2}{\sqrt{u^2+v^2}}-1$ is always ≥ 0.
We define

$$f(u,v,w,z) = \frac{r_1+r_2}{\sqrt{u^2+v^2}}-1 = \frac{\gamma^{-1}(w\sin^2 z)+\gamma^{-1}(w\cos^2 z)}{\sqrt{u^2+v^2}}-1.$$

Then

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{2}{\sqrt{u^2+v^2}}\left(\frac{w\sin z\cos z}{\gamma'(\gamma^{-1}(w\sin^2 z))}-\frac{w\sin z\cos z}{\gamma'(\gamma^{-1}(w\cos^2 z))}\right) \\ &= \frac{2}{\sqrt{u^2+v^2}}\sqrt{\gamma(r_1)\gamma(r_2)}\left(\frac{1}{\gamma'(r_1)}-\frac{1}{\gamma'(r_2)}\right)\le 0, \end{aligned}$$

since $\gamma'' > 0$ and $r_1 \geq r_2$. Equality holds if, and only if, $z = \frac{\pi}{4}$. It follows that f is a decreasing function of z and so there is at most one zero of f, call it z_0. In the event that f has no zero in z we take $z_0 = \frac{\pi}{2}$.

Thus (12) now becomes

$$\sup_{u,v,w} \int_{\pi/4}^{z_0} \frac{\sqrt{\epsilon_k}}{\sqrt{f(u,v,w,z)}} \left(\frac{\gamma(r_1)\gamma(r_2)}{r_1 r_2 \gamma'(r_1)\gamma'(r_2)} \right)^{1/4} dz \leq C.$$

The observation that the region of integration stops at z_0, and not $\frac{\pi}{2}$ was made in [9]. We now note that, by (4),

$$-\frac{\partial^2 f}{\partial z^2}(u,v,w,z) =$$
$$\frac{4}{\sqrt{u^2+v^2}} \left[\frac{\gamma\gamma''}{\gamma'^2}(r_1) \frac{\gamma(r_2)}{\gamma'(r_1)} + \frac{\gamma\gamma''}{\gamma'^2}(r_2) \frac{\gamma(r_1)}{\gamma'(r_2)} - \frac{(\gamma(r_1)-\gamma(r_2))}{2} \left(\frac{1}{\gamma'(r_2)} - \frac{1}{\gamma'(r_1)} \right) \right]$$
$$\geq 0.$$

It follows that $-\frac{\partial f}{\partial z}$ is increasing in z. Then $f(u,v,w,z) \geq -\int_z^{z_0} \frac{\partial f}{\partial z}(s)ds \geq -\frac{\partial f}{\partial z}(z)(z_0 - z)$ and hence $\sqrt{-\frac{\frac{\partial f}{\partial z}}{f(u,v,w,z)}} \leq \frac{1}{\sqrt{z_0 - z}}$. Moreover we claim that

$$-\frac{\partial f}{\partial z} \geq C\epsilon_k \sqrt{\frac{\gamma(r_1)\gamma(r_2)}{r_1 r_2 \gamma'(r_1)\gamma'(r_2)}} (z - \frac{\pi}{4}). \tag{13}$$

Assuming for a moment that (13) is indeed true, we have

$$\begin{aligned} &\int_{\pi/4}^{z_0} \frac{\sqrt{\epsilon_k}}{\sqrt{f(u,v,w,z)}} \left(\frac{\gamma(r_1)\gamma(r_2)}{r_1 r_2 \gamma'(r_1)\gamma'(r_2)} \right)^{1/4} dz \\ &\leq C \int_{\pi/4}^{z_0} \sqrt{-\frac{\frac{\partial f}{\partial z}}{f(u,v,w,z)}} \frac{1}{\sqrt{z - \frac{\pi}{4}}} dz \\ &\leq C \int_{\pi/4}^{z_0} \frac{1}{\sqrt{z_0 - z}} \frac{1}{\sqrt{z - \frac{\pi}{4}}} dz \\ &\leq C. \end{aligned}$$

Thus it remains to prove (13). We have

$$\frac{\gamma\gamma''}{\gamma'^2}(r) \geq \epsilon_k \Longrightarrow \frac{\gamma''}{\gamma'}(r) \geq \epsilon_k \frac{\gamma'}{\gamma}(r) \Longrightarrow \frac{\gamma'(r_1)}{\gamma'(r_2)} \geq \left(\frac{\gamma(r_1)}{\gamma(r_2)} \right)^{\epsilon_k} = (\tan z)^{2\epsilon_k}.$$

Then

$$
\begin{aligned}
-\frac{\partial f}{\partial z} &= \frac{2\sqrt{\gamma(r_1)\gamma(r_2)}}{\sqrt{u^2+v^2}}\left(\frac{1}{\gamma'(r_2)}-\frac{1}{\gamma'(r_1)}\right)\\
&\geq C\sqrt{\frac{\gamma(r_1)\gamma(r_2)}{r_1r_2\gamma'(r_1)\gamma'(r_2)}}\frac{\gamma'(r_1)-\gamma'(r_2)}{\gamma'(r_1)+\gamma'(r_2)}\frac{\gamma'(r_1)+\gamma'(r_2)}{\sqrt{\gamma'(r_1)\gamma'(r_2)}}\\
&\geq C\sqrt{\frac{\gamma(r_1)\gamma(r_2)}{r_1r_2\gamma'(r_1)\gamma'(r_2)}}\frac{\frac{\gamma'(r_1)}{\gamma'(r_2)}-1}{1+\frac{\gamma'(r_1)}{\gamma'(r_2)}}\\
&\geq C\sqrt{\frac{\gamma(r_1)\gamma(r_2)}{r_1r_2\gamma'(r_1)\gamma'(r_2)}}\frac{(\tan z)^{2\epsilon_k}-1}{1+(\tan z)^{2\epsilon_k}}\\
&\geq C\epsilon_k\sqrt{\frac{\gamma(r_1)\gamma(r_2)}{r_1r_2\gamma'(r_1)\gamma'(r_2)}}\frac{\tan z-1}{1+\tan z} \qquad (14)\\
&\geq C\epsilon_k\sqrt{\frac{\gamma(r_1)\gamma(r_2)}{r_1r_2\gamma'(r_1)\gamma'(r_2)}}(z-\frac{\pi}{4}),
\end{aligned}
$$

□

where (14) follows since $\epsilon_k \leq \frac{4}{3}$, for all k. (See Remark 1b) after Theorem1.2.)

3. Universal theorems

We recall that, in Remark 2, we claimed that universal restriction theorems could be obtained by making certain monotonicity assumptions. In this section we justify this claim in detail. The following lemma is the key.

Lemma 3.1. *a) If $\gamma \in C^2$ is convex and increasing, and $\gamma(0) = 0$, then*

$$\frac{1}{r}\int_0^r \frac{\gamma(u)\gamma''(u)}{\gamma'(u)^2}du \leq 1.$$

b) If $h \geq 0$, and $\frac{1}{r}\int_0^r h(u)du \leq 1$, then there is some γ such that $\gamma(0) = 0, \gamma'' \geq 0, \gamma' \geq 0$, and $h = \frac{\gamma\gamma''}{\gamma'^2}$. In fact,

$$\gamma(r) = \gamma(1)exp\left(-\int_r^1 \frac{1}{s-\int_0^s h(u)du}ds\right). \qquad (15)$$

Proof. a) $\int_0^r \frac{\gamma\gamma''}{\gamma'^2}(u)du = \int_0^r 1 - \left(\frac{\gamma}{\gamma'}\right)'(u)du = r - \frac{\gamma(r)}{\gamma'(r)}$ and so, in particular, $\frac{1}{r}\int_0^r \frac{\gamma\gamma''}{\gamma'^2}(u)du \leq 1$.

b) The calculation is straightforward.

□

Corollary 3.1. *a) If $\gamma \in C^2$ is convex and increasing, $\gamma(0) = 0$, and $\frac{\gamma\gamma''}{\gamma'^2}$ is monotone-increasing then $\frac{\gamma(r)\gamma''(r)}{\gamma'(r)^2} \leq 2$, for all $r \geq 0$.*

b) If $\gamma \in C^2$ is convex and increasing, $\gamma(0) = 0$, and $\frac{\gamma\gamma''}{\gamma'^2}$ is monotone-decreasing then $\frac{\gamma(r)\gamma''(r)}{\gamma'(r)^2} \leq 1$, for all $r \geq 0$.

Proof. a) Since $\frac{\gamma(r)\gamma''(r)}{\gamma'(r)^2}$ is increasing, we use Lemma 3.1a) to obtain $\frac{\gamma(r)\gamma''(r)}{\gamma'(r)^2} \leq \frac{1}{r}\int_r^{2r} \frac{\gamma(u)\gamma''(u)}{\gamma'(u)^2}du \leq \frac{1}{r}\int_0^{2r} \frac{\gamma(u)\gamma''(u)}{\gamma'(u)^2}du \leq 2$.

b) We again use Lemma 3.1a). We have $\frac{\gamma(r)\gamma''(r)}{\gamma'(r)^2} \leq \frac{1}{r}\int_0^{r} \frac{\gamma(u)\gamma''(u)}{\gamma'(u)^2}du \leq 1$. □

We are now in a position to give the justification of Remark 2.

Proposition 3.1. *Suppose that* $\gamma \in C^1[0,b), C^2(0,b), \gamma(0) = 0,\ \gamma' \geq 0$ *on* $[0,b)$, $\gamma'' > 0$ *on* $(0,b)$.

a) If $\frac{\gamma(r)\gamma''(r)}{\gamma'(r)^2}$ *is continuous, monotone (either increasing or decreasing), and* $\lim_{r\longrightarrow 0} \frac{\gamma(r)\gamma''(r)}{\gamma'(r)^2} = \alpha$, *with* $\alpha > 0$, *then* (3) *and* (4) *hold, for* k *sufficiently large.*

b) If $\frac{\gamma(r)\gamma''(r)}{\gamma'(r)^2}$ *is continuous,monotone-increasing and* $\lim_{r\longrightarrow 0} \frac{\gamma(r)\gamma''(r)}{\gamma'(r)^2} = 0$, *then* (4) *holds, for all* $\left(\frac{5}{4}\right)^{-k-1} \leq r_2 \leq r_1 \leq \left(\frac{5}{4}\right)^{-k+1}$, *and for* k *sufficiently large.*

We remark that a change of scale is needed in b). This is a technicality and is of no consequence since the Littlewood-Paley theory used in the proof of Theorem 1.2 can be done with any $\lambda > 1$.

The proof of this proposition is a calculus exercise, which relies on Lemma 3.1.

Finally, we observe that, by Proposition 3.1, we need h to oscillate if (4) is to fail. An example of a curve for which (4) *fails* is the curve given by

$$\frac{\gamma\gamma''}{\gamma'^2} = \begin{cases} \epsilon & 2^{-k} \leq s \leq 2^{-k}(1+2^{-k}) \\ 1-2^{-k} & 2^{-k}(1+2^{-k}) < s \leq 2^{-k+1}. \end{cases}$$

We note that, by Corollary 3.1, it suffices to define the quotient $\frac{\gamma\gamma''}{\gamma'^2}$, since we can then recover γ via (15).

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