LOCALISATION AND WEIGHTED INEQUALITIES FOR SPHERICAL FOURIER MEANS

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Abstract. In this work we establish certain equivalences between the localisation properties with respect to spherical Fourier means of the support of a given Borel measure and the L^2-rate of decay of the Fourier extension operator associated to it. This, in turn, is intimately connected with the property that the X-ray transform of the measure be uniformly bounded. Geometric properties of sets supporting such a measure are studied.

§1. Introduction.

In this paper we continue with the work initiated in [CS2] and [CS3] concerning the geometric properties of the sets where the localisation property for the spherical Fourier means in $\mathbb{R}^n$ holds. Let us start by recalling some basic definitions and notation. For a suitable function f defined on $\mathbb{R}^n$ we denote its Fourier transform by $\widehat{f}$, and for $R > 0$ we define

$$S_R f(x) = \int_{|\xi|<R} \widehat{f}(\xi)\, e^{2\pi i x\cdot\xi}\, d\xi.$$

One of the most interesting and difficult problems in harmonic analysis is that of determining whether we can recover the values of every function f in $L^2(\mathbb{R}^n)$ from the pointwise limit of its spherical means $S_R f$. That is, whether or not we have

$$\lim_{R\to\infty} S_R f(x) = f(x) \qquad a.e., \qquad \forall f \in L^2(\mathbb{R}^n).$$

The result is known to be true in dimension $n = 1$; this is the extension to the real line of the celebrated theorem of Carleson (see [C], [KT]). The problem however remains open for $n \geq 2$.

On the other hand, Riemann's localisation principle says that for any $f \in L^1(\mathbb{R})$ we have

$$\lim_{R\to\infty} S_R f(x) = 0$$

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at **every point** x off the support of f. In fact the convergence is uniform on compact subsets of $(\operatorname{supp} f)^c$. In [CS1] the first two authors considered the related question of localisation in higher dimensions. They proved the following:

Theorem [CS1]. *For every* $0 \leq \gamma < 1$ *there exists a constant* C_γ *so that*

$$\int_{|x|\leq 1} \sup_{R>1} \left| S_R \left(f \mathcal{X}_{|\cdot|>2} \right)(x) \right|^2 dx \leq C_\gamma \int |f(x)|^2 \frac{dx}{|x|^\gamma}. \tag{1}$$

Taking $\gamma = 0$ in the above theorem and using standard approximation arguments, we conclude (after an appropriate rescaling of the problem) that for $f \in L^2(\mathbb{R}^n)$ the set

$$\{x \notin \operatorname{supp} f : \{S_R f(x)\}_R \text{ does not converge to } 0 \text{ as } R \to \infty\}$$

has measure 0. In particular, for every $f \in L^2(\mathbb{R}^n)$, $\lim_{R\to\infty} S_R f(x) = 0$ **almost everywhere** off the support of f. Moreover, since $L^p(\mathbb{R}^n) \subset L^2(\mathbb{R}^n, dx) + L^2(\mathbb{R}^n, \frac{dx}{|x|^\gamma})$ whenever $2 \leq p < \frac{2n}{n-\gamma}$, the same result about localisation holds for every function $f \in L^p(\mathbb{R}^n)$, if $2 \leq p < \frac{2n}{n-1}$. This range of p's is optimal.

As is well known, (see for example Il'in [I], Pinsky [P]), there are functions in $L^2(\mathbb{R}^n)$, $n \geq 2$, for which the localisation principle fails at at least one point. For $n \geq 3$ we have in fact the following simple example: if $\mathcal{X}_1$ denotes the characteristic function of the unit ball then

$$S_R \mathcal{X}_1(0) = c_n \int_0^R \frac{J_{\frac{n-2}{2}}(t)}{t^{n/2}} t^{n-1} dt = c_n \int_0^R t^{1/2} J_{\frac{n-2}{2}}(t) t^{\frac{n-3}{2}} dt,$$

which clearly diverges as $R \to \infty$ if $n \geq 3$. ($J_{\frac{n-2}{2}}(t)$ is the Bessel function of order $\frac{n-2}{2}$.) Now, if ϕ is a smooth function with compact support so that $\phi \equiv 1$ on $\{|x| \leq 1\}$ then 0 is not a point of localisation for the function $\phi - \mathcal{X}_1$. In this respect, the conclusion of Theorem [CS1] about the localisation 'only' in the almost everywhere sense, is optimal. A natural question is therefore raised: how big may the sets of divergence be? or more precisely, how can we determine them? This was the topic of study in [CS2] and [CS3]. We recall that a measurable set $E \subset \mathbb{B} = \{|x| < 1\}$ is said to be a set of divergence for the localisation problem (**SDLP**) if there exists a function $f \in L^2(\mathbb{R}^n)$ with $\operatorname{supp} f \cap \mathbb{B} = \emptyset$ so that $\{S_R f(x)\}_R$ diverges at every $x \in E$.

Examples of SDLP's, in addition to singletons, include spheres and with more generality any collection of concentric spheres indexed by a set $F \subset (0,1)$ whenever the Hausdorff dimension of F is less than $1/2$. Also, any set of the form $(A \times \mathbb{R}) \cap \mathbb{B}$, where $A \subset \mathbb{R}^{n-1}$ has $n-1$ dimensional Lebesgue measure zero is an SDLP. ([CS3].) We return to the matter of SDLP's in Section 4 below.

In this work we continue studying the problem of localisation but from a different point of view. Our approach here will be specifically the more positive one of determining, in analogy with (1) above, for which positive, finite Borel measures μ supported in the unit ball of $\mathbb{R}^n$ one has the following inequality

$$\text{(A)} \qquad \int \sup_{R>1} \left| S_R \left(f \mathcal{X}_{|\cdot|>2} \right)(x) \right|^2 d\mu(x) \le C_\mu \int |f(x)|^2 \, dx, \quad \forall f \in L^2(\mathbb{R}^n),$$

for a certain constant $C_\mu < \infty$. Whenever this holds, then, for every function $f \in L^2(\mathbb{R}^n)$ with $\mathrm{supp} f \subset \{|x| \ge 2\}$ there exists at least one point $x \in \mathrm{supp}\, \mu$ (in fact a set of full μ-measure) for which $\{S_R f(x)\}_R$ converges. Hence, $\mathrm{supp}\, \mu$ cannot be an SDLP.

One of the things that was observed in [CS3] is that the maximal condition (A) is in fact equivalent to a uniform estimate for each of the spherical means; that is,

$$\text{(B)} \qquad \sup_{R>1} \int \left| S_R \left(f \mathcal{X}_{|\cdot|>2} \right)(x) \right|^2 d\mu(x) \le C_\mu \int |f(x)|^2 \, dx.$$

An indication of this equivalence was given using heuristic arguments, via the "folk-calculation" of C. Fefferman. This calculation in turn leads to a direct formulation of the problem in terms of the order of decay of the L^2-average over the unit sphere of the Fourier transform of $g d\mu$, for every $g \in L^2(d\mu)$. In other words, both (A) and (B) hold if and only if the following holds:

$$\text{(C)} \qquad \left(\int_{S^{n-1}} \left| \widehat{g d\mu}(R\omega) \right|^2 d\sigma(\omega) \right)^{1/2} \le \frac{C}{R^{\frac{n-1}{2}}} \left(\int |g|^2 d\mu \right)^{1/2},$$

with C independent of $g \in L^2(d\mu)$ and $R > 1$. In Theorems 1 and 2 below we give formal proofs of all these results.

The dual statement of (C) is the condition that

$$\text{(C*)} \qquad \left(\int \left| \widehat{h d\sigma}(Ry) \right|^2 d\mu(y) \right)^{1/2} \le \frac{C}{R^{\frac{n-1}{2}}} \left(\int_{S^{n-1}} |h|^2 d\sigma \right)^{1/2}, \qquad \forall h \in L^2(S^{n-1}).$$

This condition was considered independently by Barceló, Ruiz and Vega in [BRV] in connection with weighted inequalities for solutions to the Helmholtz equation. We shall come back to this point later.

Condition (B) can be interpreted as an L^2-weighted inequality for each operator S_R. This is reminiscent of a problem posed by E.M. Stein (see [St1]) during the famous conference on harmonic analysis at Williamstown, in 1978. The proposed problem was to determine the operator or operators which control the L^2-inequalities for the Bochner-Riesz means. To be more precise, if $\mathcal{S}_\delta$ denotes the (smooth) Fourier multiplier operator associated to the annulus $\{\xi : 1 - \delta < |\xi| < 1\}$ then the question is to find a pairing which associates to a given weight u another weight U so that the following holds

$$\int |\mathcal{S}_\delta f(x)|^2 \, u(x) \, dx \le C \int |f(x)|^2 \, U(x) \, dx.$$

After the work of C. Fefferman [F] and A. Córdoba [Co], it was natural to conjecture that the pairing $u \to U$ should be given by some appropriate modification of the maximal function, $\mathcal{M}_N$, on integral averages over rectangles of eccentricity N with $N = \delta^{-1/2}$. The limiting case of the Bochner-Riesz means corresponds to the disc multiplier and so the question here is whether or not the limiting operator for the $\mathcal{M}_N$'s, the Besicovitch/Kakeya maximal function $\mathcal{M}_\infty$, controls the L^2-weighted inequalities for the disc multiplier. A good indication that this may be the case is hinted at by condition (C*). For if we take h to be the characteristic function of a spherical cap $\mathcal{O}$, then the absolute value of $\widehat{hd\sigma}$ is (essentially) constant on a "tube" passing through the origin in the direction of the centre of $\mathcal{O}$. Then (C*) tells us that certain averages of μ over such a set are uniformly bounded. Since the problem is invariant under translations, the tubes may be in any position in space. We give an alternative proof of the necessity of this condition based on the assumption (B) (see Theorem 1 below). This was also shown by different arguments in [BRV].

In [CRS] it was proved that Stein's conjecture is correct for radial weights:

Theorem [CRS]. *Let w be a positive, radial and locally integrable function in $\mathbb{R}^n$. Then for each $\alpha > 1$ there exists C_α so that*

$$\int_{\mathbb{R}^n} |S_R f(x)|^2 \, w(x) \, dx \leq C_\alpha \int_{\mathbb{R}^n} |f(x)|^2 \left(\mathcal{M}_\infty w^\alpha(x)\right)^{1/\alpha} dx.$$

The original theorem was only stated for the case $R = 1$, the disc multiplier, but the invariance of $\mathcal{M}_\infty$ obviously gives the result for all $R > 0$.

A quick look at the arguments in the proof of this theorem shows that if w is supported in the unit ball and $f \equiv 0$ in $\{x : |x| \leq 2\}$ then

$$\int_{|x|\leq 1} |S_R f(x)|^2 \, w(x) \, dx \leq C_\epsilon \int_{|x|\geq 2} |f(x)|^2 \, |x|^\epsilon \mathcal{M}_\infty w(x) \, dx, \tag{2}$$

for every $\epsilon > 0$ if $R \geq 1$. In this form the inequality makes sense for measures also. An explicit expression of the Kakeya maximal function for radial weights, which was used to prove the correct boundedness estimates for this operator on $L^n_{rad}(\mathbb{R}^n)$, was given in [CHS]:

Theorem [CHS]. *If w is as above and we denote by w_0 is radial projection on $\mathbb{R}_+$ then*

$$M_\infty w(x) \sim \sup_{0<r<|x|} \frac{1}{\sqrt{|x|^2 - r^2}} \int_r^{|x|} w_0(s) \frac{s\,ds}{\sqrt{s^2 - r^2}} + \sup_{h>0} \frac{1}{h} \int_{|x|}^{|x|+h} w_0(s)\,ds.$$

In particular, if μ is rotationally invariant, has compact support, say in the unit ball, and we take $x \in \mathbb{R}^n$ with $|x| \geq 2$ then

$$M_\infty(d\mu)(x) \sim \frac{1}{|x|} \sup_{0<r<1} \int_r^1 \frac{s\,d\mu_0(s)}{\sqrt{s^2 - r^2}}, \tag{3}$$

where μ_0 denotes the radial projection of μ (see Theorem 3). Observe that the second term in the right hand side of (3), the "sup" term, is just a constant which only depends on μ, call it $|||\mu_0|||$. Then, (2) becomes in the case of a rotationally invariant measure μ

$$\int_{|x|\leq 1} |S_R f(x)|^2 \, d\mu(x) \leq C_\epsilon |||\mu_0||| \int_{|x|\geq 2} |f(x)|^2 \, \frac{dx}{|x|^{1-\epsilon}}. \tag{4}$$

The constant $|||\mu_0|||$ is easily seen to be equivalent, after a change of variables to polar coordinates, to the L^∞ norm of the averages on tubes that we mentioned above for general measures. So, one of the consequences of (2) is that in the case of rotationally invariant measures the maximal estimate (A) follows from, and in fact is equivalent to the finiteness of $|||\mu_0|||$. We give all the details in Theorem 3 below.

In [BRV] the authors study conditions on the radial weight V for the inequality

$$\int_{\mathbb{R}^n} |u(x)|^2 V(x) dx \leq C_V \int_{\mathbb{R}^n} |(\Delta+1)u(x)|^2 V^{-1}(x) dx$$

to hold. Their result is that this is true if and only if $|||V|||$ is finite; this is referred to as the "radial Mizohata-Takeuchi" condition. In fact one can take $C_V \sim |||V|||^2$. If u satisfies the so-called Sommerfeld outgoing radiation condition then u is given by the convolution of $f = (\Delta+1)u$ with a kernel which is more singular than that of S_R. However, for radial weights the problems are, as we see, equivalent.

The paper is organised as follows. In the next section we give a proof of the equivalence between the conditions (A), (B), (C) described in this introduction. Section 3 addresses the problem for the special case of rotationally invariant measures, for which we have a complete solution. In the last section we analyse the geometry of those measures for which the "norm" $|||\cdot|||$ is finite and further explore the relation between sets supporting such a measure and SDLP's. We would like to thank Laura Wisewell for a number of illuminating and very helpful discussions on the material of the last section.

§2. The maximal localisation theorem for general measures.

For a general finite Borel measure μ we want to investigate the connection between the localisation property that it inherits, the L^2 behaviour of the spherical means S_R with respect to it and the decay of the restriction of its Fourier transform to spheres. To this end, let us look at the conditions

$$\left\| S_R \left(f \mathcal{X}_{|\cdot|\geq 2} \right) \right\|_{L^2(d\mu)} \leq C \, \|f\|_2 \tag{B}$$

and

$$\left\| \widehat{g d\mu}(R\cdot) \right\|_{L^2(S^{n-1})} \leq \frac{C}{R^{\frac{n-1}{2}}} \, \|g\|_{L^2(d\mu)} \, , \tag{C}$$

as well as the two seemingly weaker conditions

$$\text{(B}'\text{)} \qquad \int_{|x|\geq 2} \left|(S_{R+1} - S_R)\,(gd\mu)\right|^2 dx \leq C \int |g|^2 \, d\mu,$$

and

$$\text{(C}'\text{)} \qquad \int_R^{R+1} \int_{S^{n-1}} \left|\widehat{gd\mu}(t\omega)\right|^2 d\sigma(\omega) t^{n-1} dt \leq C \int |g|^2 \, d\mu.$$

In all the cases, the constant C is understood to be uniform in $R \geq 1$ and independent of the functions $f, g, \ldots$ considered.

Theorem 1. *Let μ be any positive finite Borel measure whose support lies in the unit ball of $\mathbb{R}^n$. The four conditions $(B), (B'), (C)$ and (C') stated above are equivalent.*

Moreover, if we denote by $T_\epsilon(x, \omega), x \in \mathbb{S}^n, \omega \in S^{n-1}$ the translation to x and rotation by ω of the infinite tube $\{(y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |y'| < \epsilon\}$, then the condition

$$|||\mu||| := \sup_{\epsilon > 0, x, \omega} \frac{1}{\epsilon^{n-1}} \int_{T_\epsilon(x,\omega)} d\mu < \infty$$

is necessary for any of the above to hold.

In what follows, we sometimes refer to $|||\cdot|||$ as the "triple norm".

Proof. Let us begin with the observation that, by duality, condition (B) is equivalent to

$$\text{(B*)} \qquad \int_{|x|\geq 2} |S_R(gd\mu)|^2 \, dx \leq C \int |g|^2 \, d\mu,$$

and that Plancherel's identity shows that (C$'$) is the same as

$$\text{(B}''\text{)} \qquad \int_{\mathbb{R}^n} \left|(S_{R+1} - S_R)\,(gd\mu)\right|^2 dx \leq C \int |g|^2 \, d\mu,$$

Clearly (B*) => (B$'$), so the first part of the theorem will follow from the proof of the chain of implications (B$'$) => (C$'$) => (C) => (B*).

(**C**) => (**B***). We will prove first the following:

Lemma 1. *Let ψ be a function in the Schwartz class with $\psi(0) = 1$. Then, if we denote by δ_0 the Dirac delta at the point 0 and if B_R is the ball centred at the origin with radius R, we have*

$$\left|\left(\delta_0 - \widehat{\psi}\right) * \mathcal{X}_{B_R}(x)\right| \leq 2\int_{|y|\geq ||x|-R|} \left|\widehat{\psi}(y)\right| dy.$$

Proof. Set $I(x) = \left(\delta_0 - \widehat{\psi}\right) * \mathcal{X}_{B_R}(x) = \int_{|x-y|<R} \left(\delta_0 - \widehat{\psi}(y)\right) dy$ and consider the two cases:
Case 1: $|x| > R$. Then $|I(x)| \leq \int_{|x-y|<R} \left|\widehat{\psi}(y)\right| dy \leq \int_{|y|\geq|x|-R} \left|\widehat{\psi}(y)\right| dy$.

Case 2: $|x| \leq R$. Observe that $|I(x)| \leq \left|\int_{|y|<R-|x|} \left(\delta_0 - \widehat{\psi}(y)\right) dy\right| + \int_{|y|\geq R-|x|} \left|\widehat{\psi}(y)\right| dy$ $= I + II$.

Now, using the hypothesis $\psi(0) = \int \widehat{\psi} = 1$ we obtain

$$I = \left|1 - \int_{|y|<R-|x|} \widehat{\psi}(y)dy\right| = \left|\int_{|y|\geq R-|x|} \widehat{\psi}(y)dy\right| \leq II.$$

Therefore, $|I(x)| \leq 2II = 2\int_{|y|\geq R-|x|} \left|\widehat{\psi}(y)\right| dy$. Q.E.D.

Now take a smooth function ψ with supp$\psi \subset \{|x| \leq 1\}$ and $\psi(0) = 1$ and let K_R be the kernel of the operator S_R (i.e., $\widehat{K_R} = \mathcal{X}_{B_R}$). Put $K_R = \psi K_R + (1-\psi)K_R = K_R^1 + K_R^2$. Now, if μ has support in the unit ball then $K_R^1 * (gd\mu) \cap \{|x| \geq 2\} = \emptyset$. Therefore, from the lemma we get

$$\begin{aligned}
\int_{|x|\geq 2} |K_R * (gd\mu)|^2 \, dx &\leq \int |(1-\psi)K_R * (gd\mu)|^2 \, dx \\
&= \int \left|\left(\delta_0 - \widehat{\psi}\right) * \mathcal{X}_{B_R}(\xi)\widehat{gd\mu}(\xi)\right|^2 d\xi \\
&\leq \int \left(2\int_{||\xi|-R|\leq|y|} \left|\widehat{\psi}(y)\right| dy\right)^2 \left|\widehat{gd\mu}(\xi)\right|^2 d\xi.
\end{aligned}$$

Using Cauchy-Schwarz in the y-integral and writing the integral in ξ in polar coordinates we obtain from (C)

$$\begin{aligned}
\int_{|x|\geq 2} |K_R * (gd\mu)|^2 \, dx &\leq 4||\widehat{\psi}||_1 \int \left|\widehat{\psi}(y)\right| \int_{|t-R|\leq|y|} t^{n-1} \int_{S^{n-1}} \left|\widehat{gd\mu}(t\omega)\right|^2 d\sigma(\omega)\, dt\, dy \\
&\leq C\int \left|\widehat{\psi}(y)\right| |y| dy \int |g|^2 d\mu \sim C' \int |g|^2 d\mu.
\end{aligned}$$

(C′) => (C). Fix $R \geq 1$ and assume that (C′) holds for all functions in $L^2(d\mu)$. Take any g in that class and set

$$A(t) = \int_{S^{n-1}} \left| \widehat{gd\mu}(t\omega) \right|^2 d\sigma(\omega).$$

By hypothesis, there exists at least one $t_0 \in [R, R+1]$ so that $A(t_0) \leq \frac{C}{t_0^{n-1}} \int |g|^2 d\mu$. Observe that

$$\frac{d}{dt}\left(\widehat{gd\mu}\right)(t\omega) = 2\pi i \int (\omega \cdot y)\, g(y) e^{2\pi i t\omega\cdot y} d\mu(y).$$

Therefore,

$$\int_{S^{n-1}} \left| \frac{d}{dt}\left(\widehat{gd\mu}\right)(t\omega) \right|^2 d\sigma(\omega) \leq \sum_{j=1}^{n} |A_j(t)|^2 = \mathcal{A}(t),$$

where each A_j is defined as A but with $g(y)$ replaced by $y_j g(y)$, if $y = (y_1, \ldots, y_n)$. Since

$$A(R) = A(t_0) - \int_R^{t_0} A'(t)dt \leq A(t_0) + 2\int_R^{R+1} (A(t))^{1/2} (\mathcal{A}(t))^{1/2} dt,$$

we conclude, using (C′) again, that

$$R^{n-1}A(R) \leq t_0^{n-1}A(t_0) + 2\left(\int_R^{R+1} t^{n-1}A(t)dt \int_R^{R+1} t^{n-1}\mathcal{A}(t)dt\right)^{1/2} \leq C\int |g|^2 d\mu.$$

In the last inequality we have used the support condition on μ.

(B′) => (C′). Fix $g \in L^2(d\mu)$ and $R \geq 1$. Define $h(x) = (S_{R+\epsilon} - S_R)(gd\mu)(x)$. We first claim that for a sufficiently small $\epsilon > 0$ (depending only on the dimension) we have

$$\int_{\mathbb{R}^n} |h(x)|^2 dx \leq 2\int_{|x|\geq 2} |h(x)|^2 dx.$$

This, after rescaling, will do the job, (using (B′) and the equivalence of (B″) with (C′)). Take a function ψ in the Schwartz class, so that $\psi(x) \equiv 1$ on $\{|x| \leq 2\}$. Then

$$\left(\int_{|x|\geq 2} |h|^2\right)^{1/2} \geq \left(\int |h(1-\psi)|^2\right)^{1/2} \geq \left(\int |h|^2\right)^{1/2} - \left(\int |h\psi|^2\right)^{1/2}.$$

We show that $\int |h\psi|^2 \leq \frac{1}{2}\int |h|^2$, and that will prove our claim. Now, we observe that supp $\widehat{h} \subset \{\xi : R \leq |\xi| \leq R+\epsilon\}$ and that

$$\int |h\psi|^2 = \int |\widehat{h} * \widehat{\psi}|^2 \leq \left(\int |\widehat{h}|^2\right) ||\widehat{\psi}||_1 \left(\sup_x \int_{R\leq|x-y|\leq R+\epsilon} |\widehat{\psi}(y)|dy\right).$$

Put $D(x) = \{y : R \le |x-y| \le R+\epsilon\}$. Then,

$$\int_{R\le|x-y|\le R+\epsilon} |\widehat{\psi}(y)|dy = \int_{D(x)\cap\{|y|\le 1\}} |\widehat{\psi}(y)|dy + \sum_{k\ge 0}\int_{D(x)\cap\{|y|\sim 2^k\}} |\widehat{\psi}(y)|dy$$
$$\le C_n C_n'\epsilon + \sum_{k\ge 0} C_n'\epsilon\, 2^{k(n-1)} C_n 2^{-kn} = C_n''\epsilon,$$

where we have used that $|\widehat{\psi}(y)| \le C_n \frac{1}{1+|y|^n}$ for certain constant C_n. It suffices to take then $\epsilon = \frac{1}{2||\widehat{\psi}||_1 C_n''}$.

Finally, we show that if (B) holds then we must have $|||\mu|||$ finite. To do that, let us recall the construction given in Lemma 1 of [CS3]. In fact, we only need the first part of it:

Lemma 2. *Let ϕ be a smooth bump function supported in $\{|x| \le 1\}$. Given $R >> 1$ we consider the function $\phi_R(x) = e^{2\pi i R x_1}\phi(x_1 - 3, R^{1/2}x')$, where $x = (x_1, x') \in \mathbb{R}\times\mathbb{R}^{n-1}$. Then $|S_R\phi_R(x)| \ge c$ on the set $\{|x_1| \le 1,\ |x'| \le R^{-1/2}\}$, while for each $k \in \mathbb{N}$, there is a constant C_k so that $|S_R\phi_R(x)| \le C_k\{1 + R^{1/2}|x'|\}^{-k}$ for all x. Moreover, for each $k \in \mathbb{N}$, there is a constant C_k so that $|S_{R'}\phi_R(x)| \le C_k(R'/R)^{(n-1)/2}\max(R,R')^{-k}$ when $x \in \mathbb{B}$ and $R/R' \notin [1/2, 2]$.*

With our notation, what the lemma says is that $|S_R\phi_R(x)| \ge c$ on the set $T_\epsilon(\bar{0}, \bar{e_1}) \cap \{|x| \le 1\}$, where $\epsilon = R^{-1/2}$, $\bar{0}$ denotes the origin and $\bar{e}_1$ is just the vector $(1, 0, \dots, 0)$. Therefore, if μ is supported in the unit ball and (B) holds we must have

$$\int_{T_\epsilon(\bar{0},\bar{e_1})} d\mu \le C\int |\phi_R|^2 \sim C\,R^{-\frac{n-1}{2}}.$$

Since we can translate and rotate $S_R\phi_R$ as we wish, without changing the size of ϕ_R or the shape of its support, we obtain then

$$|||\mu||| \le C.$$

This finishes the proof of Theorem 1. Q.E.D.

We now prove the equivalence of the above conditions with (A). The proof is similar to the one used to prove (C) => (B*) but rather more technical. This is why we have decided to present it separately.

Theorem 2. *Let μ be a finite Borel measure μ whose support lies in the unit ball of $\mathbb{R}^n, n \geq 2$. Then, if μ satisfies condition (C) it satisfies the maximal condition (A) too. In particular, we obtain that the set of conditions (A), (B), (C), as well as (B') and (C'), are all equivalent.*

Proof. We follow the same line of arguments as in Theorem 2.2 of [CS1]. Take ϕ a smooth function supported in the unit ball so that $\phi \equiv 1$ on $\{|x| \leq 1/2\}$. Define $\psi(x) = \phi(\frac{x}{2}) - \phi(x)$ and $\psi_j(x) = \psi(\frac{x}{2^j})$ for $j = 0, 1, \ldots$ In this way we obtain $\phi(x) + \sum_{j\geq 0} \psi_j(x) \equiv 1$.

Now, if $f \in L^2(\mathbb{R}^n)$ with supp $f \subset \{|x| \geq 2\}$ we have for all $|x| \leq 1$

$$K_R * f(x) = \sum_{j=0}^{\infty} \sum_{k=j-2}^{k=j+2} (\psi_j K_R) * (\psi_k f)(x).$$

Writing $K_R^j = (\psi_j K_R)$, it is clear that it suffices to prove the estimate

$$\int \sup_{R\geq 1} \left| K_R^j * g(x) \right|^2 d\mu(x) \leq C\, 2^{-j} \int |g(x)|^2 dx, \qquad \forall g \in L^2(\mathbb{R}^n), \tag{5}$$

under the assumption

$$\int_{S^{n-1}} \left| \widehat{hd\mu}(r\omega) \right|^2 d\omega \leq \frac{C}{r^{n-1}} \int |h|^2 d\mu, \qquad \forall r > 0, \quad \forall h \in L^2(d\mu). \tag{6}$$

Observe that in (C) we consider only the case $r \geq 1$. However (6) is true also for $0 < r < 1$ always if $||\mu||$ (the total variation of μ) is finite since

$$\int_{S^{n-1}} \left| \widehat{hd\mu}(r\omega) \right|^2 d\omega \leq ||\widehat{hd\mu}||_\infty^2 \leq ||\mu|| \int |h|^2 d\mu.$$

We will not require in the rest of the proof the support condition on μ. Using the Fundamental Theorem of Calculus we obtain

$$\left| K_R^j * g(x) \right|^2 \leq \left| K_1^j * g(x) \right|^2 + 2 \int_1^R \left| K_t^j * g(x) \frac{d}{dt} K_t^j * g(x) \right| dt.$$

Hence, using the Cauchy-Schwarz inequality

$$\begin{aligned} &\int \sup_{R\geq 1} \left| K_R^j * g(x) \right|^2 d\mu(x) \leq \int \left| K_1^j * g(x) \right|^2 d\mu(x) \\ &+ 2 \left(\int \int_1^\infty \left| K_t^j * g(x) \right|^2 dt\, d\mu(x) \right)^{1/2} \left(\int \int_1^\infty \left| \frac{d}{dt} K_t^j * g(x) \right|^2 dt\, d\mu(x) \right)^{1/2} \\ &= I \,+\, 2 II_1 \cdot II_2. \end{aligned}$$

We will show that $I, II_1 \le C2^{-j}\|g\|_2$ while $II_2 \le C\|g\|_2$. This will prove (5). By duality, these estimates are equivalent, respectively, to

$$\int_{\mathbb{R}^n} \left|K_1^j * (hd\mu)(x)\right|^2 dx \le C\,2^{-j} \int |h|^2 d\mu$$
$$\int_{\mathbb{R}^n} \left|\int_1^\infty K_t^j * [f(\cdot,t)d\mu](x)dt\right|^2 dx \le C\,2^{-2j} \int\int_1^\infty |f(x,t)|^2 dt\, d\mu(x)$$
$$\int_{\mathbb{R}^n} \left|\int_1^\infty \left(\frac{d}{dt}K_t^j\right) * [f(\cdot,t)d\mu](x)dt\right|^2 dx \le C \int\int_1^\infty |f(x,t)|^2 dt\, d\mu(x)$$

and, by Plancherel's theorem, to

$$E_1 = \int_{\mathbb{R}^n} \left|m_1^j(\xi)\widehat{hd\mu}(\xi)\right|^2 d\xi \le C\,2^{-j} \int |h|^2 d\mu$$
$$E_2 = \int_{\mathbb{R}^n} \left|\int_1^\infty m_t^j(\xi)[f(\cdot,t)d\mu]\widehat{\ }(\xi)dt\right|^2 d\xi \le C\,2^{-2j} \int\int_1^\infty |f(x,t)|^2 dt\, d\mu(x)$$
$$E_3 = \int_{\mathbb{R}^n} \left|\int_1^\infty \left(\frac{d}{dt}m_t^j\right)(\xi)[f(\cdot,t)d\mu]\widehat{\ }(\xi)dt\right|^2 d\xi \le C \int\int_1^\infty |f(x,t)|^2 dt\, d\mu(x).$$

Here m_t^j denotes the Fourier transform of K_t^j, that is, $m_t^j(\xi) = \mathcal{X}_{B_t} * \widehat{\psi_j}(\xi)$. Using the fact that $\psi_j(0) = \int \widehat{\psi_j} = 0$ we have, as in Lemma 1 above,

$$\left|m_t^j(\xi)\right| \le 2\int_{||\xi|-t|\le 2^{-j}|y|} \left|\widehat{\psi}(y)\right| dy.$$

In particular, $|m_t^j(\xi)| \le C$ and $\int_0^\infty |m_t^j(\xi)|\,dt \le C\,2^{-j}$. Thus, the first estimate follows from

$$E_1 \le C\,2^{-j}||\widehat{hd\mu}||_\infty^2 \le C\,2^{-j}||\mu|| \int |h|^2 d\mu$$

since $\int_{\mathbb{R}^n} |m_1^j(\xi)|^2 d\xi \le 2^{-j}$. For the second we use our hypothesis (6)

$$\begin{aligned} E_2 &\le \int\left(\int_1^\infty \left|m_t^j(\xi)\right| dt\right) \int_1^\infty \left|m_t^j(\xi)\right| \left|[f(\cdot,t)d\mu]\widehat{\ }(\xi)\right|^2 dt\, d\xi \\ &\le C\,2^{-j}\int_1^\infty \int \left|\widehat{\psi}(y)\right| \int_{|r-t|\le 2^{-j}|y|} r^{n-1} \int_{S^{n-1}} \left|[f(\cdot,t)d\mu]\widehat{\ }(r\omega)\right|^2 d\omega\, dr\, dy\, dt \\ &\le C\,2^{-2j} \int\int_1^\infty |f(x,t)|^2 dt\, d\mu(x). \end{aligned}$$

Finally, observe that

$$\frac{d}{dt}\left(m_t^j(\xi)\right) = \frac{n}{t}\, m_t^j(\xi) + 2^j \int_{|2^j\xi - y|<2^j t} \frac{(y-2^j\xi)}{2^j t} \cdot \nabla\widehat{\psi}(y)\, dy.$$

As in Lemma 1 we obtain here the bound for $t \geq 1$

$$\left| \frac{d}{dt} \left(m_t^j(\xi) \right) \right| \leq C\, 2^j \int_{||\xi|-t| \leq 2^{-j}|y|} \left(\left| \widehat{\psi}(y) \right| + \left| \nabla \widehat{\psi}(y) \right| \right) \, dy.$$

So, the same argument as for E_2 gives us $E_3 \leq C$. Q.E.D.

In the spirit of Stein's conjecture as discussed in the Introduction, one may ask whether finiteness of $|||\mu|||$ implies the condition (C) and its equivalents. For the purposes of this paper we shall label the assertion that this is indeed the case as the triple norm conjecture. In the next section we examine this question in the case of rotationally invariant measures μ.

§3. The case of rotationally invariant measures.

As we have said in the Introduction, this is the case for which we have the complete characterisation of condition (C) and its equivalents in terms of the finiteness of the norm $||| \cdot |||$. (Thus the triple norm conjecture is true in the rotationally invariant case.)

Theorem 3. *Consider a finite Borel measure μ which is invariant under rotations and whose support lies in the unit ball of $\mathbb{R}^n, n \geq 2$. Let us denote by μ_0 its radial projection on $[0, \infty)$, that is*

$$\int_{\mathbb{R}^n} \phi \, d\mu = \int_0^\infty \left(\int_{S^{n-1}} \phi(r, \theta) \, d\sigma(\theta) \right) r^{n-1} d\mu_0(r).$$

Then the following conditions are equivalent:
(a) For all $f \in L^2(\mathbb{R}^n)$ with supp$f \subset \{|x| \geq 2\}$

$$\left\| \sup_{R \geq 1} |S_R f(x)| \right\|_{L^2(d\mu)} \leq C \, \|f\|_2 \, .$$

(b) For all $f \in L^2(\mathbb{R}^n)$ with supp$f \subset \{|x| \geq 2\}$

$$\sup_{R \geq 1} \|S_R f\|_{L^2(d\mu)} \leq C \, \|f\|_2 \, .$$

(c) For all g,

$$\left\| \widehat{g d\mu}(R \cdot) \right\|_{L^2(S^{n-1})} \leq \frac{C}{R^{\frac{n-1}{2}}} \, \|g\|_{L^2(d\mu)}$$

(d) μ satisfies

$$|||\mu||| \sim \sup_{r > 0} \int_r^\infty \frac{s^{\frac{1}{2}} d\mu_0(s)}{(s-r)^{\frac{1}{2}}} < \infty.$$

Obviously, the equivalence between (a), (b) and (c) follows from the results of the previous section. We will not discuss condition (a) here. However, we do consider (b) and (c) because the arguments that we will use are of a different nature (spherical harmonics and theory of Bessel functions) and have their own interest. The equivalence between (c) and (d) appears also in [BRV].

Proof. We will prove first that (b) and (c) are equivalent to the condition

$$\text{(e)} \qquad \sup_{l\in\frac{1}{2}\mathbb{N}}\sup_{R>1}\int_0^\infty \left|\tilde{J}_l(Rr)\right|^2 d\mu_0(r)<\infty,$$

where $\tilde{J}_l(t)=t^{1/2}J_l(t)$ and J_l is the Bessel function of order l. We will show later the equivalence between (d) and (e).

We begin with condition (c). Let us fix an orthonormal basis $\{\mathcal{Y}_k\}_k$, of spherical harmonic in $L^2(S^{n-1})$. Given $g\in L^2(d\mu)$ we consider its expansion with respect to that basis $g\sim\sum_k g_k(\cdot)\mathcal{Y}_k(\cdot)$ (the convergence in the L^2-sense). Then, the expansion of $\widehat{gd\mu}(R\omega)$ is

$$\widehat{gd\mu}(R\omega)=\frac{1}{R^{\frac{n-1}{2}}}\sum_k\left(\int_0^\infty g_k(t)\tilde{J}_{k'}(Rt)t^{\frac{n-1}{2}}d\mu_0(t)\right)\mathcal{Y}_k(\omega),$$

where $k'=\frac{n-2}{2}+\text{degree}(\mathcal{Y}_k)$ (see [SW]). Therefore, the statement in (c) holds if and only if

$$\left|\int_0^\infty g_k(t)\tilde{J}_{k'}(Rt)t^{\frac{n-1}{2}}d\mu_0(t)\right|^2\leq C\int\left|g_k(t)t^{\frac{n-1}{2}}\right|^2 d\mu_0(t),$$

with constant C independent of R and k. Since we can now assume that $g_k(t)t^{\frac{n-1}{2}}$ is any arbitrary function in $L^2(d\mu_0)$, the Riesz representation theorem tells us that the above holds if and only if

$$\int_0^\infty\left|\tilde{J}_l(Rr)\right|^2 d\mu_0(r)\leq C,$$

(uniformly in R and l) which is precisely (e).

Let us now see the equivalence between (b) and (e). This is the only place where we will consider the support condition on the measure μ.

Recall that if $f\sim\sum_k f_k\mathcal{Y}_k$ then $S_Rf\sim\sum_k(T^R_{k'}f_k)\mathcal{Y}_k$, where

$$T^R_l h(r)=\frac{C}{r^{\frac{n-1}{2}}}\int_0^\infty h(s)s^{\frac{n-1}{2}}RK_l(Rr,Rs)ds,$$

$$K_l(r,s)=\frac{r}{r^2-s^2}\tilde{J}_l(s)\widetilde{J'}_l(r)+\frac{s}{r^2-s^2}\tilde{J}_l(r)\widetilde{J'}_l(s)=K^1_l(r,s)+K^2_l(r,s)$$

$$\Big(\text{here } \widetilde{J'}_l(t) = t^{1/2} J'(t)\Big)$$

and $k' = \frac{n-2}{2} + \text{degree}(\mathcal{Y}_k)$ as above. (See [W], [CRS].) So, the statement in (b) holds if and only if

$$\text{(b}'\text{)} \qquad \int_0^1 \left| \int_2^\infty h(s) R K_l(Rr, Rs) ds \right|^2 d\mu_0(r) \le C \int_2^\infty |h(s)|^2 ds,$$

(with C independent of $R > 1$ and l). Since $|RK_l(Rr, Rs)| \le C_l \frac{1}{s}$, we can always assume that l is large.

The estimates about Bessel functions that we will use are summarised as follows (see [CRS] and [BRV]):

$$\text{(i)} \quad |\tilde{J}_l(t)| \le C \left[\left(\frac{|l+t|}{|l-t|} \right)^{1/4} \wedge l^{1/6} \right] := C\tau_l(t), \qquad |\widetilde{J'}_l(t)| \le C, \qquad l \ge 1, \forall t > 0$$

$$\text{(ii)} \qquad |\tilde{J}_l(t)| \le C \frac{l^{1/2}}{(l-t)}, \qquad \text{if } l/2 \le t \le l - l^{1/3}.$$

(C represents an absolute constant.)

Now, we point out that (b$'$) always holds if we replace K_l with K_l^1 since then the left hand side is majorised by

$$C \int_0^1 \left(\int_2^\infty |h(s)| \frac{\tau_l(Rs)}{s^2} ds \right)^2 d\mu_0(r) \le C ||\mu_0|| \left(\int_2^\infty |h(s)|^2 ds \right) \left(\int_2^\infty \frac{|\tau_l(Rs)|^2}{s^2} ds \right),$$

and the last integral is bounded uniformly in R. Therefore, (b) follows if and only if (b$'$) follows with K_l replaced by K_l^2, that is, if

$$\int_0^1 \left| \int_2^\infty h(s) \frac{s}{s^2 - r^2} \widetilde{J'}_l(Rs) ds \right|^2 |\tilde{J}(Rr)|^2 d\mu_0(r) \le C \int_2^\infty |h(s)|^2 ds,$$

which is easily seen to be equivalent to (e).

We finally show the equivalence between (d) and (e). That (e) implies (d) follows from the additional fact that in (i) one has $|\tilde{J}_l(t)| \sim \tau_l(t)$ in the region $l + l^{1/3} \le t \le \alpha l$, for some $\alpha > 1$.

To prove that (d) implies (e) we define for a fixed R the measure μ_0^R as

$$\int_0^\infty F(s)\, d\mu_0^R(s) = \int_0^\infty F(Rs)\, d\mu_0(s).$$

We make the observation that if (d) holds for μ_0 so does for μ_0^R with the same value, independent of R. Therefore, in proving (e) we can assume without loss of generality that $R = 1$. Let us split the integral as

$$\int_0^\infty \left|\tilde{J}_l(t)\right|^2 d\mu_0(t) = \left(\int_0^{l/2} + \int_{l/2}^{l-l^{1/3}} + \int_{l-l^{1/3}}^{l+l^{1/3}} + \int_{l+l^{1/3}}^\infty\right) \left|\tilde{J}_l(t)\right|^2 d\mu_0(t) = \sum_{j=1}^4 I_j.$$

From estimate (i) we have that $I_1 \leq C||\mu_0||$ and

$$I_4 \leq C \int_{l+l^{1/3}}^\infty \left(\frac{|l+t|}{|l-t|}\right)^{1/2} d\mu_0(t) \leq C \int_l^\infty \frac{t^{\frac{1}{2}} d\mu_0(t)}{(t-l)^{\frac{1}{2}}} \leq C.$$

We also get $I_3 \leq Cl^{1/3}\mu_0\left([l - l^{1/3}, l + l^{1/3}]\right)$. To estimate this, as well as I_2, we observe that if $I = [a, b]$ then (d) implies

$$\mu_0(I) = \int_a^{a+|I|} d\mu_0 \leq \left(\frac{|I|}{a}\right)^{1/2} \int_a^{a+|I|} \left(\frac{s}{s-a}\right)^{1/2} d\mu_0(t) \leq C \left(\frac{|I|}{a}\right)^{1/2}.$$

This shows that $I_3 \leq C$. Now, the above observation and (ii) gives

$$\begin{aligned} I_2 &\leq C \sum_{\{k\geq 0: 2^k \leq l^{2/3}\}} \int_{(l-t)\sim l/2^k} \frac{l}{(l-t)^2} d\mu_0(t) \\ &\leq C \sum_{2^k \leq l^{2/3}} \frac{l}{(l/2^k)^2} \mu_0\left([l - l/2^k, l]\right) \leq C \sum_{2^k \leq l^{2/3}} \frac{2^{2k}}{l} \frac{1}{l^{1/2}} \left(\frac{l}{2^k}\right)^{1/2} \sim C. \end{aligned}$$

Q.E.D.

Recall that the triple norm conjecture may be generated by testing (c) in Theorem 3 on g's which are essentialy characteristic functions of spherical caps of radius $R^{-1/2}$ and then letting R tend to ∞. One may be tempted to formulate similar conjectures for each scale R separately. In [BBC] it has recently been shown that in even in the rotationally invariant case, such conjectures fail. One should therefore perhaps exercise caution in one's belief in the veracity of such conjectures in general.

§4. SDLP's and geometric measure theory.

Throughout this section the s-dimensional Hausdorff measure on $\mathbb{R}^n$ will be denoted by $\mathcal{H}^s$, and the Hausdorff dimension of a set E by $\dim_{\mathcal{H}}(E)$. A *tube* T is taken to mean the intersection of an infinite tube with the unit ball $\mathbb{B}$. The $(n-1)$-dimensional measure of a cross section of a tube T is denoted by $w(T)$ and is called (somewhat unconventionally) the *width* of T. Our main result in this section, Theorem 4, links SDLP's with the notion of tube-null sets. In the remainder of the section we explore some elementary relations between tube-nullity and other geometric measure theoretic notions.

§4.1 Tube-nullity and SDLP's.

As mentioned in the Introduction, it was proved in [CS3] that a subset E of $\mathbb{B}$ which, for each $\epsilon > 0$, can be covered by countably many **parallel** tubes with total width $< \epsilon$, is an SDLP. It was also noted in [CS3], without proof, that a countable union of SDLP's is an SDLP. (The proof is modelled on the classical one for divergence of one-dimensional Fourier series which may be found, for example, in [K], pp. 55-57.) It is therefore natural to expect that there is a condition similar to the one above, but which is stable under countable unions, which is sufficient for being an SDLP.

Definition. *A subset E of $\mathbb{B}$ is* **tube − null** *if, for every $\epsilon > 0$, there exists a countable collection of tubes $\{T_j\}_{j=1}^{\infty}$ covering E such that $\sum_{j=1}^{\infty} w(T_j) < \epsilon$.*

Note that this definition is stable under countable unions, that tube-nullity implies (classical) nullity and that if a subset E of $\mathbb{B}$ satisfies $\mathcal{H}^{n-1}(E) = 0$, then E is tube-null. Nevertheless the notion of tube-nullity is not very well understood geometrically. For example, it seems not to be known whether there exist tube-null sets containing a unit line segment in each direction. Csörnyei has recently shown ([Cs]) that there do exist tube-null sets containing, in almost every direction, a unit line segment less a possible set of one-dimensional measure zero. Preiss has subsequently shown ([Pr]) that *all* $\mathcal{H}^2$-null minimal Besicovitch–Kakeya sets have the property that for each $\epsilon > 0$, there exists a collection of tubes of total width less than ϵ, for which, for almost every direction, the part of the line with that direction not covered by the tubes is of one-dimensional measure zero.

A consequence of Csörnyei's result is that there exist sets A with $\mathcal{H}^1(A) < \infty$ but for which it is *not* true that for almost every $x \in A$, almost every line through x meets A in a finite set. This answers a question in Mattila's book ([M] Remark 10.11, p.145) in the negative. However the question remains open for purely unrectifiable sets, where its interest is that it would clarify the analysis of the Besicovitch–Federer projection theorem. (Specifically, a positive answer to this question for purely unrectifiable sets would show that the third alternative in [M], Lemma 18.7, p. 253 was not needed. See also Remark 18.10, p. 258 of [M].)

Theorem 4. *Let $E \subset \mathbb{B}$ be tube-null. Then E is an SDLP.*

(By [CS1], any SDLP is automatically null.)

Proof of Theorem 4. The proof follows the broad lines of Theorem 5 of [CS3].

We first of all note that it is immaterial whether we use tubes or rectangles in the definition of tube-nullity. Let (α_m) be a positive sequence tending in a slowly varying manner towards ∞ and let (γ_m) be a positive sequence such that $\sum_{m=1}^{\infty} \alpha_m \gamma_m^{1/2}$ converges. Let now $\mathcal{F} = \{T_j\}$ be a countable family of rectangles with $\sum_{j=1}^{\infty} w(T_j) \le \epsilon\gamma_1$, (where ϵ is a small dimensional constant) and such that $\mathcal{F}$ covers E infinitely often. Now for all M sufficiently large we may assume (at the expense of altering ϵ by a dimensional constant) that that each T_j has size $M^{-k} \times M^{-k} \cdots \times M^{-k} \times 1$ for some $k \in \mathbb{N}$ (where of course k depends on the rectangle). Moreover we may assume that if two distinct such rectangles of the same size have parallel long directions then they are disjoint, and their cross-sections have the same orientation (i.e. are translates of each other). Note that there are at most $M^{2k(n-1)}$ rectangles of size $M^{-k} \times M^{-k} \cdots \times M^{-k} \times 1$.

Claim. *We may further assume that*

(i) $w(T_{j+1}) \leq w(T_j)$ *for all* j*, and*

(ii) if T_j *and* T_{j+1} *are not parallel, then* $w(T_j) \geq M^{(n-1)} w(T_{j+1})$.

Proof of Claim. Rename $T_1 := S_1^1$. Consider now T_2. If $w(T_2) < w(S_1)$, rename T_2 as S_2^1. If $w(T_2) > w(S_1)$, or if $w(T_2) = w(S_1)$, and T_2 and S_1 are not parallel, decompose T_2 into $M^{n-1}\frac{w(T_2)}{w(S_1)}$ parallel rectangles of smaller size with common width $M^{-(n-1)}w(S_1)$. Call these rectangles $\{S_2^i\}$. Now consider T_3. If $w(T_3) <$ (the common value of) $w(S_2^i)$, rename T_3 as S_3^1. If $w(T_3) >$ (the common value of) $w(S_2^i)$, or if $w(T_3) = w(S_2^i)$ and T_3 is not parallel to the (parallel) rectangles S_2^i, decompose T_3 into parallel rectangles of smaller size with common width $M^{-(n-1)}w(S_2^i)$. Call these rectangles $\{S_3^i\}$.

Proceeding inductively, we arrive at T_{j+1}. We compare it with the parallel congruent S_j^i's from the previous step. If $w(T_{j+1}) < w(S_j^i)$, rename T_{j+1} as S_{j+1}^1. If $w(T_{j+1}) > w(S_j^i)$, or if $w(T_{j+1}) = w(S_j^i)$ and T_{j+1} is not parallel to the S_2^i, decompose T_{j+1} into many parallel rectangles of smaller size with common width $M^{-(n-1)}w(S_j^i)$. Call these rectangles $\{S_{j+1}^i\}$.

Finally, we reorder the $\{S_j^i\}$ lexicographically (i.e. $\{S_1^1, S_1^2, \cdots, S_1^{i_1}, S_2^1, S_2^2, \cdots\}$) and rename this sequence as $\{W_1, W_2, \cdots\}$. Then $w(W_{j+1}) \leq w(W_j)$ for all j, and if W_j and W_{j+1} are not parallel, then $w(W_j) > w(W_{j+1})$, (and hence $w(W_j) \geq M^{(n-1)}w(W_{j+1})$). By construction the $\{W_j\}$ continue to cover E infinitely often, and we have not altered the sum of the measures of the rectangles in the collection. Q.E.D.

Let $D >> 1$ be given. We partition $\mathcal{F}$ into subfamilies $\mathcal{F}_l$ $(1 \leq l \leq D^{(n-1)})$ with the property that parallel congruent rectangles in any $\mathcal{F}_l$ of any given size $R^{-1/2} \times \cdots \times R^{-1/2} \times 1$ are $DR^{-1/2}$ -separated. Let $E^{(l)}$ be the set of those $x \in E$ which are in infinitely many members of $\mathcal{F}_l$; since every $x \in E$ is in infinitely many members of $\mathcal{F}$, each $x \in E$ belongs to some $E^{(l)}$. Since finite unions of SDLP's are SDLP's, it suffices to show that each $E^{(l)}$ is an SDLP. Rename a typical $E^{(l)}$ as E.

Summarising, given $D >> 1$, we may assume that for all M sufficiently large, E is covered infinitely often by members of a family $\mathcal{F}$ of rectangles which are of size $M^{-k} \times \cdots \times M^{-k} \times 1$ for various $k \in \mathbb{N}$, which satisfies the conclusion of the Claim, for which parallel congruent rectangles are separated by a factor D times their cross-sectional diameter and for which $\sum_{T \in \mathcal{F}} w(T) \leq \gamma_1$.

Now let T be any rectangle of size $R^{-1/2} \times \cdots \times R^{-1/2} \times 1$ (where $R = R(T)$), and denote by ϕ^T the translation and rotation of the function $e^{2\pi i R x_1}\phi(x_1 - 3/2, R^{1/2}x')$ (from Lemma 2 above) which is adapted to T.

Suppose we have a family $\mathcal{G}$ of parallel congruent rectangles of size $R^{-1/2} \times \cdots \times R^{-1/2} \times 1$

such that distinct tubes are separated by $DR^{-1/2}$. Lemma 2 implies that there is a fixed absolute constant D_0 depending only on the dimension n such that if $D \geq D_0$, then

$$(*) \qquad |\sum_{T\in\mathcal{G}} \pm S_R\phi^T(x)| \geq C \text{ for all } x \in \bigcup_{T\in\mathcal{G}} T,$$

where C is an absolute constant independent of the choice of $\pm$.

For R fixed and $\mathcal{F}$ as above, let $\mathcal{F}^*$ be a subfamily of $\mathcal{F}$ consisiting of those rectangles of sizes either larger than $\rho^{-1/2} \times \cdots \times \rho^{-1/2} \times 1$ when $\rho << R$, or smaller than $\rho^{-1/2} \times \cdots \times \rho^{-1/2} \times 1$ when $\rho >> R$. Taking $k > n-1$ in Lemma 2 and using the fact that the number of rectangles of size $M^{-l} \times \cdots \times M^{-l} \times 1$ is at most $M^{2l(n-1)}$ implies that

$$(**) \qquad |\sum_{T\in\mathcal{F}^*} \pm S_R\phi^T(x)| \leq C_k \frac{R^{(n-1)/2}\rho^{(n-1)/2}}{(R+\rho)^k} \text{ for all } x \in \mathbb{B},$$

where C_k is an absolute constant independent of $\pm$. In particular, when ρ has the same order of magnitude as R, and $\mathcal{F}^*$ consists of rectangles of size different from $R^{-1/2} \times \cdots \times R^{-1/2} \times 1$, we have the estimate $|\sum_{T\in\mathcal{F}^*} \pm S_R\phi^T(x)| \leq c$ where $c << C$, by the conclusions of the Claim.

Now let (j_m) be a subsequence of $\mathbb{N}$ such that $\sum_{j=j_m}^{\infty} w(T_j) \leq \gamma_m$. Let $\mathcal{A}_m = \{j_m, j_m+1, \ldots, j_{m+1}-1\}, \mathcal{T}_m = \{T_j : j \in \mathcal{A}_m\}$, and $E_m = \bigcup_{j\in\mathcal{A}_m} T_j$. By replacing the sequence (j_m) by one tending more rapidly to infinity, (recall that the $w(T_j)$ are decreasing), we can arrange it so that no two congruent (hence parallel) rectangles lie in different $\mathcal{T}_m$'s.

Fix m. Observe that there is a choice of $\pm$ such that

$$\|\sum_{T\in\mathcal{T}_m} \pm\phi^T\|_2^2 \leq \sum_{T\in\mathcal{T}_m} \|\phi^T\|_2^2 \leq C \sum_{T\in\mathcal{T}_m} w(T) \leq C\gamma_m.$$

Set $f_m = \sum_{T\in\mathcal{T}_m} \pm\phi^T$ for that choice of $\pm$. Then for each $x \in E_m$, there is a T_x such that $x \in T_x \in \mathcal{T}_m$, T_x has size $R^{-1/2} \times \cdots \times R^{-1/2} \times 1$ for some R depending on x and m and, by (*), (**) and the conclusion of the Claim we have $|S_R f_m(x)| \geq C$. (Consider the contributions arising from rectangles congruent to T_x under (*) and the remainder under (**).) Moreover, if $p \neq m$, then the members of $\mathcal{T}_p$ differ in cross-sectional diameter from those of $\mathcal{T}_m$ by a multiplicative factor of at least $M^{|p-m|}$. Hence by (**), for $x \in \mathbb{B}$ and for each $k > n-1$, if $p > m$

$$|S_R(f_p)(x)| \leq C_k R^{-(k-n+1)} M^{-(p-m)(n-1)/2}$$

while if $p < m$,

$$|S_R(f_p)(x)| \leq C_k R^{-(k-n+1)} M^{-(m-p)(k-(n-1)/2)}.$$

Thus for $x \in E_m$, and its associated R, $S_R f_p(x)$ is negligible for $p \neq m$.

Set $f = \sum_{p=1}^{\infty} \alpha_p f_p$; then $f \in L^2$ and *supp* $f \cap \mathbb{B} = \emptyset$. Moreover by the remark at the end of the previous paragraph, if $x \in E_m$, there is an R such that $|S_R f(x)| \sim \alpha_m |S_R f_m(x)| \geq C\alpha_m$, provided the sequence (α_p) does not behave too wildly.

Now if $x \in E$, there is a sequence m_k such that $x \in \bigcap_{k=1}^{\infty} E_{m_k}$; thus there is a sequence R_{m_k} (depending on x), such that $|S_{R_{m_k}} f(x)| \geq C\alpha_{m_k}$ for all k, and so $\limsup_{R\to\infty} |S_R f(x)| = \infty$. Hence $S_R f(x)$ diverges on E, and E is therefore an SDLP.

Q.E.D.

One might ask whether the converse is true, i.e. whether being an SDLP implies tube-nullity. At present, all known SDLP's are also tube-null.

§**4.2 Tube-nullity and measure conditions.**

There is a strong connection between whether a set E is an SDLP, is tube-null and whether or not it supports a measure μ with $|||\mu|||$ finite. It was asserted without proof in Theorem 9 of [CS3] that a radial set which supports no measure μ with $|||\mu|||$ finite is an SDLP; Proposition 6(a) below, in conjunction with Theorem 4 provides evidence for this assertion, but we have not proved it in general.

Nevertheless, for now we wish to focus on two conditions motivated by this discussion and which are related to tube-nullity. We recall that a *tube* is the intersection of an infinite tube with the unit ball $\mathbb{B}$. The following discussion is elementary.

Definition. *A subset E of $\mathbb{B}$ is* **outer measure tube – null** *(OM tube-null) if for every outer measure μ^* satisfying $\mu^*(T) \leq Cw(T)$ for all tubes T, E is μ^* - null. A subset E of $\mathbb{B}$ is* **Borel measure tube – null** *(BM tube-null) if for every positive Borel measure μ satisfying $\mu(T) \leq Cw(T)$ for all tubes T, E is μ - null.*

Thus a set E is BM tube-null if and only if it supports no positive Borel measure μ with $|||\mu||| < \infty$.

Recall that in the case of s-dimensional Hausdorff measure $\mathcal{H}^s$, the notions $\mathcal{H}^s$-null, null for every outer measure μ^* satisfying $\mu^*(B) \leq Cdiam(B)^s$ and null for every Borel measure μ satisfying $\mu(B) \leq Cdiam(B)^s$ (B being an arbitrary ball) all coincide. (This is a consequence of Frostman's Lemma.) A weak analogue for tube-nullity is as follows:

Proposition 5. *Consider the following conditions on a Borel subset E of $\mathbb{B}$:*
(a) E is tube-null
(b) E is OM tube-null
(c) E is BM tube-null
(d) E is null (i.e. $\mathcal{H}^n$-null)

Then $(a) \iff (b) \implies (c) \implies (d)$.

Moreover if the triple norm conjecture is true, then E an SDLP $\implies$ E is BM tube-null.

Proof.
$(a) \implies (b)$.
Let μ^* be an outer measure satisfying $\mu^*(T) \leq Cw(T)$ for all tubes T. Let $\epsilon > 0$. Cover E by tubes T_j with $\sum w(T_j) < \epsilon$; then

$$\mu^*(E) \leq \mu^*(\cup T_j) \leq \sum \mu^*(T_j) < \epsilon$$

so that $\mu^*(E) = 0$.

$(b) \implies (a)$.
Define an outer measure ν^* by $\nu^*(A) = \inf_{A \subset \cup T_j} \sum w(T_j)$. If E is not tube-null, then $\nu^*(E) > 0$, and, as is easily checked, $\nu^*(T) = w(T)$ for all tubes T.

$(a), (b) \implies (c)$.
If E is not BM tube-null, then there exists a Borel measure μ with $\mu(T) \leq Cw(T)$ for all tubes T but $\mu(E) > 0$. Hence E is not tube-null (and indeed ν^* as above provides an outer measure for which E is not null).

$(c) \implies (d)$.
Take μ to be Lebesgue measure restricted to E: certainly $|||\mu||| < \infty$, and $\mu(E) = 0$, so that E is null.

Finally, if the triple norm conjecture is true and if E supports a Borel measure μ with $|||\mu||| < \infty$, then by Theorem 2 above, $S_R f(x)$ converges to zero μ-almost everywhere on E when $f \in L^2$ is supported outside $\mathbb{B}$, and so E is not an SDLP. Q.E.D.

We shall see below in the discussion of the rotationally invariant case that (d) does not imply (c), but we do not know whether (c) implies (a),(b). The problem in trying to show that (c) implies (a) is that the measurable sets for the outer measure ν^* may be very few and far between. In particular, Borel sets E need not be ν^*–measurable as is seen by the simple example $E = [0, 1/2] \times [-1/2, 1/2]$ and $A = [-1/2, 1/2] \times [0, \delta]$. While $\nu^*(A) = \delta$, both $\nu^*(E \cap A)$ and $\nu^*(E^c \cap A)$ are both equal to δ too. Thus E is not ν^*-measurable. It is not clear whether there are *any* ν^*–measurable sets for which the set and its complement have positive measure. Wisewell [Wi] has shown (when $n = 2$) that if a ν^*-measurable set is Lebesgue measurable, then either it or its complement is Lebesgue-null.

§4.3 The rotationally invariant case.

Consider a set of the form $E = E' \times [-1, 1] \cap \mathbb{B}$ with $E' \subset \mathbb{R}^{n-1}$. Then the conditions tube-nullity, OM tube-nullity, BM tube-nullity and being an SDLP are all equivalent and are equivalent to $\mathcal{H}^n$ nullity of E or to $\mathcal{H}^{n-1}$ nullity of E'.

More interesting is the rotationally invariant case as discussed in Section 3 above. In accordance with the notation there, for a set $F \subset [1/2, 1]$ let $E = E(F) = \{x : |x| \in F\}$. Once again, the discussion is elementary.

Proposition 6.
(a) If F is $\mathcal{H}^{1/2}$-null, then E is tube-null.

(b) E is BM tube-null if and only if F is null for every positive Borel measure μ_0 satisfying

$$\sup_{r>0} \int_r^\infty \frac{s^{\frac{1}{2}} d\mu_0(s)}{(s-r)^{\frac{1}{2}}} < \infty. \tag{7}$$

(c) If E is tube-null then $\dim_{\mathcal{H}}(F) \leq 1/2$.

(d) E is null if and only if $\mathcal{H}^1(F) = 0$.

Proof.

(a) Let $\epsilon > 0$. Cover F by intervals I such that $\sum_I |I|^{1/2} < \epsilon$. Then, for each I, $E(I)$ can be covered by $O(|I|^{-n+3/2})$ tubes each of width $|I|^{n-1}$ because of the curvature of the sphere. So E can be covered by tubes of total width $\sum_I |I|^{-n+3/2}|I|^{n-1} = \sum_I |I|^{1/2} < \epsilon$. Thus E is tube-null.

(b) The forward implication is obvious as (7) is the triple norm of the radial extension of μ_0. If E is not BM tube-null, then there is a Borel measure with $|||\mu||| < \infty$ and $\mu(E) > 0$. By averaging μ over rotations, there will be a radial such μ. Thus F is not null for some Borel measure μ_0 satisfying (7).

(c) If E is tube-null, it is also BM tube-null by Proposition 5, and thus by (b), F is null for every μ_0 satisfying (7). Hence F has dimension at most 1/2.

(d) This is obvious by Fubini's theorem.

Q.E.D.

If we take a set $F \subset [1/2, 1]$ which is $\mathcal{H}^1$-null but of Hausdorff dimension strictly larger than 1/2, then it will support a measure μ_0 with $\sup_{r>0} \int_r^\infty \frac{s^{\frac{1}{2}} d\mu_0(s)}{(s-r)^{\frac{1}{2}}} < \infty$. Thus the implication $(d) \implies (c)$ in Propositon 5 fails.

§4.4 Hausdorff dimension and tube-nullity.

As mentioned in the Introduction, it was proved in [CS3] that sets of Hausdorff dimension less than $n-1$ are SDLP's. Here we improve this to include sets of σ-finite $(n-1)$ - dimensional Hausdorff measure, and show that such sets are in fact tube-null.

Proposition 8. *Let $E \subset \mathbb{B}$ a be set of σ-finite $(n-1)$ - dimensional Hausdorff measure. Then E is tube-null.*

Proof. Without loss of generality we may assume that $\mathcal{H}^{n-1}(E) < \infty$.

Note first that by the structure theorem (see [M], p.205), if $\mathcal{H}^{n-1}(E) < \infty$ then there is a decomposition of $E = E_1 \cup E_2$, where E_1 is an $(n-1)$-rectifiable set and E_2 is purely $(n-1)$-unrectifiable.

Also, by [M] Theorem 16.2 on p.222, or by Theorem 18.1 on p.250, there is an $(n-1)$–hyperplane V such that $\mathcal{H}^{n-1}(P_V(E_2)) = 0$, where P_V is the orthogonal projection onto V. Thus E_2 is tube null.

Moreover, by Theorem 15.21, p.214 in [M], there is a countable collection of $(n-1)$–dimensional $\mathcal{C}^1$ submanifolds, $M_1, M_2, \ldots$, such that

$$\mathcal{H}^{n-1}(E_1 \setminus \bigcup_{i=1}^{\infty} M_i) = 0,$$

Since $\mathcal{H}^{n-1}$–null sets are tube-null, it is therefore enough to prove the proposition for a set E which is an $(n-1)$-dimensional manifold in $\mathbb{R}^n$.

We can assume that E is the graph of a $\mathcal{C}^1$ function ϕ; thus $E = \{(\bar{x}, \phi(\bar{x})) = \Phi(\bar{x}) : \bar{x} \in U \subset \mathbb{R}^{n-1}\}$ where U is bounded and where we may assume that ϕ and its derivatives are bounded on U. Let $\epsilon > 0$. By the definition of the derivative of ϕ and the Littlewood principles, there is an exceptional subset U' of U with $\mathcal{H}^{n-1}(U') < \epsilon$ such that for $\bar{x} \in U \setminus U'$, $E \cap B_r(\Phi(\bar{x}))$ is contained in an $r\epsilon$ – neighbourhood of the tangent plane at $\Phi(\bar{x})$ whenever r is sufficiently small. This is uniform in $\bar{x} \in U \setminus U'$.

Now $\mathcal{H}^{n-1}(\Phi(U')) < C\epsilon$ and thus $\Phi(U')$ can be covered by tubes of total width less than $C\epsilon$.

For $x \notin U'$, $E \cap B_r(\Phi(\bar{x}))$ can be covered by tubes of total width $C\epsilon r^{n-1}$ and so $\Phi(U \setminus U')$ can be covered by tubes of total width $C\epsilon r^{n-1} \times r^{-(n-1)} = C\epsilon$.

So E is tube-null and we are done.

Q.E.D.

Since there exist tube-null sets of Hausdorff dimension n (take $E' \times [-1, 1]$ with E' of dimension $n-1$ but $\mathcal{H}^{n-1}(E') = 0$), there is no interesting smallness consequence (in terms of Hausdorff dimension) of being tube-null. The radial case tells us that for each $s \in (n - 1/2, n)$, every radial set E of positive finite s-dimensional Hausdorff measure is not tube-null and indeed supports a positive Borel measure μ (which we can take to be $\mathcal{H}^s|_E$) with $|||\mu||| < \infty$.

Thus

$$n - 1 \leq \sup\{s : \dim_{\mathcal{H}}(E) \leq s \implies E \ \text{ tube-null}\} \leq n - 1/2.$$

It would be of interest to determine the middle number in this chain of inequalities. It is hoped to address this question in a future work.

REFERENCES

[BRV] J.A. Barceló, A. Ruiz and L. Vega, *Weighted estimates for the Helmholtz equation and some applications*, Jour. Funct. Anal. **150, no.2** (1997), 356-382.

[BBC] J.A. Barceló, J. Bennett and A. Carbery, *A note on localised weighted estimates for the extension operator*, preprint (2005).

[CHS] A. Carbery, E. Hernández and F. Soria, *Estimates for the Kakeya maximal operator on radial functions in* $\mathbb{R}^n$, ICM-90 Satellite Conference Proc. on Harmonic Analysis (1991), 41-50.

[CRS] A. Carbery, E Romera and F. Soria, *Radial weights and mixed norm inequalities for the disc multiplier*, Jour. Funct. Anal. **109** (1992), 52-75.

[CS1] A. Carbery and F. Soria, *Almost everywhere convergence of Fourier integrals for functions in Sobolev spaces and an* L^2*-localisation principle*, Revista Matemática Iberoamericana **4 (2)** (1988), 319-337.

[CS2] A. Carbery and F. Soria, *Sets of divergence for the localization problem for Fourier integrals*, Comptes Rendus Acad. Sci. Paris **325, I** (1997), 1283-1286.

[CS3] A. Carbery and F. Soria, *Pointwise Fourier Inversion and Localisation in* $\mathbb{R}^n$, Jour. Fourier Anal. Appl. **3** (1997), 846-858.

[C] L. Carleson, *On convergence and growth of partial sums of Fourier series*, Acta Math. **116** (1966), 135-157.

[Co] A. Córdoba, *The Kakeya maximal function and spherical summation multipliers*, Amer. J. Math **99** (1977), 1-22.

[Cs] M. Csörnyei, *Personal communication.*

[F] C. Fefferman, *A note on spherical summation multipliers*, Israel J. Math. **15** (1973), 44-52.

[I] V.A. Il'in, *The problem of localisation and convergence of Fourier Series with respect to the fundamental system of functions of the Laplace operator*, Russian Math. Surv. **23 (2)** (1968), 59-116.

[K] Y. Katznelson, *An Introduction to Harmonic Analysis* (Dover, ed.), 1976.

[KT] C. Kenig and P. Tomas, *Maximal operators defined by Fourier multipliers*, Studia Math. **68** (1980), 79-83.

[M] P. Mattila, *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability* (Cambridge Studies in Advanced Mathematics, 44. Cambridge University Press, ed.), 1995.

[P] M. Pinsky, *Pointwise Fourier inversion in several variables*, Notices of A.M.S. **42, 3** (1995), 330-334.

[Pr] D. Preiss, *Personal communication.*

[St1] E. M. Stein, *Some problems in Harmonic Analysis*, Proc. Symp. Pure Math., Amer. Math. Soc. **35 (1)** (1979), 3-20.

[St2] E.M. Stein, *Harmonic Analysis: Real-variable methods, orthogonality, and oscillatory integrals* (Princeton U. Press, ed.), 1993.

[StW] E.M. Stein and G. Weiss, *Introduction to Harmonic Analysis on Euclidean Spaces* (Princeton U. Press, ed.), 1971.

[W] G. N. Watson, *Theory of Bessel functions* (Cambridge U. Press, ed.), 1922.

[Wi] L. Wisewell, *Personal communication.*

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