EXISTENCE AND UNIQUENESS FOR A SEMILINEAR ELLIPTIC PROBLEM ON LIPSCHITZ DOMAINS IN RIEMANNIAN MANIFOLDS

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Abstract. For a general class of semilinear equations $\Delta u - F(x,u) = 0$ in a Lipschitz domain $\Omega \subset M$, where M is a smooth compact Riemannian manifold, we establish the existence and uniqueness of solutions to Dirichlet and Neumann boundary problems. The main contribution of this paper is that we consider 'rough' boundary data that are typically just $L^p(\partial\Omega)$ functions. Results of this type for the linear equation are typically obtained using singular integral techniques.

1. Introduction

The gap between linear and nonlinear partial differential equations is quite substantial. Often, sharp results proven for linear equations have no equivalent when we move to more general nonlinear setting.

The main aim of this paper is to bridge this gap for a class of semilinear elliptic equations

$$\Delta u - F(x,u) = 0 \quad \text{in } \Omega, \tag{1.1}$$

where Ω is a bounded connected Lipschitz domain in a Riemannian manifold M of dimension $n \geq 3$. The restriction $n \geq 3$ is a convenience, it is connected to the fact that the fundamental solution to the Laplace equation has a different form when $n = 2$. However, there is no fundamental difficulty in extending our treatment to two dimensions. We do not consider closely this case only because we do not want to further complicate our arguments.

The linear version of this equation with $F(x, u) = V(x)u$ (and also with $F(x, u) = 0$) has generated substantial interest with the aim of establishing existence and uniqueness of Dirichlet and Neumann boundary problems on Ω with 'rough' boundary data. Here 'rough' means that typically the solution u on the boundary should be just an element of $L^p(\partial\Omega)$. A first result of this nature for Dirichlet problem is due to Dahlberg [8], who used estimates on harmonic measure. A breakthrough came after A. P. Calderón established a result on singular integrals on Lipschitz curves [2]. This paper was followed by the work of Coifman, McIntosh and Meyer who generalized Calderón's result in [5]. Soon, utilizing singular integral technique, Verchota in [27], Fabes, Jodeit and Rivère in [11], Dahlberg and Kenig in [9] and others established existence and uniqueness for Dirichlet and Neumann problem on Lipschitz and C^1 domains. Let us remark, that all mentioned work was done for a flat $\mathbb{R}^n$ space with the standard Laplace operator.

On the other hand, Isakov and Nachman [14] recently considered the two dimensional version of the equation (1.1) on a bounded planar Lipschitz domain Ω with $F(x, 0) = 0$. Their approach is purely variational with use of maximum principle. The result on the Dirichlet problem obtained by them requires bounded boundary data and having at least half derivative, i.e., $H^{1/2,2}(\partial\Omega)$. Here $H^{s,p}$ stands for a standard Sobolev space of L^p integrable functions $1 \leq p \leq \infty$, with s derivatives, $s \in \mathbb{R}$. For the nonlinear equation (1.1) in dimension greater than two one interesting result we know about is in the book by Gilbarg and Trudinger [13]. Their assumption on the boundary data is $H^{(2p-1)/p,p}$ with $p > n$. They also require the boundary to be at least $C^{2,\alpha}$. Another result on existence of positive solutions can be found in the paper by Chen, Williams and Zhao [6]. These authors assume that the boundary data are in L^∞ and small.

Recently, further development has been made for the linear equation. Mitrea and Taylor [23], [24] and [25] managed to substantially generalize previous results. Namely, they brought the whole subject into much more general variable coefficient setting - on Riemannian manifolds. The author using their work later considered C^1 domains on Riemannian manifolds in [7]. It turned out that again L^p boundary data are enough to establish uniqueness and existence results.

Still, the gap between the linear and nonlinear equation (1.1) remained. The problem was that the main tool used to establish solution of (1.1) is usually a certain version of fixed point theorem used in some background Banach space. Up to now all attempts tried to make use of Sobolev spaces on Ω. This approach does not seem work for the results we would like to obtain. In particular, if we prescribe $L^p(\partial\Omega)$ Dirichlet boundary data and try to look for a solution in $H^{p,1/p}(\Omega)$ we fail, since we do not have the trace theorem for the space $H^{p,1/p}(\Omega)$.

In the work [7] we introduced Banach spaces that seem to be very convenient to work with. Interior regularity results for (1.1) guarantee that the solution u is inside Ω quite regular (of the class $C^{1+\beta}_{\text{loc}}(\Omega)$ for any $\beta < \alpha$), where α is the regularity of the metric tensor. On the other hand, having L^p boundary data on $\partial\Omega$ means that this regularity cannot be preserved up to the boundary. Hence this fact has to be dealt with. As we will see in the next section the introduced spaces $\mathcal{D}^{s,p}$ are exactly the 'right' ones, which allow us to reconcile interior smoothness with rougher behavior when approaching boundary $\partial\Omega$.

The use of these newly defined spaces will allow us to establish results for the equation (1.1) equivalent to those for the linear case. Moreover, it seems very likely that the work presented here could be pushed further for more general nonlinear terms $a(x, u)$. This will be the topic of our next paper, where we relax the condition (1.6) and allow polynomial growth of the function a in the variable u.

The organization of this paper is following. After we define and establish properties of the Banach spaces $\mathcal{D}^{s,p}$ in Section 2, we study actions of the linear operator $L = \Delta - V$ in Section 3. In Section 4 we prove certain uniform estimates which do not appear in [23]-[25] or [7]. The fifth section is devoted to the equation (1.1) and its variants for boundary data in L^p and the last section for boundary data in the Hardy space $\hbar^1$, the Hölder space C^β and the space bmo.

We treat the equation (1.1) in slightly different form. If the function

$F(x,u)$ is differentiable in u we can write

$$F(x,u) = a(x,u)u + f, \tag{1.2}$$

where $f(x) = F(x,0)$ and

$$a(x,u) = \int_0^1 \frac{\partial}{\partial u} F(x,tu)\ dt. \tag{1.3}$$

Hence, we can consider instead of (1.1) the equation

$$\Delta u - a(x,u)u = f \quad \text{in } \Omega. \tag{1.4}$$

The main results we obtain are the following. If $a(x,u) \in L^\infty(\Omega \times \mathbb{R})$ and $a(x,u) \geq 0$ then for $2-\varepsilon < p \leq \infty$ the Dirichlet problem (1.4) with boundary data $u\big|_{\partial\Omega} = g \in L^p(\partial\Omega)$ has a solution for any f in a Banach space X such as described in Theorem 5.5. If in addition for $b(x,u) = a(x,u)u$ we have

$$\frac{\partial}{\partial u} b(x,u) \in L^\infty(\Omega \times \mathbb{R}) \quad \text{and} \quad \frac{\partial}{\partial u} b(x,u) \geq 0, \tag{1.5}$$

then the solution is unique. Moreover, if $g \in C^\beta(\partial\Omega)$, then $u \in C^\beta(\overline{\Omega})$ for some $\beta > 0$ small. Also if $g \in \text{bmo}(\partial\Omega)$ then the exponential of the maximal operator $a\mathcal{M}^0 u$ is integrable on $\partial\Omega$ for some a small (depending on the norms of f and g).

Similarly, the Dirichlet regularity problem is solvable for any $1 < p < 2+\varepsilon$ and $g \in H^{1,p}(\partial\Omega)$ under essentially the same assumptions. For $p = 1$ the same is true for $g \in H^{1,1}(\partial\Omega)$, where $H^{1,1}$ is the Hardy-Sobolev space.

The Neumann problem (1.4) with $\partial_\nu u\big|_{\partial\Omega} = g \in L^p(\partial\Omega)$ is solvable for $1 < p < 2 + \varepsilon$, provided for some function $q \geq 0$ on Ω and $q > 0$ on a set of positive measure in Ω we have:

$$a(x,u) \in L^\infty(\Omega \times \mathbb{R}), \quad \inf_{u \in \mathbb{R}} a(.,u) \geq q(.). \tag{1.6}$$

If in addition for $b(x,u) = a(x,u)u$ we have

$$\frac{\partial}{\partial u} b(x,u) \in L^\infty(\Omega \times \mathbb{R}) \quad \text{and} \quad \inf_{u \in \mathbb{R}} \frac{\partial}{\partial u} b(.,u) \geq q(.), \tag{1.7}$$

then the solution is unique.

In Section 6 we also establish an endpoint result for the Neumann problem for $p = 1$. In such case we replace $L^1(\partial\Omega)$ by the Hardy space $\hbar^1(\partial\Omega)$. The claim is that given $g \in \hbar^1(\partial\Omega)$ and f in $L^r(\Omega)$ or in $\mathcal{D}^{0,1}$ (see the definition in section 2) the solution to the equation (1.4) exists, provided (1.6) is true. Uniqueness is guaranteed if (1.7) holds.

If $f = 0$ and $g \in L^\infty(\partial\Omega)$ we can relax the condition on the function a. In such case we can assume that

$$\text{for any } M \in (0,\infty) \text{ we have:} \quad \sup_{\substack{u\in[-M,M]\\ x\in\Omega}} |a(x,u)| < \infty, \quad a(x,u) \geq 0. \tag{1.8}$$

Then the Dirichlet problem with $u\big|_{\partial\Omega} = g$ has at least one solution. A condition similar to (1.8), namely

$$\text{for any } M \in (0,\infty)\text{:} \quad \sup_{\substack{u\in[-M,M]\\ x\in\Omega}} \left|\frac{\partial}{\partial u} b(x,u)\right| < \infty \quad \text{and} \quad \frac{\partial}{\partial u} b(x,u) \geq 0, \tag{1.9}$$

guarantee uniqueness. Here $b(x,u) = a(x,u)u$. Again we also have a regularity result, i.e., given $g \in C^\beta(\partial\Omega)$ ($\beta > 0$ small), the solution $u \in C^\beta(\overline{\Omega})$, provided (1.8) holds. If we have (1.8) and also

$$\begin{aligned} &(i) \text{ either } \lim_{u\to\infty} (\sup_{x\in\Omega} a(x,u)) < \infty \quad \text{or} \quad \limsup_{u\to\infty} (\inf_{x\in\Omega} a(x,u)) > 0, \\ &(ii) \text{ either } \lim_{u\to-\infty} (\sup_{x\in\Omega} a(x,u)) < \infty \quad \text{or} \quad \limsup_{u\to-\infty} (\inf_{x\in\Omega} a(x,u)) > 0, \end{aligned} \tag{1.10}$$

then the solution $u \in L^\infty(\overline{\Omega})$ to the Dirichlet problem (1.4) exists, provided $f \in L^\infty(\Omega)$ and $g = u\big|_{\partial\Omega} \in L^\infty(\partial\Omega)$. Also the C^β regularity result holds.

2. The Banach spaces $\mathcal{D}^{s,p}$

In this section we introduce a new class of Banach spaces $\mathcal{D}^{s,p}$, for $s \geq 0$ and $1 \leq p \leq \infty$. As we will see later, these spaces turn out to be extremely useful for the considered semilinear elliptic problem. The spaces $\mathcal{D}^{s,p}$ with $s = 0$ were introduced in [7]. Also a very useful interpolation theorem can be found there (Proposition B.7).

Our goal here is to give more general definition for any $s \geq 0$, and prove results about interpolation, embedding and traces for these spaces.

Consider the same settings as we outlined in the introduction. Let M be a smooth compact n-dimensional Riemannian manifold with a Riemannian metric tensor, which is assumed to be in Hölder class C^α, for some $\alpha > 0$. It means that, M can be covered by local coordinate charts with the components g_{jk} of the metric tensor being of the Hölder class C^α.

Let Ω be a open, connected subset of M with Lipschitz boundary. That is for any point of the boundary $\partial\Omega$, we can find a small neighborhood U of this point such that in this neighborhood there are smooth local coordinates in which

$$U \cap \Omega = \{x = (x', x_n) \in U; |x'| < c \text{ and } \varphi(x') < x_n < \varphi(x') + c\}, \qquad (2.1)$$

for some $c > 0$ small. Here $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant bounded by L. L and c does not depend on chosen point $x \in \partial\Omega$. We use notation $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$.

Given any $K > L$, consider nontangential approach regions (cones) $\gamma(z)$ to any point $z = (z', \varphi(z')) \in \partial\Omega$ such that the vertex of the cone $\gamma(z)$ at z is sharp enough. Namely, we require that any half-ray with vertex at z that lies in $\gamma(z)$ has "steepness" (absolute value of its slope) at least K. Hence

$$\gamma(z) = \{y = (y', y_n); y_n - z_n > K|y' - z'| \text{ and } y_n < \varphi(y') + c\}. \qquad (2.2)$$

Naturally, the approach region $\gamma(z)$ depends on the coordinates (2.1) and the constant K. Hence, for a different choice of coordinates or the constant K, we get different approach regions $\gamma'(z)$. Nevertheless, as we will se later, the norms we define using $\gamma(z)$ and $\gamma'(z)$, respectively, are equivalent. Thus, a particular choice of the coordinates or the constant K is not important.

Assume therefore, that we have defined the nontangential approach region $\gamma(x)$ for any $x \in \partial\Omega$ and some constant $K > L$. Clearly, there is a collar neighborhood $\mathcal{C}$ of $\partial\Omega$ (i.e., $\mathcal{C} = \{x \in M;\ \mathrm{dist}(x, \partial\Omega) < \varepsilon\}$, for some $\varepsilon > 0$), such that the union of all $\gamma(x)$ covers $\mathcal{C} \cap \Omega$ and moreover for any $z \in \mathcal{C} \cap \Omega$ the $n-1$ dimensional surface measure of the set

$$\{x \in \partial\Omega; z \in \gamma(x)\}$$

is proportional to $(\mathrm{dist}(z, \partial\Omega))^{n-1}$.

Let $f : \Omega \to \mathbb{R}$ be a function on Ω. For any $s \geq 0$ and $x \in \partial\Omega$ we consider the number

$$\mathcal{M}^s f(x) = \begin{cases} \|f\big|_{\gamma(x)}\|_{C^s(\gamma(x))}, & \text{for } s \text{ not an integer,} \\ \sum_{|\alpha|\leq s} \|D^\alpha f\big|_{\gamma(x)}\|_{L^\infty(\gamma(x))}, & \text{for integer } s, \end{cases} \tag{2.3}$$

where $f\big|_{\gamma(x)}$ means the restriction of f to the set $\gamma(x)$. The norm considered in the first line of (2.3) is the Hölder norm of the space $C^s(\gamma(x))$. Hence we have that

$$\begin{aligned} \mathcal{M}^s f(x) &= \sum_{|\alpha|\leq s} \sup_{z\in\gamma(x)} |D^\alpha u|, & \text{if } s \in \mathbb{Z}, \\ \mathcal{M}^s f(x) &= \sum_{|\alpha|\leq k} \sup_{z\in\gamma(x)} |D^\alpha u| \\ &\quad + \sum_{|\alpha|=k} \sup_{z,z'\in\gamma(x)} \frac{|D^\alpha u(z) - D^\alpha u(z')|}{\operatorname{dist}(z,z')^{s-k}}, & \text{otherwise.} \end{aligned} \tag{2.4}$$

Here k is the integer part of s, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a multiindex ($|\alpha| = \sum \alpha_i$). Of course, we allow the numbers $\mathcal{M}^s f(x)$ to be infinite. We are ready to proceed. Denote by $\widetilde{\Omega}$ the set $\Omega \setminus \{x \in M;\ \operatorname{dist}(x, \partial\Omega) \leq \frac{\varepsilon}{2}\}$. The number ε we take here is the same that appears above in the definition of the collar neighborhood $\mathcal{C}$. Hence $\widetilde{\Omega}$ was chosen such that $\widetilde{\Omega} \subset\subset \Omega$ and $\widetilde{\Omega} \cup \mathcal{C} = \Omega$.

Definition 2.1. Let $s \geq 0$ and $1 \leq p \leq \infty$. Consider the set

$$\begin{aligned} \mathcal{D}^{s,p} &= \{f : \Omega \to \mathbb{R}; \mathcal{M}^s f \in L^p(\partial\Omega) \ \&\ \|f\big|_{\widetilde{\Omega}}\|_{C^s(\widetilde{\Omega})} < \infty\} & s \notin \mathbb{Z}, \\ \mathcal{D}^{s,p} &= \{f : \Omega \to \mathbb{R}; \mathcal{M}^s f \in L^p(\partial\Omega) \ \&\ \sup_{\substack{|\alpha|\leq s \\ x\in\widetilde{\Omega}}} |D^\alpha f(x)| < \infty\} & s \in \mathbb{Z}, \end{aligned} \tag{2.5}$$

where by $L^p(\partial\Omega)$ we denoted the space of L^p integrable functions on $\partial\Omega$.

Then $\mathcal{D}^{s,p}$ equipped with the norm

$$\begin{aligned} \|f\|_{\mathcal{D}^{s,p}} &= \|\mathcal{M}^s f\|_{L^p(\partial\Omega)} + \|f\big|_{\widetilde{\Omega}}\|_{C^s(\widetilde{\Omega})}, & s \notin \mathbb{Z}, \\ \|f\|_{\mathcal{D}^{s,p}} &= \|\mathcal{M}^s f\|_{L^p(\partial\Omega)} + \sum_{|\alpha|\leq s} \|D^\alpha f\|_{L^\infty(\widetilde{\Omega})}, & s \in \mathbb{Z}, \end{aligned} \tag{2.6}$$

is a Banach space.

Remark 2.2. Notice that if $p = \infty$ then $\mathcal{D}^{s,p} = C^s(\Omega)$, for s not an integer and $\mathcal{D}^{0,p} = L^\infty(\Omega)$.

There are several issues that have to be dealt with. First of all, one need to check that (2.6) defines a norm that makes the space $\mathcal{D}^{s,p}$ complete. We skip the proof since it is quite straightforward and is based on the fact that any Cauchy sequence $(f_n)_{n\in N}$ in $\mathcal{D}^{s,p}$ is also Cauchy in $C^s_{\rm loc}(\Omega)$. This gives us a natural candidate for the limit of this sequence.

The second issue has to do with the way the space $\mathcal{D}^{s,p}$ was defined. Our definition depends on nontangential regions $\gamma(.)$, hence it seems conceivable that a different collection of these regions might yield different space. The following lemma settles this matter and shows that for $1 \le p \le \infty$, if we choose a different collection of nontangential approach regions $\{\gamma'(x); x \in \partial\Omega\}$ then the resulting spaces $\mathcal{D}^{s,p}$ and their norms are equivalent to those we obtained using the original collection $\{\gamma(x); x \in \partial\Omega\}$.

Lemma 2.3. *Consider one coordinate chart of the form (2.1). Let $K \ge K' > L$, $c, c' > 0$ and consider the nontangential regions*

$$\begin{aligned}\gamma(z) &= \{y = (y', y_n); y_n - z_n > K|y' - z'| \text{ and } y_n < \varphi(y') + c\}\\ \gamma'(z) &= \{y = (y', y_n); y_n - z_n > K'|y' - z'| \text{ and } y_n < \varphi(y') + c'\}.\end{aligned} \tag{2.7}$$

For $1 \le p \le \infty$ and $s \ge 0$, let $\mathcal{D}^{s,p}$ be the Banach space from Definition 2.1 using the collection $\{\gamma(x); x \in \partial\Omega\}$ and $\mathcal{D}'^{s,p}$ be the corresponding Banach space for the collection $\{\gamma'(x); x \in \partial\Omega\}$.

Then $\mathcal{D}^{s,p} = \mathcal{D}'^{s,p}$ and the considered norms are equivalent.

Remark. This lemma in its original version (proven by a different technique) did not cover the case $p = 1$. The level set approach which is an adaptation of the idea by Kenig [18] was pointed out to me by the referee.

Proof. First let us take care of the case when $K = K'$ and $c > c'$, i.e., the nontangential approach cones $\gamma(.)$ are 'longer' than $\gamma'(.)$. Crucially,

$$\bigcup_{x\in\partial\Omega} (\gamma(x) \setminus \gamma'(x)) \subset\subset \Omega. \tag{2.8}$$

This and the fact that regardless of what nontangential regions we use to define $\mathcal{D}^{s,p}$ we always have $\mathcal{D}^{s,p} \subset C^s_{\rm loc}(\Omega)$ give us our claim.

Consider the projection $P : \overline{\Omega} \cap U \to \mathbb{R}^{n-1}$ defined by $(x', x_n) \mapsto x'$. Let $\widetilde{P}$ be the restriction of P onto $\partial\Omega$. Hence, $\widetilde{P}$ is a 1-1 mapping between $\partial\Omega \cap U$ and some open set in $\mathbb{R}^{n-1}$. It also follows that $\widetilde{P}^{-1}(x') = (x', \varphi(x'))$, where φ is the Lipschitz function defining the domain Ω in (2.1).

Thanks to the argument in the first paragraph of this proof, we can assume that $K > K'$ and $0 < c < c'$. Here c' can be taken large enough, such that for any $z \in \gamma(x)$ we have $z \in \gamma'(\widetilde{P}^{-1}(P(z)))$.

Take first $s = 0$. We claim that there exists $C > 0$ depending only on K, K' and the Lipschitz character of the boundary such that for any $f \in L^\infty_{\mathrm{loc}}(\Omega)$

$$\sigma(\{x \in \partial\Omega; \mathcal{M}^0 f > \lambda\}) \leq C\sigma(\{x \in \partial\Omega; \mathcal{M}'^0 f > \lambda\}). \tag{2.9}$$

From this the inclusion $\mathcal{D}'^{0,p} \subset \mathcal{D}^{0,p}$ follows.

The proof of (2.9) is based on the following covering lemma which can be found in [4].

Lemma 2.4. *Let $E \subset \mathbb{R}^n$, and suppose that to each $x \in E$ a number $r(x) > 0$ is given. Assume furthermore that $\sup_{x\in E} r(x) < \infty$. Then there exists a sequence $x_i \in E$ such that the balls $B(x_i, r(x_i))$ with center at x_i and radius $r(x_i)$ are disjoint, and*

(i) $E \subset \bigcup_i B(x_i, 3r(x_i))$

(ii) For all $x \in E$ there exists x_i such that $B(x, r(x)) \subset B(x_i, 5r(x_i))$.

Now we make two simple geometrical observations: If $z = (z', z_n) \in \gamma(x)$, then there is a constant $d > 0$ (independent of x and z) such that $P(x) \in B(z', d(z_n - \varphi(z')))$. Secondly, there exists another constant $e > 0$ such that if $s' \in B(z', e(z_n - \varphi(z')))$, then $z \in \gamma'(\widetilde{P}^{-1}(s'))$. If $c > 0$ and $z_n > \varphi(z')$, we define

$$A_c(z', z_n) = \{s' \in \mathbb{R}^{n-1}; |s' - z'| < c(z_n - \varphi(z'))\}. \tag{2.10}$$

Assume that $x = (x', x_n) \in E(\lambda) \stackrel{\text{def}}{=} \{y \in \partial\Omega; \mathcal{M}^0 f(y) > \lambda\}$. It follows that for some $z = (z', z_n) \in \gamma(x)$ we have $|f(z)| > \lambda$. Hence, $x' \in A_d(z', z_n)$ and, since for every $s' \in A_e(z', z_n)$, z belongs to $\gamma'(\widetilde{P}^{-1}(s'))$, it follows that $\widetilde{P}^{-1}(A_e(z', z_n)) \subset E'(\lambda) \stackrel{\text{def}}{=} \{y \in \partial\Omega; \mathcal{M}'^0 f(y) > \lambda\}$.

For $x' \in P(E(\lambda))$ define $r(x') = (d + e)(z_n - \varphi(z'))$, where $z = (z', z_n) \in \gamma(\widetilde{P}^{-1}(x'))$ is a point for which $|f(z)| > \lambda$. By Lemma 2.4 applied to

$P(E(\lambda)) \subset \mathbb{R}^{n-1}$ there exists a sequence of points $x_i = (x_i', x_{in}) \in E(\lambda)$ such that

$$E(\lambda) \subset \bigcup_i \widetilde{P}^{-1}(B(x_i', 3r(x_i'))), \quad \text{and the balls } B(x_i', r(x_i')) \text{ are disjoint.} \tag{2.11}$$

Let $z_i = (z_i', z_{in})$ be the point associated with x_i' in the definition of the number $r(x_i')$. As $A_e(z_i', z_{in}) \subset B(x_i', r(x_i'))$ we get that all $\widetilde{P}^{-1}(A_e(z_i', z_{in}))$ are disjoint and contained in $E'(\lambda)$. Hence

$$\sum_i \sigma(\widetilde{P}^{-1}(A_e(z_i', z_{in}))) \leq \sigma(E'(\lambda)). \tag{2.12}$$

The measures of both $\widetilde{P}^{-1}(A_e(z_i', z_{in}))$ and $\widetilde{P}^{-1}(B(x_i', 3r(x_i')))$ are proportional to $(z_{in} - \varphi(z_i'))^{n-1}$, hence we get that

$$\sigma(E(\lambda)) \leq \sum_i \sigma(\widetilde{P}^{-1}(B(x_i', 3r(x_i')))) \approx \sum_i \sigma(\widetilde{P}^{-1}(A_e(z_i', z_{in}))) \leq \sigma(E'(\lambda)). \tag{2.13}$$

Naturally (2.13) is equivalent to (2.9).

Now consider $0 < s < 1$. In this case the argument given above requires slight modification. We will establish that there exist $C > 0$ and $k \in \mathbb{N}$ such that for any $f \in C^s_{\mathrm{loc}}(\Omega)$

$$\sigma(\{x \in \partial\Omega; \mathcal{M}^0 f > \lambda\}) \leq C\sigma(\{x \in \partial\Omega; \mathcal{M}'^0 f > 2^{-s}k^{s-1}\lambda\}). \tag{2.14}$$

Once again, the inclusion $\mathcal{D}'^{s,p} \subset \mathcal{D}^{s,p}$ follows from (2.14).

The key here is the choice of k. Again a simple geometrical argument can be made that if we pick k large enough, $d > 0$ also large and $e > 0$ small, then for any $x \in \partial\Omega$ and $z^1 = (z^{1\prime}, z^1_n), z^2 = (z^{2\prime}, z^2_n) \in \gamma(x)$ the following is true: Let $L = L_1 \cup L_2$ be a subset of $\gamma(x)$ consisting of two uniquely determined line segments L_1 and L_2 (understood in local coordinates on U) such that

(a) L_1 has endpoints z^1 and z^0 and L_2 has endpoints z^0 and z^2

(b) $z^0 = (z^{0\prime}, z^0_n)$ is picked such that $z^0_n = \max\{z^1_n, z^2_n\}$ and one of the line segments L_1, L_2 is parallel to the hyperplane $\{y = (y', y_n); y_n = 0\}$ and the other one is perpendicular to it (all understood in the Euclidean metric on $U \subset \mathbb{R}^n$).

Now we find $k+1$ points $p^0 = z^1, p^1, p^2, \ldots, p^{k-1}, p^k = z^2$ on L, such that when we move from p^0 along L to p^k we pass through these points in the order given by their index $i \in \{0, 1, 2, \ldots, k\}$. We also require that the Euclidean distance $D = |p^{i+1} - p^i|$, $i = 0, 1, \ldots, k-1$ does not depend on i, i.e., these points are equidistantly spaced. This implies that

$$\tfrac{1}{k}|z^2 - z^1| \le D \le \tfrac{2}{k}|z^2 - z^1|. \tag{2.15}$$

The points p^i, $i = 0, 1, \ldots, k$ are uniquely determined. Finally, take $y^1 = (y^{1\prime}, y^1_n)$, $y^2 = (y^{2\prime}, y^2_n)$ to be any pair of points p^i, p^{i+1} for some $i \in \{0, 1, 2, \ldots, k-1\}$. Then we require:

(i) $P(x) \in B(y^{i\prime}, d(y^i_n - \varphi(y^{i\prime})))$, $i = 1, 2$.

(ii) For any $s' \in B(y^{1\prime}, e(y^1_n - \varphi(y^{1\prime})))$ we have $y^i \in \gamma'(\widetilde{P}^{-1}(s'))$, $i = 1, 2$.

Given this we proceed as follows. For $x' \in P(E(\lambda))$, $(E(\lambda) \stackrel{\text{def}}{=} \{y \in \partial\Omega; \mathcal{M}^s f(y) > \lambda\})$ we take $r(x') = (d+e)(z_n - \varphi(z'))$, provided $z = (z', z_n) \in \gamma(\widetilde{P}^{-1}(x'))$ is a point for which $|f(z)| > \lambda$. If such point z does not exist, then there exist points $z^1, z^2 \in \gamma(\widetilde{P}^{-1}(x'))$ such that

$$\frac{|f(z^1) - f(z^2)|}{\mathrm{dist}(z^1, z^2)^s} > \lambda. \tag{2.16}$$

Let L be the set joining z^1 and z^2 as above. It follows that there is a pair of neighboring points $y^1 = (y^{1\prime}, y^1_n)$, $y^2 = (y^{2\prime}, y^2_n)$ taken from the set $\{p^0, p^1, p^2, \ldots, p^k\}$ defined above such that $|f(y^1) - f(y^2)| \ge \frac{1}{k}|f(z^1) - f(z^2)|$. By (2.16) and (2.15) for such pair:

$$\frac{|f(y^1) - f(y^2)|}{\mathrm{dist}(y^1, y^2)^s} > \frac{1}{2^s k^{1-s}}\lambda. \tag{2.17}$$

Hence by (ii) we get that

$$\widetilde{P}^{-1}(A_e(y^{1\prime}, y^1_n)) \subset E'(2^{-s}k^{s-1}\lambda) := \{y \in \partial\Omega; \mathcal{M}'^s f(y) > 2^{-s}k^{s-1}\lambda\}\}. \tag{2.18}$$

Finally, we take $r(x') = (d+e)(y^1_n - \varphi(y^{1\prime}))$. The rest goes as above. We eventually get the desired estimate (2.14).

This proves our lemma for $0 \le s < 1$. If $s \ge 1$ we can use induction, since

$$\|f\|_{\mathcal{D}^{s,p}} \approx \|f\|_{\mathcal{D}^{s-1,p}} + \sum_{|\alpha|=1} \|D^\alpha f\|_{\mathcal{D}^{s-1,p}}. \tag{2.19}$$

□

The next theorem outlines relations between defined spaces.

Theorem 2.5. *Let $1 \le p \le \infty$ and $s > 0$. For any $0 \le s' < s$ and any*

$$\frac{1}{p'} > \frac{1}{p} - \frac{s - s'}{n-1}, \qquad 1 \le p' \le \infty \tag{2.20}$$

we have

$$\mathcal{D}^{s,p} \subset \mathcal{D}^{s',p'}. \tag{2.21}$$

Moreover, if $sp > n - 1$ then

$$\mathcal{D}^{s,p} \subset C^{\alpha}(\overline{\Omega}), \tag{2.22}$$

with $\alpha < s - \frac{n-1}{p}$.

Also, the embeddings (2.21), (2.22) are compact.

Proof. First, let us assume that $0 < s \le 1$ and $s' = 0$. Pick any $f \in \mathcal{D}^{s,p}$. We want to consider the function $\mathcal{M}^0 f(x)$ for $x \in \partial\Omega$. Pick $x, y \in \partial\Omega$ close to each other, such that they belong to a common coordinate chart (2.1) and the nontangential approach regions $\gamma(x)$, $\gamma(y)$ overlap. We want to estimate $\mathcal{M}^0 f(x)$ using $\mathcal{M}^0 f(y)$. Pick any $z \in \gamma(x)$. Then there is a point $w \in \gamma(y) \cap \gamma(x)$ such that $\operatorname{dist}(z, w) \approx \operatorname{dist}(x, y)$. Hence using the fact, that on $\gamma(x)$ the function f is C^s Hölder with Hölder constant $\mathcal{M}^s f(x)$ we get that

$$f(z) \le f(w) + \mathcal{M}^s f(x)\operatorname{dist}(z, w)^s \le \mathcal{M}^0 f(y) + C\mathcal{M}^s f(x)\operatorname{dist}(x, y)^s. \tag{2.23}$$

In the last inequality we used the fact that $f(w) \le \mathcal{M}^0 f(y)$. Now if we take supremum over all $z \in \gamma(x)$ we get that:

$$\mathcal{M}^0 f(x) \le \mathcal{M}^0 f(y) + C\mathcal{M}^s f(x)\operatorname{dist}(x, y)^s. \tag{2.24}$$

If we exchange x, y in (2.23) we get similar estimate for $\mathcal{M}^0 f(y)$. Together we get:

$$|\mathcal{M}^0 f(x) - \mathcal{M}^0 f(y)| \le C(\mathcal{M}^s f(x) + \mathcal{M}^s f(y))\operatorname{dist}(x, y)^s. \tag{2.25}$$

Recall, that a function g belongs to the Bessov space $B^q_r(\partial\Omega)$, for $1 \le q < \infty$ and $0 < r < 1$ provided the number (norm)

$$\|u\|_{B^q_r(\partial\Omega)} = \|u\|_{L^q(\partial\Omega)} + \left(\int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x) - u(y)|^q}{\operatorname{dist}(x, y)^{n-1+rq}}\, d\sigma(x)\, d\sigma(y)\right)^{1/q} \tag{2.26}$$

is finite. If we integrate (2.25) in x and y, we get that for any $\varepsilon > 0$:

$$\int_{\partial\Omega}\int_{\partial\Omega} \frac{|\mathcal{M}^0 f(x) - \mathcal{M}^0 f(y)|^p}{\operatorname{dist}(x,y)^{n-1-\varepsilon+ps}}\, d\sigma(x)\, d\sigma(y) \leq$$

$$\leq C \int_{\partial\Omega}\int_{\partial\Omega} \left(\frac{(\mathcal{M}^s f(x))^p}{\operatorname{dist}(x,y)^{n-1-\varepsilon}} + \frac{(\mathcal{M}^s f(y))^p}{\operatorname{dist}(x,y)^{n-1-\varepsilon}} \right) d\sigma(x)\, d\sigma(y) \tag{2.27}$$

Now, since the function $1/\operatorname{dist}(x,y)^{n-1-\varepsilon}$ is integrable on $\partial\Omega$, we can conclude that (2.27) is finite. Hence we get that $\mathcal{M}^0 f \in B^p_{s-\varepsilon}(\partial\Omega)$. Moreover, the norm of this function in $B^p_{s-\varepsilon}(\partial\Omega)$ can be estimated by $C\|\mathcal{M}^s f\|_{L^p(\partial\Omega)}$. Recall also that $B^p_{s-\varepsilon}(\partial\Omega) \subset H^{s-2\varepsilon,p}(\partial\Omega)$. So we have $\mathcal{M}^0 f \in H^{s-\varepsilon,p}(\partial\Omega)$ for any $\varepsilon > 0$, as well. This and Sobolev embedding theorem prove (2.21) for $s' = 0$.

Consider now the case $0 < s' < s \leq 1$. We want estimates similar to (2.22)-(2.24). Again let $x, y \in \partial\Omega$ be close enough, such that their approach regions $\gamma(x)$, $\gamma(y)$ overlap. It is a simple geometrical exercise to show that in the local coordinates (2.1) the overlap region must contain a cut cone (2.2) with vertex at a point z inside Ω. Also $\operatorname{dist}(x,y) \approx \operatorname{dist}(x,z) \approx \operatorname{dist}(y,z)$. Hence if $t = z - x$, then for any $w \in \gamma(x)$, $w + t \in \gamma(z)$ (all in local coordinates (2.1)). Pick any $w, w' \in \gamma(x)$. There are two cases to be considered. If $\operatorname{dist}(w,w') < \operatorname{dist}(x,y)$ we get:

$$\frac{|f(w) - f(w')|}{\operatorname{dist}(w,w')^{s'}} \leq \mathcal{M}^s f(x) \operatorname{dist}(w,w')^{s-s'} \leq \mathcal{M}^s f(x) \operatorname{dist}(x,y)^{s-s'}. \tag{2.28}$$

On the other hand, if $\operatorname{dist}(w,w') \geq \operatorname{dist}(x,y)$ we use:

$$\frac{|f(w) - f(w')|}{\operatorname{dist}(w,w')^{s'}} \leq \tag{2.29}$$

$$\frac{|f(w) - f(w+t)|}{\operatorname{dist}(w,w')^{s'}} + \frac{|f(w+t) - f(w'+t)|}{\operatorname{dist}(w,w')^{s'}} + \frac{|f(w'+t) - f(w')|}{\operatorname{dist}(w,w')^{s'}}.$$

The second term of the right side of (2.29) can be estimated by $\mathcal{M}^{s'} f(y)$, the first and the third term by $C\mathcal{M}^s f(x) \operatorname{dist}(x,y)^{s-s'}$. Hence (2.29) together with (2.28) yield for any $w, w' \in \gamma(x)$:

$$\frac{|f(w) - f(w')|}{\operatorname{dist}(w,w')^{s'}} \leq \mathcal{M}^{s'} f(y) + C\mathcal{M}^s f(x) \operatorname{dist}(x,y)^{s-s'}. \tag{2.30}$$

Taking the supremum over all such pairs w, w' yields

$$\mathcal{M}^{s'} f(x) \leq \mathcal{M}^{s'} f(y) + C\mathcal{M}^{s} f(x)\mathrm{dist}(x,y)^{s-s'}. \tag{2.31}$$

Hence, as before exchanging x and y gives

$$|\mathcal{M}^{s'} f(x) - \mathcal{M}^{s'} f(y)| \leq C(\mathcal{M}^{s} f(x) + \mathcal{M}^{s} f(y))\mathrm{dist}(x,y)^{s-s'}. \tag{2.32}$$

From here we proceed as before using (2.26) to get that $\mathcal{M}^{s'} f \in L^{p'}(\partial\Omega)$ with p' determined by (2.20).

Now we are ready to prove compactness of the embedding (2.21). We do it first for $s' = 0$. It follows from the Sobolev embedding theorem, that $H^{s-\varepsilon,p}(\partial\Omega) \subset\subset L^{p'}(\partial\Omega)$ with $\frac{1}{p'} > \frac{1}{p} - \frac{s}{n-1}$. This is what we need. Consider any given bounded sequence of functions $(f_n)_{n\in N}$ in $\mathcal{D}^{s,p}$. Since $\mathcal{D}^{s,p} \subset C^{s}_{\mathrm{loc}}(\Omega)$, there is a subsequence of $(f_n)_{n\in N}$ (also denoted by $(f_n)_{n\in N}$) such that $f_n \to f$ in $C^{\delta}_{\mathrm{loc}}(\Omega)$ for any $0 \leq \delta < s$. Moreover, this subsequence can be picked such that $\mathcal{M}^{\delta} f_n \to g$ in $L^{p'}(\partial\Omega)$ for some function g and $0 < \delta < s$ small. Furthemore, it is clear that we can pointwise require that for almost every $x \in \partial\Omega$: $\mathcal{M}^{\delta} f_n(x) \to g(x)$. From this obviously: $\mathcal{M}^{\delta} f(x) \leq g(x)$ for any such x. So $f \in \mathcal{D}^{\delta,p'} \subset \mathcal{D}^{0,p'}$.

Pick $\varepsilon > 0$ small and consider $\Omega_\varepsilon = \{x \in \Omega; \mathrm{dist}(x,\partial\Omega) > \varepsilon\}$. It follows that there is $k \in N$ such that for any $n \geq k$:

$$\|(f - f_n)\big|_{\Omega_\varepsilon}\|_{C^{\delta}(\Omega_\varepsilon)} < \varepsilon. \tag{2.33}$$

Now pick any $z \in \gamma(x)\backslash\Omega_\varepsilon$. Clearly, there is $z' \in \Omega_\varepsilon \cap \gamma(x)$ and $\mathrm{dist}(z, z') < \varepsilon$. Thus:

$$\begin{aligned}|(f - f_n)(z)| &\leq |(f - f_n)(z')| + \mathcal{M}^{\delta}(f - f_n)(x)\mathrm{dist}(z,z')^{\delta} \\ &\leq \varepsilon + \varepsilon^{\delta}\mathcal{M}^{\delta}(f - f_n)(x),\end{aligned} \tag{2.34}$$

for any $n \geq k$. This estimate is crucial. It follows that

$$|\mathcal{M}^{0}(f - f_n)(x)| \leq \varepsilon + \varepsilon^{\delta}(\mathcal{M}^{\delta} f(x) + \mathcal{M}^{\delta} f_n(x)). \tag{2.35}$$

Finally, we take the $L^{p'}$ norm on both sides and let $n \to \infty$. We get:

$$\overline{\lim_{n\to\infty}} \|\mathcal{M}^{0}(f - f_n)\|_{L^{p'}(\partial\Omega)} \leq \varepsilon\sigma(\partial\Omega) + 2\varepsilon^{\delta}\|g\|_{L^{p'}(\partial\Omega)}. \tag{2.36}$$

Since $\varepsilon > 0$ was arbitrary, clearly $\lim_{n\to\infty} \|\mathcal{M}^0(f - f_n)\|_{L^{p'}(\partial\Omega)} = 0$. This together with (2.33) imply that $f_n \to f$ in $\mathcal{D}^{0,p'}$. So the compactness is established.

Now, consider $0 < s' < s < 1$. We have established above that for some $\delta = \delta(p') > 0$ small and $f \in \mathcal{D}^{s,p}$: $\mathcal{M}^{s'+\delta} f \in L^{p'}(\partial\Omega)$ for any p' given by (2.20). Consider again a bounded sequence $(f_n)_{n\in N}$ of functions in $\mathcal{D}^{s,p}$. Again we pick a subsequence such that

$$f_n \to f \quad \text{in } C^{s'+\delta}_{\text{loc}}(\Omega), \qquad \text{and} \qquad \mathcal{M}^{s'+\delta} f_n \to g \quad \text{in } L^{p'}(\partial\Omega), \tag{2.37}$$

for some $g \in L^{p'}(\partial\Omega)$. Again we also require pointwise convergence of $\mathcal{M}^{s'+\delta} f_n$ to g for almost every $x \in \partial\Omega$. It follows that $\mathcal{M}^{s'+\delta} f \le g$. Take any $\varepsilon > 0$ small. We find k such that for all $n \ge k$ we have

$$\|(f - f_n)\big|_{\Omega_\varepsilon}\|_{C^{s'+\delta}(\Omega_\varepsilon)} < \varepsilon, \quad \text{and} \quad \|(f - f_n)\big|_{\Omega_\varepsilon}\|_{C^{s'}(\Omega_\varepsilon)} < \varepsilon. \tag{2.38}$$

Pick any $z \in \gamma(x) \setminus \Omega_\varepsilon$ and $z' \in \gamma(x)$. If $\text{dist}(z, z') \ge 2\varepsilon$ we clearly have that $z' \in \Omega_\varepsilon$. So there is a third point $\widetilde{z} \in \gamma(x) \cap \Omega_\varepsilon$ such that $\text{dist}(z, \widetilde{z}) < \varepsilon$ and $\text{dist}(z, z') \approx \text{dist}(\widetilde{z}, z')$. This gives us an estimate

$$\begin{aligned} &\frac{|(f - f_n)(z) - (f - f_n)(z')|}{\text{dist}(z, z')^{s'}} \le \\ \le &\frac{|(f - f_n)(z) - (f - f_n)(\widetilde{z})|}{\text{dist}(z, z')^{s'}} + \frac{|(f - f_n)(\widetilde{z}) - (f - f_n)(z')|}{\text{dist}(z, z')^{s'}} \le \\ \le &C(\mathcal{M}^{s'+\delta}(f - f_n)(x)\varepsilon^\delta + \varepsilon). \end{aligned} \tag{2.39}$$

The second term of (2.39) we estimated using (2.38). On the other hand if $\text{dist}(z, z') < 2\varepsilon$ we get:

$$\frac{|(f - f_n)(z) - (f - f_n)(z')|}{\text{dist}(z, z')^{s'}} \le C\ \mathcal{M}^{s'+\delta}(f - f_n)(x)\varepsilon^\delta. \tag{2.40}$$

If we combine (2.39) and (2.40) together we get for any such z, z':

$$\frac{|(f - f_n)(z) - (f - f_n)(z')|}{\text{dist}(z, z')^{s'}} \le C(\varepsilon + \mathcal{M}^{s'+\delta}(f - f_n)(x)\varepsilon^\delta). \tag{2.41}$$

Combining (2.38), (2.41) and (2.35) finally gives:

$$\mathcal{M}^{s'}(f - f_n)(x) \le C(\varepsilon + \varepsilon^\delta(\mathcal{M}^{s'+\delta} f(x) + \mathcal{M}^{s'+\delta} f_n(x))). \tag{2.42}$$

Taking $L^{p'}$ norm on both sides and letting $n \to \infty$ yields

$$\overline{\lim_{n\to\infty}} \|\mathcal{M}^{s'}(f - f_n)\|_{L^{p'}(\partial\Omega)} \leq C(\varepsilon\sigma(\partial\Omega) + 2\varepsilon^{\delta}\|g\|_{L^{p'}(\partial\Omega)}). \tag{2.43}$$

From this again follows that $f_n \to f$ in $\mathcal{D}^{s',p'}$, since $\varepsilon > 0$ was arbitrary.

Finally, if $s \geq 1$ thanks to (2.19) we can use induction to establish (2.21) and (2.22). □

Now we state a result that can be interpreted as a 'trace' theorem for our spaces.

Theorem 2.6. *Let $1 \leq p \leq \infty$ and $s > 0$. Write s as $s = k + \tau$, for k integer and $0 < \tau \leq 1$. Then given $f \in \mathcal{D}^{s,p}$ for any $\varepsilon > 0$ and $|\alpha| \leq k$ there exists $C = C(s, p, \varepsilon) > 0$ such that:*

$$\begin{aligned} D^{\alpha} f\big|_{\partial\Omega} &\in H^{\tau-\varepsilon,p}(\partial\Omega), \text{ and } \|D^{\alpha} f\|_{H^{\tau-\varepsilon,p}(\partial\Omega)} \leq C\|f\|_{\mathcal{D}^{s,p}}, \\ D^{\alpha} f\big|_{\partial\Omega} &\in B^{p}_{\tau-\varepsilon}(\partial\Omega), \quad \text{and } \|D^{\alpha} f\|_{B^{p}_{\tau-\varepsilon}(\partial\Omega)} \leq C\|f\|_{\mathcal{D}^{s,p}}. \end{aligned} \tag{2.44}$$

Here, $D^{\alpha} f\big|_{\partial\Omega}$ is defined as a nontangential limit of the function $D^{\alpha} f$, i.e.,

$$D^{\alpha} f\big|_{\partial\Omega}(x) = \lim_{\substack{y\to x \\ y\in\gamma(x)}} D^{\alpha} f(y), \qquad \textit{for almost every } x \in \partial\Omega. \tag{2.45}$$

Proof. Assume s is not an integer, and let $f \in \mathcal{D}^{s,p}$. For any $x \in \partial\Omega$, for which $\mathcal{M}^s f(x) < \infty$, we have that the function $u\big|_{\gamma(x)}$ is of the Hölder class C^s on $\gamma(x)$. This means, that $f\big|_{\gamma(x)}$ can be extended to $\overline{\gamma(x)}$, so that it remains of the Hölder class C^s. It follows, that for such x the values of $D^{\alpha} f(x)$ are well defined, for any $|\alpha| \leq k = [s]$. Also in this case $\tau = s - k \in (0, 1)$. Now, our approach slightly resembles the proof of Theorem 2.5. For any $|\alpha| \leq k$, we have that $D^{\alpha} f \in \mathcal{D}^{s-|\alpha|,p}$. Clearly, $D^{\alpha} f \in \mathcal{D}^{s-|\alpha|,p} \subset \mathcal{D}^{\tau,p}$. Thus, for any $x, y \in \partial\Omega$ that are close to each other we get:

$$|D^{\alpha} f(x) - D^{\alpha} f(y)| \leq |D^{\alpha} f(x) - D^{\alpha} f(z)| + |D^{\alpha} f(z) - D^{\alpha} f(y)|, \tag{2.46}$$

where z is a point in $\gamma(x) \cap \gamma(y)$, and $\text{dist}(x, y) \approx \text{dist}(x, z) \approx \text{dist}(y, z)$. It follows that the first term of the right side of (2.46) can be estimated by $C\mathcal{M}^{\tau} f(x)\text{dist}(x, y)^{\tau}$, and the second one by $C\mathcal{M}^{\tau} f(y)\text{dist}(x, y)^{\tau}$. This yields

$$\frac{|D^{\alpha} f(x) - D^{\alpha} f(y)|}{\text{dist}(x, y)^{\tau}} \leq C(\mathcal{M}^{\tau} f(x) + \mathcal{M}^{\tau} f(y)). \tag{2.47}$$

Hence as before we have that for any $\varepsilon > 0$: $D^\alpha f \in H^{\tau-\varepsilon,p}(\partial\Omega)$ and $D^\alpha f \in B^p_{\tau-\varepsilon}(\partial\Omega)$. From this the result follows. Finally, if $s > 0$ is as integer we use the fact that for any $s' < s$ we have $\mathcal{D}^{s,p} \subset \mathcal{D}^{s',p}$. $\square$

Now we look at an interpolation result. Recall quickly a simple case of complex interpolation scheme we would like to use.

Let E, F be Banach spaces. Suppose that F is included in E and the inclusion $F \hookrightarrow E$ is continuous. If $\mathcal{O}$ is the vertical strip in the complex plane,

$$\mathcal{O} = \{z \in C; 0 < \text{ Re } z < 1\}, \tag{2.48}$$

we define

$$\begin{aligned}\mathcal{H}_{E,F}(\mathcal{O}) = \{&u(z) \text{ bounded and continuous on } \overline{\mathcal{O}} \text{ with values in } E;\\ &\text{holomorphic on } \mathcal{O}\text{: } \|u(1+iy)\|_F \text{ is bounded for } y \in \mathbb{R}\}.\end{aligned} \tag{2.49}$$

For $\theta \in [0,1]$ we put

$$[E,F]_\theta = \{u(\theta); u \in \mathcal{H}_{E,F}(\mathcal{O})\}. \tag{2.50}$$

We give $[E,F]_\theta$ the Banach space topology making it isomorphic to the quotient

$$\mathcal{H}_{E,F}(\Omega)/\{u : u(\theta) = 0\}. \tag{2.51}$$

For convenience we use the convention: $[E,F]_\theta = [F,E]_{1-\theta}$.

Theorem 2.7. *For $0 < \theta < 1$, $1 \le p_1 < p_2 \le \infty$.*

$$[\mathcal{D}^{0,p_1}, \mathcal{D}^{0,p_2}]_\theta = \mathcal{D}^{0,q}, \tag{2.52}$$

where p_1, p_2 and q are related by

$$\frac{1}{q} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}. \tag{2.53}$$

Proof. This has been established in [7]. For the sake of completeness we include the proof here. Given $f \in \mathcal{D}^{0,q}$, we can define

$$u(z) = |f(x)|^{c(\theta-z)} f(x); \tag{2.54}$$

u by convention zero when $f(x) = 0$, with c chosen such that u belongs to $\mathcal{H}_{\mathcal{D}^{0,p_1},\mathcal{D}^{0,p_2}}(\mathcal{O})$, which gives $\mathcal{D}^{0,q} \subset [\mathcal{D}^{0,p_1}, \mathcal{D}^{0,p_2}]_\theta$. This proves one inclusion.

The other inclusion follows from the following clever argument pointed out to me by the referee. Since $\mathcal{M}^0$ is sublinear and maps $\mathcal{D}^{0,p_j}$ boundedly into $L^{p_j}(\partial\Omega)$ for $j = 1, 2$, a real interpolation gives us that $\mathcal{M}^0$ maps $(\mathcal{D}^{0,p_1}, \mathcal{D}^{0,p_2})_{\theta,\infty}$ into $L^q(\partial\Omega)$, thus, $(\mathcal{D}^{0,p_1}, \mathcal{D}^{0,p_2})_{\theta,\infty} \hookrightarrow \mathcal{D}^{0,q}$. Now, according to well-known connection between the complex and the real methods of interpolation, $[\mathcal{D}^{0,p_1}, \mathcal{D}^{0,p_2}]_\theta \hookrightarrow (\mathcal{D}^{0,p_1}, \mathcal{D}^{0,p_2})_{\theta,\infty}$, hence the inclusion $[\mathcal{D}^{0,p_1}, \mathcal{D}^{0,p_2}]_\theta \subset \mathcal{D}^{0,q}$ follows. □

Remark 2.8. It has been pointed out to me by Michael Taylor, that several Moser-type estimates hold for the spaces $\mathcal{D}^{s,p}$.

For any $0 \le s \le 1$ Moser estimates applied to C^s functions on $\gamma(x)$ yield

$$\mathcal{M}^s(fg)(x) \le C\mathcal{M}^0 f(x)\mathcal{M}^s g(x) + C\mathcal{M}^s f(x)\mathcal{M}^0 g(x), \tag{2.55}$$

for all $x \in \partial\Omega$ and some $C > 0$ independent of x. Consequently,

$$\|\mathcal{M}^s(fg)\|_{L^p} \le C\|f\|_{L^\infty}\|g\|_{\mathcal{D}^{s,p}} + C\|f\|_{\mathcal{D}^{s,p}}\|g\|_{L^\infty}. \tag{2.56}$$

This together with estimates on the norm of fg on $\widetilde{\Omega}$ yield:

Corollary 2.9. *For any $0 \le s \le 1$ and $1 \le p \le \infty$ there is a constant $C = C(s)$ such that*

$$\|fg\|_{\mathcal{D}^{s,p}} \le C\|f\|_{L^\infty}\|g\|_{\mathcal{D}^{s,p}} + C\|f\|_{\mathcal{D}^{s,p}}\|g\|_{L^\infty}. \tag{2.57}$$

Other similar corollaries arise with use of Theorem 2.5.

3. The operator L^{-1} on spaces $\mathcal{D}^{s,p}$

In this section we consider a linear operator

$$L = \Delta - V, \tag{3.1}$$

that acts on functions on a Riemannian manifold M. Here Δ is the Laplace-Beltrami operator on M that can be written in a local coordinates as

$$\Delta u = g^{-1/2}\partial_j(g^{jk}g^{1/2}\partial_k u). \tag{3.2}$$

We use the summation convention, take (g^{jk}) to be the inverse matrix to (g_{jk}) and set $g = \det(g_{jk})$. As before we assume that the metric tensor (g_{jk}) is of Hölder class C^α, $0 < \alpha \leq 1$. We also assume that $V \in L^\infty(M)$, $V \geq 0$ on M and $V > 0$ on a set of positive measure in each connected component of $M \setminus \overline{\Omega}$. Here $\Omega \subset M$ is assumed to be open, connected and with Lipschitz boundary. Therefore we can consider the spaces $\mathcal{D}^{s,p}$ on Ω.

The properties of the operator L has been studied extensively in the papers [23], [24], [25] by Mitrea and Taylor. They established, that the operator L is an isomorphism

$$L : H^{r+1,p}(M) \to H^{r-1,p}(M), \tag{3.3}$$

for each $p \in (1, \infty)$ and $-\alpha < r < \alpha$. If we denote by $E(x, y)$ the integral kernel of L^{-1}, such that formally:

$$L^{-1}u(x) = \int_M E(x, y)u(y)\, d\mathrm{Vol}(y), \qquad x \in M, \tag{3.4}$$

then $E(x, y)$ is singular only on the diagonal $\{x = y\}$ and decomposes as

$$E(x, y) = g(y)^{-1/2} \left\{e_0(x - y, y) + e_1(x, y)\right\}, \tag{3.5}$$

where

$$e_0(x - y, y) = C_n \left(\sum g_{jk}(y)(x_j - y_j)(x_k - y_k)\right)^{-(n-2)/2}, \tag{3.6}$$

for a suitable constant C_n, and the residual term $e_1(x, y)$ satisfies

$$\begin{aligned} |e_1(x, y)| &\leq C_\varepsilon |x - y|^{-(n-2-\alpha+\varepsilon)}, \\ |\nabla_x e_1(x, y)| &\leq C_\varepsilon |x - y|^{-(n-1-\alpha+\varepsilon)}, \end{aligned} \tag{3.7}$$

for any $\varepsilon > 0$. (See [25] for details). Let us note here that for $n = 2$ the term $e_0(x - y, y)$ takes different form, namely it contains a logarithmic singularity of type $\log |x - y|^{-1}$. For this reason all what follow is carried out only for $n \geq 3$, although the argument can be adapted to accommodate the two dimensional case.

As shown in [7], there is actually more that can be established about $e_1(x, y)$. Namely, let $|x_0 - y| = 2\rho$, we want to estimate $e_1(x, y)$ on $\{x : |x - x_0| \leq \rho\}$. We shift coordinates so $x_0 = 0$ and introduce dilation operators

$$u_\rho(x) = u(\rho x), \qquad |x| \leq 1. \tag{3.8}$$

If $u(x) = e_1(x, y)$ for $|x| \leq \rho$ then (2.76)-(2.80) of [25] yields

$$\|u_\rho\|_{H^{s,q}(B_{1/2})} \leq C(s, q, \delta)\rho^{-(n-2-\alpha+\delta)}, \qquad \forall s < 1 + \alpha,\ q < \infty,\ \delta > 0. \tag{3.9}$$

Hence for any $\delta > 0$

$$\|\nabla_x u_\rho\|_{C^{\alpha-\delta}(B_{1/2})} \leq C_\delta \rho^{-(n-2-\alpha+\delta)}. \tag{3.10}$$

This means that for $|x - x_0| \leq \frac{1}{4}|x_0 - y|$ we get

$$|\nabla_x e_1(x, y) - \nabla_x e_1(x_0, y)| \leq C_\delta |x_0 - y|^{-(n-1)}|x - x_0|^{\alpha-\delta}. \tag{3.11}$$

This is equivalent to the statement that for any $\varepsilon > 0$ and $|x - x_0| \leq \frac{1}{4}|x_0 - y|$

$$\frac{|\nabla_x e_1(x, y) - \nabla_x e_1(x_0, y)|}{|x - x_0|^{\alpha-\varepsilon}} \leq C_\varepsilon |x_0 - y|^{-(n-1)}. \tag{3.12}$$

On the other hand if $|x - x_0| > \frac{1}{4}|x_0 - y|$ it follows from (3.7) that

$$\frac{|\nabla_x e_1(x, y) - \nabla_x e_1(x_0, y)|}{|x - x_0|^{\alpha-\varepsilon}} \leq C_\varepsilon \max\{|x_0 - y|^{-(n-1)}, |x - y|^{-(n-1)}\}. \tag{3.13}$$

A direct computation for $e_0(x - y, y)$ gives:

$$\begin{aligned} |e_0(x - y, y)| &\leq C|x - y|^{-(n-2)}, \\ |\nabla_x e_0(x - y, y)| &\leq C|x - y|^{-(n-1)}, \\ |\nabla_x^2 e_0(x - y, y)| &\leq C|x - y|^{-n}. \end{aligned} \tag{3.14}$$

It follows that for any $0 < \beta < 1$ and $|x - x_0| \leq \frac{1}{4}|x_0 - y|$ we have

$$\frac{|\nabla_x e_0(x, y) - \nabla_x e_0(x_0, y)|}{|x - x_0|^{\beta}} \leq C|x_0 - y|^{-(n-1+\beta)}. \tag{3.15}$$

Similarly, for $|x - x_0| > \frac{1}{4}|x_0 - y|$ we get that

$$\frac{|\nabla_x e_0(x, y) - \nabla_x e_0(x_0, y)|}{|x - x_0|^{\beta}} \leq C \max\{|x_0 - y|^{-(n-1+\beta)}, |x - y|^{-(n-1+\beta)}\}. \tag{3.16}$$

Hence, if we put things together we get for any $0 < \beta < \alpha$:

$$\begin{aligned} |E(x, y)| &\leq C|x - y|^{-(n-2)}, \\ |\nabla_x E(x, y)| &\leq C|x - y|^{-(n-1)}, \\ \frac{|\nabla_x E(x, y) - \nabla_x E(x_0, y)|}{|x - x_0|^{\beta}} &\leq C|x_0 - y|^{-(n-1+\beta)}, \quad \text{for } |x - x_0| \leq \frac{1}{4}|x_0 - y|, \\ \frac{|\nabla_x E(x, y) - \nabla_x E(x_0, y)|}{|x - x_0|^{\beta}} &\leq C \max\{|x_0 - y|^{-(n-1+\beta)}, |x - y|^{-(n-1+\beta)}\}, \\ &\qquad \text{for } |x - x_0| > \frac{1}{4}|x_0 - y|. \end{aligned} \tag{3.17}$$

We are ready to establish the following important lemma.

Lemma 3.1. *Assume that the metric tensor on M belongs to the Hölder class C^α, $0 < \alpha < 1$. Let $x \in M$ be an arbitrary point and $r > 0$. Consider a geodesic ball $B_r(x)$ of radius r around x and assume that $f \in L^\infty(M)$ is a given function with support in $B_r(x)$ and bounded in absolute value by one on M. Let u solve*

$$Lu = f \quad \text{in } M, \qquad \text{i.e.,} \qquad u = L^{-1} f. \tag{3.18}$$

Then there exist constants C, C_β independent on f and x such that for any $y, y' \in M$

$$\begin{aligned} |u(y)| &\le C \frac{r^{n-1}}{(r + |x-y|)^{n-3}}, \\ |\nabla u(y)| &\le C \frac{r^{n-1}}{(r + |x-y|)^{n-2}}, \\ \frac{|\nabla u(y) - \nabla u(y')|}{|y-y'|^\beta} &\le C_\beta \frac{r^{n-1}}{(r + \min\{|x-y|, |x-y'|\})^{n-2+\beta}}, \end{aligned} \tag{3.19}$$

where $0 < \beta < \alpha$.

Proof. We prove (3.19) in two steps. First we take $y \in B_{2r}(x)$. We estimate $|u(y)|$ and $|\nabla u(y)|$ using (3.4) and (3.17). Assume for simplicity that $r > 0$ is small enough so that we can consider just one geodesic coordinate chart centered at x that contains the ball $B_{2r}(x)$. In this chart we can also assume that x is the origin. We integrate over $(n-1)$-dimensional shells $S_\rho = \partial B_\rho(y)$ centered at y. Simple estimate using (3.17) gives

$$\begin{aligned} |u(y)| &\le C \int_0^{3r} \int_{S_\rho} \frac{1}{|z-y|^{n-2}} \, d\sigma(z) \, d\rho, \\ |\nabla u(y)| &\le C \int_0^{3r} \int_{S_\rho} \frac{1}{|z-y|^{n-1}} \, d\sigma(z) \, d\rho. \end{aligned} \tag{3.20}$$

Since the surface area of S_ρ is of the order of ρ^{n-1} from (3.20) we get

$$|u(y)| \le C \int_0^{3r} \rho \, d\rho \le C r^2, \qquad |\nabla u(y)| \le C r \tag{3.21}$$

By possibly enlarging the constant C in (3.21) we can see that (3.21) and (3.19) are equivalent for $y \in B_{2r}(x)$.

Similarly, pick $y, y' \in B_{2r}(x)$ and denote by δ the distance between them, i.e., $\delta = |y - y'| < 2r$. We have

$$\frac{|\nabla u(y) - \nabla u(y')|}{|y - y'|^\beta} \leq \int_M \frac{|\nabla_y E(y,z) - \nabla_y E(y',z)|}{|y - y'|^\beta}\, d\mathrm{Vol}(z). \tag{3.22}$$

Using (3.17) we see that different situations arise for $|y-z| \leq 4\delta$ and $|y-z| > 4\delta$. Consider first the integral (3.22) over the set $|y - z| \leq 4\delta$. Obviously

$$\max\{|y-z|^{-(n-1+\beta)}, |y'-z|^{-(n-1+\beta)}\} \leq |y-z|^{-(n-1+\beta)} + |y'-z|^{-(n-1+\beta)}, \tag{3.23}$$

and therefore:

$$\begin{aligned}&\int_{\{|y-z|\leq 4\delta\}} \frac{|\nabla_y E(y,z) - \nabla_y E(y',z)|}{|y - y'|^\beta}\, d\mathrm{Vol}(z)\\ \leq\ & C\int_0^{4\delta}\int_{S\rho} \frac{1}{|y-z|^{n-1+\beta}}\, d\sigma(z)\, d\rho + C\int_0^{5\delta}\int_{S'\rho} \frac{1}{|y'-z|^{n-1+\beta}}\, d\sigma(z)\, d\rho,\end{aligned} \tag{3.24}$$

where $S_\rho = \partial B_\rho(y)$, $S'_\rho = \partial B_\rho(y')$. Obviously, the right hand side of (3.24) can be estimated by $C\delta^{1-\beta} \leq Cr^{1-\beta}$. On the other hand, integrating (3.24) over the set $|y - z| > 4\delta$ yields:

$$\begin{aligned}&\int_{\{|y-z|> 4\delta\}} \frac{|\nabla_y E(y,z) - \nabla_y E(y',z)|}{|y - y'|^\beta}\, d\mathrm{Vol}(z)\\ \leq\ & C\int_{4\delta}^{3r}\int_{S\rho} \frac{1}{|y-z|^{n-1+\beta}}\, d\sigma(z)\, d\rho\end{aligned} \tag{3.25}$$

Again (3.25) is bounded by $Cr^{1-\beta}$. Hence we see that, (3.19) is indeed true for $y, y' \in B_{2r}(x)$.

Now we consider y outside the ball $B_{2r}(x)$. Denote by ε the distance between y and $B_r(x)$. We integrate the same way is we did in the first part over S_ρ. However now it is clear that S_ρ intersects the support of f only for $\rho \geq \varepsilon$. Moreover, the surface measure of such intersection can be estimated by Cr^{n-1}. This leads to

$$\begin{aligned}|u(y)| \leq & C\int_\varepsilon^\infty \int_{S_\rho \cap B_r(x)} \frac{1}{|z-y|^{n-2}}\, d\sigma(z)\, d\rho \leq\\ \leq & C\int_\varepsilon^\infty r^{n-1}\frac{1}{\rho^{n-2}}\, d\rho \leq C\frac{r^{n-1}}{\varepsilon^{n-3}},\\ |\nabla u(y)| \leq & C\int_\varepsilon^\infty \int_{S_\rho \cap B_r(x)} \frac{1}{|z-y|^{n-1}}\, d\sigma(z)\, d\rho \leq\\ \leq & C\int_\varepsilon^\infty r^{n-1}\frac{1}{\rho^{n-1}}\, d\rho \leq C\frac{r^{n-1}}{\varepsilon^{n-2}}.\end{aligned} \tag{3.26}$$

Now since $\varepsilon \approx |x-y|-r$ we get that for $|x-y| \geq 2r$ (i.e $y \notin B_{2r}(x)$)

$$\varepsilon \approx |x-y|+r. \tag{3.27}$$

This implies that the estimate (3.19) works for such y.

Finally, consider the case when either y or y' are not in $B_{2r}(x)$. If $|y-y'| \geq \frac{r}{5}$, then the last estimate (3.19) follows from (3.26) and (3.27). Therefore we have to consider only the case when $\delta = |y-y'| < \frac{r}{5}$. If such case however for any $z \in$ supp f we get that $|y-z| > 4\delta$. Thus as in (3.26) using (3.17) we get:

$$\begin{aligned} \frac{|\nabla u(y) - \nabla u(y')|}{|y-y'|^\beta} \leq & C \int_\varepsilon^\infty \int_{S_\rho \cap B_r(x)} \frac{1}{|z-y|^{n-1+\beta}} \, d\sigma(z) \, d\rho \leq \\ \leq & C \int_\varepsilon^\infty r^{n-1} \frac{1}{\rho^{n-1+\beta}} \, d\rho \leq C \frac{r^{n-1}}{\varepsilon^{n-2+\beta}}. \end{aligned} \tag{3.28}$$

This concludes the proof. $\square$

Now we are ready to establish the following:

Proposition 3.2. *Assume that $r > 0$ is small. Let $f \in L^\infty(M)$ be a function on M bounded in absolute value by one with support in $B_r(x) \cap \Omega$, where x is a point from the boundary $\partial\Omega$. Let u be as before the solution to $Lu = f$ in M, i.e., $u = L^{-1}f$. We have the following estimates for the maximal operator $\mathcal{M}^s u$:*

$$\begin{aligned} \|\mathcal{M}^0 u\|_{L^p(\partial\Omega)} &\leq C r^{n-1}, \text{ for } 1 \leq p < (n-1)/(n-3), \\ \|\mathcal{M}^1 u\|_{L^p(\partial\Omega)} &\leq C r^{n-1}, \text{ for } 1 \leq p < (n-1)/(n-2), \\ \|\mathcal{M}^{1+\beta} u\|_{L^p(\partial\Omega)} &\leq C r^{n-1}, \text{ for } 1 \leq p < (n-1)/(n-2+\beta), \end{aligned} \tag{3.29}$$

where $0 < \beta < \alpha \leq 1$. The constant in the last estimate of (3.29) $C = C(\beta)$ depends on β.

Proof. Since $r > 0$ is small, we can find a small neighborhood U of x such that in this neighborhood there are smooth local coordinates in which

$$U \cap \Omega = \{x = (x', x_n) \in U : x_n > \varphi(x')\}, \tag{3.30}$$

where φ is a Lipschitz function with Lipschitz constant bounded by L. Here L does not depend on chosen point $x \in \partial\Omega$. We will consider nontangential approach regions $\gamma(z)$ to any point $z = (z', \varphi(z')) \in \partial\Omega$ such that the vertex of $\gamma(z)$ at z is sharp enough. Namely, we require that any half-ray with vertex at z that lies whole in $\gamma(z)$ has "steepness" (absolute value of its slope) at least $2L$.

From this it follows that there exists a universal constant k (independent) on r such that we can split points $z \in \partial\Omega$ into two distinct sets. If $z = (z', \varphi(z')) \in \partial\Omega$ and $|z' - x'| \le kr$ then $\gamma(z)$ might intersect $B_r(x)$. At such point by (3.19) we have:

$$\mathcal{M}^0 u(z) \le Cr^2, \quad \mathcal{M}^0(\nabla u)(z) \le Cr, \quad \sup_{y,y' \in \gamma(z)} \frac{|\nabla u(y) - \nabla u(y')|}{|y - y'|^\beta} \le Cr^{1-\beta}. \tag{3.31}$$

On the other hand if $|z' - x'| > kr$ then the distance between any point $w \in \gamma(z)$ and x is greater or equal to $\frac{1}{k}|z' - x'|$. This means that for such z we get from (3.19):

$$\begin{aligned}
\mathcal{M}^0 u(z) &\le C \frac{r^{n-1}}{(r + k^{-1}|z' - x'|)^{n-3}}, \\
\mathcal{M}^0(\nabla u)(z) &\le C \frac{r^{n-1}}{(r + k^{-1}|z' - x'|)^{n-2}}, \\
\mathcal{T}^\beta u(z) \overset{\text{def}}{=} \sup_{y,y' \in \gamma(z)} \frac{|\nabla u(y) - \nabla u(y')|}{|y - y'|^\beta} &\le C \frac{r^{n-1}}{(r + k^{-1}|z' - x'|)^{n-2+\beta}}
\end{aligned} \tag{3.32}$$

Now we can estimate the L^p norms of $\mathcal{M}^0 u$, $\mathcal{M}^0(\nabla u)$ and $\mathcal{T}^\beta u$. On $B_{kr}(x) \cap \partial\Omega$ we get

$$\begin{aligned}
&\int_{B_{kr}(x) \cap \partial\Omega} (\mathcal{M}^0 u(y))^p \, d\sigma(y) \le Cr^{n-1} r^{2p} = Cr^{n+2p-1} \\
&\int_{B_{kr}(x) \cap \partial\Omega} (\mathcal{M}^0(\nabla u)(y))^p \, d\sigma(y) \le Cr^{n-1} r^{p} = Cr^{n+p-1} \\
&\int_{B_{kr}(x) \cap \partial\Omega} (\mathcal{T}^\beta u(y))^p \, d\sigma(y) \le Cr^{n-1} r^{p(1-\beta)} = Cr^{n+p(1-\beta)-1}.
\end{aligned} \tag{3.33}$$

Similarly, off $B_{kr}(x) \cap \partial\Omega$ we get:

$$\begin{aligned}
\int_{\partial\Omega \setminus B_{kr}(x)} (\mathcal{M}^0 u(y))^p \, d\sigma(y) \le& C \int_{kr}^A \int_{y \in S_\rho} \left(\frac{r^{n-1}}{(r + k^{-1}\rho)^{n-3}} \right)^p \, d\sigma(y) \, d\rho \le \\
\le& Cr^{p(n-1)} \int_0^A \frac{\rho^{n-2}}{(r + k^{-1}\rho)^{p(n-3)}} \, d\rho.
\end{aligned} \tag{3.34}$$

Here S_ρ is a $(n-2)$-dimensional shell defined by $S_\rho = \partial\Omega \cap \partial B_\rho(x)$. In (3.34) we have also used that $(n-2)$-dimensional surface area of such shell is of the order ρ^{n-2}. We further estimate the integral in (3.34):

$$\int_0^A \frac{\rho^{n-2}}{(r+k^{-1}\rho)^{p(n-3)}}\, d\rho \leq C \int_0^A \rho^{n-2-p(n-3)}\, d\rho. \tag{3.35}$$

If $p < (n-1)/(n-3)$ then $n-2-p(n-3) > -1$, hence (3.35) is finite (and independent of r). Similarly, using the same technique we get

$$\begin{aligned} &\int_{\partial\Omega\setminus B_{kr}(x)} (\mathcal{M}^0(\nabla u)(y))^p\, d\sigma(y) \leq Cr^{p(n-1)} \int_0^A \rho^{n-2-p(n-2)}\, d\rho, \\ &\int_{\partial\Omega\setminus B_{kr}(x)} (\mathcal{T}^\beta u(y))^p\, d\sigma(y) \leq Cr^{p(n-1)} \int_0^A \rho^{n-2-p(n-2+\beta)}\, d\rho. \end{aligned} \tag{3.36}$$

Hence the last integral in the first line of (3.36) is finite for $p < (n-1)/(n-2)$, and the integral in the second line for $p < (n-1)/(n-2+\beta)$. Finally, we put (3.33)-(3.36) together to get

$$\begin{aligned} &\|\mathcal{M}^0 u\|_{L^p(\partial\Omega)} \leq C(r^{n-1+2p} + r^{p(n-1)})^{1/p} \leq C(r^{2+(n-1)/p} + r^{n-1}) \leq Cr^{n-1}, \\ &\|\mathcal{M}^0(\nabla u)\|_{L^p(\partial\Omega)} \leq C(r^{n-1+p} + r^{p(n-1)})^{1/p} \leq Cr^{n-1}, \\ &\|\mathcal{T}^\beta u\|_{L^p(\partial\Omega)} \leq C(r^{n-1+p(1-\beta)} + r^{p(n-1)})^{1/p} \leq Cr^{n-1}. \end{aligned} \tag{3.37}$$

In the first line of (3.37) we used the fact that for $p < (n-1)/(n-3)$, $r^{2+(n-1)/p} \leq Cr^{n-1}$ for $|r|$ bounded, similar estimate we also used in the second line for $p < (n-1)/(n-2)$ and in the third line for $p < (n-1)/(n-2+\beta)$. From (3.37) we immediately have (3.29). This concludes our proof. $\square$

Let $z = (z', z_n)$ be any point from the considered coordinate chart (3.30). We put

$$\widetilde{\gamma(z)} = \{w = (w', w_n) \in U : w_n < z_n \text{ and } |z' - w'| < 2L|z_n - w_n|\}. \tag{3.38}$$

Here L is as before a bound on the Lipschitz constant of $\partial\Omega$. So our region $\widetilde{\gamma(z)}$ is an open downward opening cone with vertex at z. Recall, that we defined $\gamma(z)$ by (2.2) and in the current setting we assume that $K = 2L$ in this definition. Hence, if we compare $\gamma(z)$ and $\widetilde{\gamma(z)}$ we can see that these two cones are symmetric with respect to the hyperplane $x_n = z_n$ in U.

Definition 3.3. Consider a coordinate chart (3.30) and a set $A \subset \partial\Omega$ on this chart which is open in $\partial\Omega$. We say that a set $\mathcal{A} \subset \Omega$ is a P-image of A and write

$$\mathcal{A} = \mathrm{Pim}(A), \tag{3.39}$$

provided the set $\mathcal{A}$ satisfies the following conditions:

(a) The set $\{z = (z', \varphi(z')) \in \partial\Omega : \exists w = (z', w_n) \in \mathcal{A}\}$ (a projection of $\mathcal{A}$ onto $\partial\Omega$) is A.

(b) $z = (z', z_n) \in \mathcal{A}$ if and only if $z \in \Omega$ and $\widetilde{\gamma(z)} \cap \partial\Omega \subset A$.

Remark. The property (a) follows from (b). It also follows that if $z \in \mathcal{A}$ then $\widetilde{\gamma(z)} \cap \Omega \subset \mathcal{A}$.

Now we are ready to establish a connection between the set $\mathcal{A}$ from the previous definition and Proposition 3.2.

Proposition 3.4. *Let the set A be as in Definition 3.3 and let $\mathcal{A} = Pim(A)$. Consider a solution $u = L^{-1}f$ to the problem $Lu = f$ where $f \in L^\infty(M)$ is a function on M bounded in absolute value by one with support in $\mathcal{A}$. Then we have:*

$$\begin{aligned}
\|\mathcal{M}^0 u\|_{L^p(\partial\Omega)} &\le C_p \sigma(A), && \text{for } 1 \le p < (n-1)/(n-3),\\
\|\mathcal{M}^1 u\|_{L^p(\partial\Omega)} &\le C_p \sigma(A), && \text{for } 1 \le p < (n-1)/(n-2),\\
\|\mathcal{M}^{1+\beta} u\|_{L^p(\partial\Omega)} &\le C_{p,\beta} \sigma(A), && \text{for } 1 \le p < (n-1)/(n-2+\beta),
\end{aligned} \tag{3.40}$$

where $0 < \beta < \alpha \le 1$ and $\sigma(A)$ is the $(n-1)$-dimensional (surface) area of A on $\partial\Omega$.

Proof. We will do our proof in several steps. Given any point $x \in A$, we can assign to it a positive "height" number $h(x) > 0$ as follows. In the coordinate chart we can write x as (x', x_n). Define

$$h(x) = \sup\{t \in \mathbb{R}^+ : y = (x', x_n + t) \in \mathcal{A}\}. \tag{3.41}$$

Now clearly $h(x) > 0$ and the whole set $\Omega \cap \widetilde{\gamma}(x', x_n + h(x))$ is in $\mathcal{A}$. Here $\widetilde{\gamma(.)}$ is the cone defined by (3.38). The set

$$V = \partial\Omega \cap \widetilde{\gamma}(x', x_n + h(x)) \tag{3.42}$$

is an open neighborhood of x on $\partial\Omega$. Also clearly $V \subset A$. Using the fact that the surface $\partial\Omega$ is Lipschitz we can also establish a relation between $h(x)$ and

$\sigma(V)$. Namely there are two positive constants c_1 and c_2 depending only on the Lipschitz constant of $\partial\Omega$ such that

$$c_1(h(x))^{n-1} \leq \sigma(V) \leq c_2(h(x))^{n-1}. \tag{3.43}$$

An immediate consequence of this observation is that the number

$$H = \sup_{x \in A} h(x) \tag{3.44}$$

must be finite, since the surface measure of A is finite. Denote by $\mathcal{H}$ the hyperplane

$$\{x = (x', 0) : x' \in \mathbb{R}^{n-1}\}.$$

Consider on the chart (3.30) a projection $P : U \to \mathcal{H}$ which assigns to any point $x = (x', x_n)$ the point $P(x) = (x', 0) \in \mathcal{H}$.

Now consider on $\mathcal{H}$ a grid created by $(n-2)$-dimensional hyperplanes such that $\mathcal{H}$ is divided into $(n-1)$-dimensional cubes with sides $2H/L$. Let us denote this collection of cubes by $\mathcal{C}_1$. Let $\mathcal{D}_1 \subset \mathcal{C}_1$ be a collection of all cubes from $\mathcal{C}_1$ that contain a point $\widetilde{x} \in \mathcal{H}$ for which there is an $x \in A$ such that $P(x) = \widetilde{x}$ and $h(x) > H/2$.

Each cube from $\mathcal{C}_1 \setminus \mathcal{D}_1$ we split further so that we get 2^{n-1} cubes with side H/L. We denote the collection of all such cubes by $\mathcal{C}_2$. Now we define $\mathcal{D}_2 \subset \mathcal{C}_2$ to be a collection of all cubes from $\mathcal{C}_2$ that contain a point $\widetilde{x} \in \mathcal{H}$ for which there is an $x \in A$ such that $P(x) = \widetilde{x}$ and $h(x) > H/4$. From here we continue inductively. At each stage we split all cubes from $\mathcal{C}_n \setminus \mathcal{D}_n$ into 2^{n-1} new cubes with sides half of the previous one. Then we put into $\mathcal{D}_n$ are cubes from $\mathcal{C}_n$ that contain a point $\widetilde{x} \in \mathcal{H}$ for which there is an $x \in A$ such that $P(x) = \widetilde{x}$ and $h(x) > H/2^n$.

Now denote by $\mathcal{D}$ the union of all $\mathcal{D}_n$, i.e., a cube belongs to $\mathcal{D}$ if and only if it was selected at certain stage of the process defined above. The set $\mathcal{D}$ is countable and therefore we can order all cubes there into a sequence $D_1, D_2, D_3, \dots$. Denote by C_i the P-preimage of a cube D_i on $\partial\Omega$, i.e.,

$$C_i = \partial\Omega \cap P^{-1}(D_i). \tag{3.45}$$

The collection of all C_i we will denote by $\mathcal{C}$.

There are several important observations we would like make. The first one is that the collection $\mathcal{C}$ covers A. From this obviously

$$\sigma(A) \leq \sigma(\cup C_i) = \sum_{i=1}^{\infty} \sigma(C_i). \tag{3.46}$$

Also, since $\partial\Omega$ is Lipschitz there are positive constants c_3 and c_4 such that for all $i \in N$

$$c_3 \lambda^{n-1}(D_i) \leq \sigma(C_i) \leq c_4 \lambda^{n-1}(D_i). \tag{3.47}$$

Here λ^{n-1} is the $(n-1)$-dimensional Lebesgue measure on $\mathcal{H}$.

Our other remark is that the inequality sign in (3.46) can also be reversed. Fix $i \in N$. Denote by r the length of the side of D_i. From our construction it follows that there is $x \in C_i \cap A$ such that

$$h(x) > r\tfrac{L}{2}. \tag{3.48}$$

This means that the whole part of a cone $\widetilde{\gamma}(x', \varphi(x') + r\frac{L}{2})$ (here $x = (x', \varphi(x'))$) that lies in Ω belongs to $\mathcal{A}$, and also the set

$$V = \partial\Omega \cap \widetilde{\gamma}(x', x_n + r\tfrac{L}{2}) \tag{3.49}$$

is a subset of A. From a simple geometric argument it follows that the surface measure of the intersection of V with C_i could be estimated from bellow by Cr^{n-1} where the constant $C > 0$ depends only on Lipschitz constant L of $\partial\Omega$. This yields

$$\sigma(A) \geq C\sigma(\cup C_i) = C\sum_{i=1}^{\infty} \sigma(C_i). \tag{3.50}$$

The estimate (3.50) is crucial. Now for each i we define a set E_i. Let r_i be the length of the side of D_i. Let $\widetilde{x}_i$ be the center of the $(n-1)$ dimensional cube D_i in $\mathcal{H}$. We lift this point onto $\partial\Omega$ such that we get $x_i = (x_i', \varphi(x_i')) \in \partial\Omega$ and $P(x_i) = \widetilde{x}_i$. Finally, we set

$$E_i = \{y = (y', y_n) : |y' - x_i'| \leq Lr_i \text{ and } |y_n - \varphi(x_i')| \leq Lr_i\}, \tag{3.51}$$

so that E_i is an n-dimensional 'cube' (naturally just in our coordinates) with center at x_i and side of length $2Lr_i$. This 'cube' was carefully picked such that

$$\mathcal{A}_i = \{w \in \mathcal{A} : P(w) \in D_i\} \subset E_i. \tag{3.52}$$

In particular the union of all E_i covers $\mathcal{A}$. Finally, we pick a ball B_i with center at x such that $E_i \subset B_i$. Clearly this all can be done such that

$$\mathrm{Vol}(B_i) \approx r_i^n. \tag{3.53}$$

Now we can define a functions f_i as follows:

$$f_i = f\mathcal{X}_{\mathcal{A}_i} \qquad i = 1, 2, 3, \dots. \tag{3.54}$$

Here the set $\mathcal{A}_i$ comes from (3.52) and $\mathcal{X}_{\mathcal{A}_i}$ is the characteristic function of the set $\mathcal{A}_i$. Obviously

$$f = \sum_{i=1}^{\infty} f_i. \tag{3.53}$$

Now we put $u_i = L^{-1} f_i$. Since $\mathcal{A}_i \subset B_i$ and f_i satisfies assumptions of Proposition 3.2 we get an L^p estimates on $\mathcal{M}^0 u_i$, $\mathcal{M}^1 u_i$ and $\mathcal{M}^{1+\beta} u_i$ for corresponding p:

$$\begin{aligned} &\|\mathcal{M}^0 u_i\|_{L^p(\partial\Omega)} \le C R_i^{n-1}, \qquad \|\mathcal{M}^1 u_i\|_{L^p(\partial\Omega)} \le C R_i^{n-1}, \\ &\|\mathcal{M}^{1+\beta} u_i\|_{L^p(\partial\Omega)} \le C R_i^{n-1}, \end{aligned} \tag{3.56}$$

where R_i is the radius of B_i. Since $r_i \approx R_i$ and $r_i^{n-1} \approx \sigma(C_i)$ we get that

$$\|\mathcal{M}^0 u_i\|_{L^p(\partial\Omega)} \le C\sigma(C_i). \tag{3.57}$$

Finally, since $u = L^{-1} f$ can be written as $u = \sum u_i$ combining (3.57) with (3.50) we get

$$\|\mathcal{M}^0 u\|_{L^p(\partial\Omega)} \le \sum_{i=1}^{\infty} \|\mathcal{M}^0 u_i\|_{L^p(\partial\Omega)} \le C \sum_{i=1}^{\infty} \sigma(C_i) \le C\sigma(A). \tag{3.58}$$

We also get similar estimates for $\mathcal{M}^1 u$ and $\mathcal{M}^{1+\beta} u$. This concludes our proof. □

Now we are ready to prove the following.

Theorem 3.5. *Let the metric tensor on M be of class C^α, $0 < \alpha \le 1$. Assume, that function $f : \Omega \to \mathbb{R}$ belongs to $\mathcal{D}^{0,p}$ for some $1 \le p \le \infty$. Consider an extension of the function f onto whole the M, by putting $f(x) = 0$, for all $x \in M \setminus \overline{\Omega}$. Let u be the solution to*

$$Lu = f \quad \text{in } M, \qquad \text{i.e.,} \qquad u = L^{-1} f. \tag{3.59}$$

For any $0 < \beta < \alpha$ the nontangential maximal function of $\mathcal{M}^{1+\beta}u$ belongs to $L^p(\partial\Omega)$ and there exists a constant $C_p = C(p, \beta, M, \Omega) > 0$ such that

$$\|\mathcal{M}^{1+\beta}u\|_{L^p(\partial\Omega)} \leq C_p\|\mathcal{M}^0 f\|_{L^p(\partial\Omega)}. \tag{3.60}$$

It follows that the map:

$$L^{-1} : \mathcal{D}^{0,p} \to \mathcal{D}^{1+\beta,p} \tag{3.61}$$

is continuous and compact for any $1 \leq p \leq \infty$ and $0 < \beta < \alpha$. We also have that the maps

$$\begin{aligned} &L^{-1} : \mathcal{D}^{0,p} \to \mathcal{D}^{0,q}, \quad &&\begin{cases} q < \frac{n-1}{(n-1)/p-2}, & \text{for } p \leq (n-1)/2, \\ q = \infty, & \text{otherwise,} \end{cases} \\ &L^{-1} : \mathcal{D}^{0,p} \to \mathcal{D}^{1,q}, \quad &&\begin{cases} q < \frac{n-1}{(n-1)/p-1}, & \text{for } p \leq n-1, \\ q = \infty, & \text{otherwise,} \end{cases} \end{aligned} \tag{3.62}$$

are continuous and compact for any $1 \leq p \leq \infty$.

Proof. Notice that if the metric tensor g on M is Lipschitz, then (3.62) follows directly from (3.61) and Theorem 2.5. On the other hand, if we have the metric tensor only Hölder continuous of some class C^α, $\alpha < 1$, then we will not get (3.62) directly from (3.61). First we concentrate on (3.61). If $p = \infty$ the claim is obvious and follows from (3.3). For $1 \leq p < \infty$, consider the sets

$$A_i = \{x \in \partial\Omega : \mathcal{M}^0 f(x) > i^{1/p}\}, \qquad i = 1, 2, \dots. \tag{3.63}$$

Here, if we want to be completely precise we should consider a partition of unity on $\partial\Omega$ and sets A_i is each coordinate chart corresponding to this partition separately. This is because on two different charts the nontangential approach region $\gamma(x)$ to a point $x \in \partial\Omega$ might slightly differ. This also means that the sets A_i would slightly differ on such two charts. Nevertheless the definition (3.63) up to this small simplification is correct.

Since we took open nontangential approach regions $\gamma(.)$, it follows that each set A_i is open. The fact that $\mathcal{M}^0 f \in L^p(\partial\Omega)$ is equivalent to

$$\sum_{i=1}^{\infty} \sigma(A_i) < \infty, \quad \& \quad \sum_{i=1}^{\infty} \sigma(A_i) \leq \|\mathcal{M}^0 f\|^p_{L^p(\partial\Omega)} \leq \sum_{i=0}^{\infty} \sigma(A_i). \tag{3.64}$$

Now we want to decompose the function f on M as infinite an sum $\sum g_i$ with functions g_i defined as follows.

$$g_0(x) = \begin{cases} f(x), & \text{if } -1 \le f(x) \le 1, \\ 1, & \text{if } 1 < f(x), \\ -1, & \text{if } 1 < -f(x), \end{cases} \tag{3.65}$$

$$g_i(x) = \begin{cases} 0, & \text{if } |f(x)| \le i^{1/p}, \\ f(x) - i^{1/p}, & \text{if } i^{1/p} < f(x) \le (i+1)^{1/p}, \\ f(x) + i^{1/p}, & \text{if } i^{1/p} < -f(x) \le (i+1)^{1/p}, \quad i = 1, 2, \dots \\ (i+1)^{1/p} - i^{1/p}, & \text{if } (i+1)^{1/p} < f(x), \\ -(i+1)^{1/p} + i^{1/p}, & \text{if } (i+1)^{1/p} < -f(x). \end{cases}$$

If we put:

$$f_0 = g_0, \qquad f_i = \frac{g_i}{(i+1)^{1/p} - i^{1/p}}, \quad i = 1, 2, \dots, \tag{3.66}$$

then for each f_i we have $|f_i| \le 1$, and

$$f = f_0 + \sum_{i=1}^{\infty} \left[(i+1)^{1/p} - i^{1/p} \right] f_i. \tag{3.67}$$

Now we want to find connection between the support set of f_i and the set A_i for $i = 1, 2, \dots$. The claim is that

$$\text{supp } f_i \subset \text{Pim}(A_i). \tag{3.68}$$

Seeing this is quite easy. Consider one coordinate chart (3.30). If $x = (x', x_n) \in$ supp f_i then clearly $|f(x)| > i^{1/p}$. Take any point z from the intersection of $\partial\Omega$ with downward opening cone $\widetilde{\gamma(x)}$. The claim is that such point is in A_i. Really, since $x \in \gamma(z)$ we have that $\mathcal{M}^0 f(z) \ge |f(x)| > i^{1/p}$. From this the fact that $x \in \text{Pim}(A_i)$ follows immediately (see Definition 3.3).

Now we can proceed. Define $u_i = L^{-1} f_i$ for $i = 0, 1, 2, \dots$. We can use Proposition 3.4 to estimate $\mathcal{M}^{1+\beta} u_i$ for $i = 1, 2, \dots$. This yields

$$\|\mathcal{M}^{1+\beta} u_i\|_{L^{1+\varepsilon}(\partial\Omega)} \le C\sigma(A_i), \tag{3.69}$$

for some $\varepsilon > 0$, small. On the other hand (3.3) yields

$$\|\mathcal{M}^{1+\beta} u_i\|_{L^\infty(\partial\Omega)} \leq C. \tag{3.70}$$

By interpolation, for any $1+\varepsilon < p < \infty$, (3.69) and (3.70) yields

$$\|\mathcal{M}^{1+\beta} u_i\|_{L^p(\partial\Omega)} \leq C\sigma(A_i)^{1/p+\delta}, \tag{3.71}$$

for some $\delta = \delta(p, \varepsilon) > 0$. By (3.69) we see that (3.71) actually holds for any $1 < p < \infty$.

To estimate $\mathcal{M}^{1+\beta} u_0$, we use (3.3) to get that $\|\mathcal{M}^{1+\beta} u_0\|_{L^p(\partial\Omega)} \leq C$. This finally gives for $p > 1$:

$$\begin{aligned} \|\mathcal{M}^{1+\beta} u\|_{L^p} &\leq \sum \left[(i+1)^{1/p} - i^{1/p}\right] \|\mathcal{M}^{1+\beta} u_i\|_{L^p} \\ &\leq C + C \sum i^{(1-p)/p} \sigma(A_i)^{1/p+\delta}. \end{aligned} \tag{3.72}$$

Here we used the fact that $(i+1)^{1/p} - i^{1/p} \approx i^{(1-p)/p}$ and (3.71).

If $p = 1$, we have instead:

$$\begin{aligned} \|\mathcal{M}^{1+\beta} u\|_{L^1} &\leq \sum \|\mathcal{M}^{1+\beta} u_i\|_{L^1} \leq C + C \sum \sigma(A_i) \\ &\leq C + C \int_{\partial\Omega} \mathcal{M}^0 f \, d\sigma = C(1 + \|\mathcal{M}^{1+\beta} f\|_{L^1(\partial\Omega)}). \end{aligned} \tag{3.73}$$

If $p > 1$, we use Hölder's inequality to estimate the sum on the right hand side of (3.72). This yields

$$\sum i^{(1-p)/p} \sigma(A_i)^{1/p+\delta} \leq \left(\sum \frac{1}{i^{1+\varepsilon}}\right)^{1/\widetilde{q}} \left(\sum \sigma(A_i)\right)^{1/\widetilde{p}}, \tag{3.74}$$

where $\widetilde{p} = \frac{1}{1/p+\delta} < p$, $\widetilde{q} = \widetilde{p}/(\widetilde{p}-1) > p/(p-1) = q$ and $\varepsilon > 0$. It follows that the right hand side of (3.74) is bounded by $C(1 + \|\mathcal{M}^0 f\|_{L^p(\partial\Omega)})$. This 'almost' establishes (3.60), baring the term '1+' appearing here and also in (3.73). But we can get rid of it by a simple scaling argument. Since L^{-1} is linear for any $K > 0$ we have:

$$\begin{aligned} \|\mathcal{M}^{1+\beta} u\|_{L^p} &= \|\mathcal{M}^{1+\beta}(L^{-1} f)\|_{L^p} = \tfrac{1}{K} \|\mathcal{M}^{1+\beta}(L^{-1}(Kf))\|_{L^p} \\ &\leq \tfrac{C}{K}(1 + \|\mathcal{M}^0(Kf)\|_{L^p(\partial\Omega)}) = \tfrac{C}{K} + C\|\mathcal{M}^0 f\|_{L^p(\partial\Omega)}. \end{aligned} \tag{3.75}$$

Limiting $K \to \infty$ clearly yields (3.60).

Finally, (3.61) follows from (3.60), one just have to realize that for any $\widetilde{\Omega} \subset\subset \Omega$ we have that $f \in L^\infty(\widetilde{\Omega})$. Hence, for $g = f\mathcal{X}_{\widetilde{\Omega}}$ we get $L^{-1}g \in C^{1+\beta}(M)$. So we can bound the $\mathcal{D}^{1+\beta,p}$ norm of $u = L^{-1}f$ using the $C^{1+\beta}(M)$ norm of $L^{-1}g$ and (3.60). Compactness of the map $L^{-1} : \mathcal{D}^{0,p} \to \mathcal{D}^{1+\beta,p}$ follows from the fact that for any $\beta < \beta' < \alpha$ the embedding $\mathcal{D}^{1+\beta',p} \subset \mathcal{D}^{1+\beta,p}$ is compact (Theorem 2.5). This concludes the first part of our proof.

Now, we concentrate on (3.62). The main idea is very similar to what we did before. Therefore we will be brief. Consider first that $\|f\|_{L^\infty(M)} \leq 1$ and supp f $\subset \mathcal{A}$. Here $\mathcal{A} = \mathrm{Pim}(A)$ for some $A \subset \partial\Omega$ open. Then we have

$$|\nabla u(x)| \leq \int_\Omega |\nabla_x E(x,y) f(y)| \, d\mathrm{Vol}(y) \leq \int_\mathcal{A} |\nabla_x E(x,y)| \, d\mathrm{Vol}(y). \qquad (3.76)$$

By (3.17) we have that $|\nabla_x E(x,y)|^q \in L^1(M)$ for any $1 \leq q < n/(n-1)$. Hence by Hölder inequality we can further estimate (3.76). This gives:

$$\begin{aligned} |\nabla u(x)| &\leq \left(\int_M |\nabla E_x(x,y)|^q \, d\mathrm{Vol}(y) \right)^{1/q} \left(\int_\mathcal{A} 1 \, d\mathrm{Vol}(y) \right)^{1/p} \\ &\leq C(q) \mathrm{Vol}(\mathcal{A})^{1/p}. \end{aligned} \qquad (3.77)$$

Here $1/p + 1/q = 1$, hence (3.77) is true for any $n < p < \infty$. Finally, if $\mathcal{A} = \mathrm{Pim}(A)$, then $\mathrm{Vol}(\mathcal{A}) \leq C\sigma(A)^{n/(n-1)}$. This inequality follows from the procedure that has been described in details in the proof of Proposition 3.4. Here the set A was decomposed into a disjoint countable union of sets C_i (essentially $n-1$ dimensional 'cubes'), such that for each C_i there is a n-dimensional ball B_i with the property $\mathrm{diam}(C_i) \approx \mathrm{diam}(B_i)$ and $\mathcal{A} \subset \bigcup B_i$. From this

$$\begin{aligned} \mathrm{Vol}(\mathcal{A}) &\leq C \sum \mathrm{diam}(B_i)^n \approx C \sum \mathrm{diam}(C_i)^n \leq C \sum \sigma(C_i)^{n/(n-1)} \\ &\leq C\sigma \left(\bigcup C_i \right)^{n/(n-1)} = C\sigma(A)^{n/(n-1)}. \end{aligned} \qquad (3.78)$$

Combining (3.77) and (3.78) finally yields

$$\|\nabla u\|_{L^\infty(M)} \leq C(p) \, \sigma(A)^{n/(np-p)}, \qquad \text{for any } n < p < \infty. \qquad (3.79)$$

We want to further improve (3.79) by estimating $C^{\delta}(M)$ (for $\delta > 0$ small) norm of ∇u. Fix $p > n$. Then we can find $\delta = \delta(p) > 0$ such that the function $y \mapsto |y|^{-(n-1-\delta)}$ belongs to $L^q(M)$ with q as before. This and (3.17) gives:

$$\begin{aligned}\frac{|\nabla u(x) - \nabla u(x_0)|}{|x - x_0|^{\delta}} &\leq C \int_{\mathcal{A}} (|x-y|^{-(n-1-\delta)} + |x_0 - y|^{-(n-1-\delta)})\, d\mathrm{Vol}(y) \\ &\leq C\ \mathrm{Vol}(\mathcal{A})^{1/p}. \end{aligned} \tag{3.80}$$

The last estimate in (3.80) follows from the Hölder inequality, exactly as in (3.77). Hence we conclude that for any given $n < p < \infty$ there is $\delta = \delta(p)$ such that

$$\|\nabla u\|_{C^{\delta}(M)} \leq C(p)\sigma(A)^{n/(np-p)}. \tag{3.81}$$

Equivalently, it follows that for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon)$ such that:

$$\|\mathcal{M}^{1+\delta}u\|_{L^{\infty}(\partial\Omega)} \leq C(\varepsilon)\sigma(A)^{1/(n-1)-\varepsilon}. \tag{3.82}$$

On the other hand, (3.40) gives for the same function u an estimate

$$\|\mathcal{M}^{1+\delta}u\|_{L^{\frac{n-1}{n-2}-\varepsilon}(\partial\Omega)} \leq C\sigma(A). \tag{3.83}$$

Hence, interpolation between (3.82) and (3.83) gives for any $(n-1)/(n-2) \leq s \leq \infty$:

$$\|\mathcal{M}^{1+\delta}u\|_{L^{s}(\partial\Omega)} \leq C\sigma(A)^{\frac{1}{s}+\frac{1}{n-1}-\varepsilon}, \tag{3.84}$$

for any $\varepsilon > 0$ and some $\delta = \delta(s, \varepsilon) > 0$.

Consider any $1 < p < \infty$ and let $f \in \mathcal{D}^{0,p}$. We again define the sets A_i as in (3.63), and we consider the decomposition (3.65)-(3.67) of f. We can apply the estimate (3.84) to each function $u_i = L^{-1}f_i$, $i = 1, 2, \ldots$. Then (3.67) gives:

$$\begin{aligned}\|\mathcal{M}^{1+\delta}u\|_{L^{s}(\partial\Omega)} \leq\ & C\sum i^{(1-p)/p}\|\mathcal{M}^{1+\delta}u_i\|_{L^{s}(\partial\Omega)} \\ \leq\ & C\sum i^{(1-p)/p}\sigma(A_i)^{\frac{1}{s}+\frac{1}{n-1}-\varepsilon}. \end{aligned} \tag{3.85}$$

Using Hölder's inequality to estimate the last term of (3.85) yields:

$$\begin{aligned}&\|\mathcal{M}^{1+\delta}u\|_{L^{s}(\partial\Omega)} \leq \\ &C\left(\sum i^{\frac{(1-p)s(n-1)}{p((s-1)(n-1)-s)}+\varepsilon'}\right)^{\frac{s-1}{s}-\frac{1}{n-1}+\varepsilon''}\left(\sum \sigma(A_i)\right)^{\frac{1}{s}+\frac{1}{n-1}-\varepsilon},\end{aligned} \tag{3.86}$$

for some $\varepsilon' = \varepsilon'(\varepsilon) > 0$ small (i.e., $\varepsilon' \to 0+$ as $\varepsilon \to 0+$). Since we want the number on the right hand side of (3.86) to be finite we need:

$$\frac{(1-p)s(n-1)}{p((s-1)(n-1)-s)} + \varepsilon' < -1. \tag{3.87}$$

We instead solve

$$\frac{(1-p)s(n-1)}{p((s-1)(n-1)-s)} < -1, \text{ which is equivalent to } \quad s < \frac{n-1}{(n-1)/p-1}. \tag{3.88}$$

It follows from (3.88) that for any $1 < p < \infty$ and any $\frac{n-1}{n-2} < s < \frac{n-1}{(n-1)/p-1}$, (we take $s = \infty$ if $p > n-1$) we can find $\varepsilon' > 0$ such that (3.87) is true. Hence, by (3.86) for such s

$$\|\mathcal{M}^{1+\delta}u\|_{L^s(\partial\Omega)} < \infty. \tag{3.89}$$

It follows that L^{-1} maps $\mathcal{D}^{0,p}$ into $\mathcal{D}^{1+\delta,s}$ for any $s < \frac{n-1}{(n-1)/p-1}$, provided $p \le n-1$, $s = \infty$ otherwise. Here $\delta = \delta(s,p) > 0$. From this the second line of (3.62) follows, and we also have compactness of the map L^{-1}. The first line of (3.62) could be obtained from the second line and Theorem 2.5. There are two special cases we did not consider here. If $p = \infty$, (3.62) is a trivial consequence of (3.3). if $p = 1$ we use (3.40) instead of (3.86). We again get that L^{-1} maps $\mathcal{D}^{0,1}$ into $\mathcal{D}^{1+\delta,s}$ for any $s < (n-1)/(n-2)$ and some $\delta = \delta(s,p) > 0$. $\square$

Combining Theorem 3.5 with the results of [23]-[25] and [7] yields following:

Theorem 3.6. *Let M be a compact n-dimensional Riemannian manifold whose metric tensor has regularity C^α, $\alpha > 0$. Assume that $V \in L^\infty(\Omega)$ and $V \ge 0$. Let $\Omega \subset M$ be a connected open subset of M with Lipschitz boundary. Let $g \in L^p(\partial\Omega)$ for some $1 < p \le \infty$. Consider the solution u to the Dirichlet problem:*

$$Lu = (\Delta - V)u = f \quad \text{in } \Omega, \qquad u\big|_{\partial\Omega} = g, \qquad \mathcal{M}^0 u \in L^p(\partial\Omega), \tag{3.90}$$

where $f \in \mathcal{D}^{0,q}$ for some $1 \le q \le \infty$. (Denote the solution u by $u = L^{-1}_{Dg} f$).

There is $\varepsilon > 0$ such that for any $2 - \varepsilon < p \le \infty$ the solution u to the Dirichlet problem (3.90) exists, is unique and the operator L^{-1}_{Dg}:

$$L^{-1}_{Dg} : \mathcal{D}^{0,q} \to \mathcal{D}^{0,p}, \tag{3.91}$$

is continuous and compact for any $q > \frac{(n-1)p}{n-1+2p}$*, provided* $p \geq \frac{n-1}{n-2}$*,* $q \geq 1$ *otherwise. Moreover, we have the following estimate on the norm of* $L_{Dg}^{-1}f$*:*

$$\|L_{Dg}^{-1}f\|_{\mathcal{D}^{0,p}} \leq C(p,\Omega)(\|g\|_{L^p(\partial\Omega)} + \|f\|_{\mathcal{D}^{0,q}}). \tag{3.92}$$

If Ω *is a domain with* C^1 *boundary, (3.90) is solvable for any* $1 < p \leq \infty$*. Also the estimate (3.92) remains true in this case.*

Proof. Define a function F on M by extending f onto the whole M, i.e.,

$$F(x) = \begin{cases} f(x), & \text{for } x \in \Omega \\ 0, & \text{otherwise.} \end{cases} \tag{3.93}$$

Clearly $\mathcal{M}^0 F = \mathcal{M}^0 f$. Let $U = L^{-1}(F)$. By Theorem 3.5, on Ω clearly

$$L\,U = f \qquad \text{and} \qquad \|U\|_{\mathcal{D}^{\delta,p}} \leq C(p)\|f\|_{\mathcal{D}^{0,q}}, \tag{3.94}$$

for some $\delta > 0$ small. Thus, we have that $U\big|_{\partial\Omega} \in L^p(\partial\Omega)$, by the 'trace' Theorem 2.6.

Consider now the following boundary problem

$$Lw = 0 \text{ on } \Omega, \quad w\big|_{\partial\Omega} = g - U\big|_{\partial\Omega} \in L^p(\partial\Omega), \quad \mathcal{M}^0 w \in L^p(\partial\Omega). \tag{3.95}$$

(3.95) is solvable for all $1 < p \leq \infty$, if Ω has a C^1 boundary by Theorem 3.1 of [7]. If $\partial\Omega$ in Lipschitz (3.95) is solvable for $2 - \varepsilon < p \leq \infty$ (see [23], [24] and [25]). Moreover, the solution to (3.95) satisfies the following estimate on $\mathcal{M}^0 w$:

$$\begin{aligned} \|\mathcal{M}^0 w\|_{L^p(\partial\Omega)} \leq C\|g + U\big|_{\partial\Omega}\|_{L^p(\partial\Omega)} &\leq C(\|g\|_{L^p(\partial\Omega)} + \|\mathcal{M}^0 U\|_{L^p(\partial\Omega)}) \\ &\leq C(\|g\|_{L^p(\partial\Omega)} + \|f\|_{\mathcal{D}^{0,q}}). \end{aligned} \tag{3.96}$$

Now clearly $u = U + w$ solves (3.90) and (3.91) follows from (3.94) and (3.96).

The compactness of L_{Dg}^{-1} can be proved as follows: Let $(f_n)_{n\in N}$ be a bounded sequence in $\mathcal{D}^{0,q}$. Extend as in (3.93) each f_n onto M. Let $U_n = L^{-1}F_n$. By Theorem 3.5 L^{-1} is a compact operator from $\mathcal{D}^{0,q}$ to $\mathcal{D}^{\delta,p}$, hence there is a subsequence, that we will again denote by $(U_n)_{n\in N}$, convergent in $\mathcal{D}^{\delta,p}$. Let U be the limit of this subsequence. Clearly, we also have that

$U_n\big|_{\partial\Omega}$ converges to $U\big|_{\partial\Omega}$ in $L^p(\partial\Omega)$. Now we consider the following Dirichlet problems.

$$
\begin{aligned}
Lw_n = 0 \text{ on } \Omega, \quad w\big|_{\partial\Omega} = g - U_n\big|_{\partial\Omega} \in L^p(\partial\Omega), \quad \mathcal{M}^0 w_n \in L^p(\partial\Omega), \\
Lw = 0 \text{ on } \Omega, \quad w\big|_{\partial\Omega} = g - U\big|_{\partial\Omega} \in L^p(\partial\Omega), \quad \mathcal{M}^0 w \in L^p(\partial\Omega).
\end{aligned} \tag{3.97}
$$

The results from [23]-[25], [7] guarantee that the sequence $(w_n)_{n\in N}$ converges to w in $\mathcal{D}^{0,p}$. Thus $u_n = U_n + w_n$ is also convergent in this space (to $U + w$). □

We have a similar proposition for the Neumann problem.

Theorem 3.7. *Let M be a compact n-dimensional Riemannian manifold whose metric tensor has regularity C^α, $\alpha > 0$. Assume that $V \in L^\infty(\Omega)$, $V \geq 0$ and $V > 0$ on a set of positive measure in Ω. Let $\Omega \subset M$ be a connected open subset of M with Lipschitz boundary. Let $g \in L^p(\partial\Omega)$ for some $1 < p < \infty$. Consider the solution u to the Neumann problem:*

$$
Lu = (\Delta - V)u = f \quad \text{in } \Omega, \qquad \partial_\nu u\big|_{\partial\Omega} = g, \qquad \mathcal{M}^1 u \in L^p(\partial\Omega), \tag{3.98}
$$

where $f \in \mathcal{D}^{0,q}$. (Denote the solution u by $u = L^{-1}_{Ng} f$).

There is $\varepsilon > 0$ such that for any $1 < p < 2 + \varepsilon$ the solution u to the Neumann problem (3.98) exists, is unique and the operator L^{-1}_{Ng}:

$$
L^{-1}_{Ng} : \mathcal{D}^{0,q} \to \mathcal{D}^{1,p}, \tag{3.99}
$$

is continuous and compact for any $q > \frac{(n-1)p}{n-1+p}$, provided $p \geq \frac{n-1}{n-2}$, $q \geq 1$ otherwise. Moreover, we have the following estimate on the norm of $L^{-1}_{Ng} f$:

$$
\|L^{-1}_{Ng} f\|_{\mathcal{D}^{1,p}} \leq C(p,\Omega)(\|g\|_{L^p(\partial\Omega)} + \|f\|_{\mathcal{D}^{0,q}}). \tag{3.100}
$$

If Ω is a domain with C^1 boundary, (3.98) is solvable for any $1 < p < \infty$. Also the estimate (3.100) remains true in this case.

Proof. Because, the proof of this theorem is essentially same as that of Theorem 3.6, we omit it. □

Theorem 3.8. *Let M be a compact n-dimensional Riemannian manifold whose metric tensor has regularity $C^{1+\alpha}$, $\alpha > 0$. Assume that $V \in L^\infty(\Omega)$ and $V \geq 0$. Let $\Omega \subset M$ be a connected open subset of M with Lipschitz boundary. Let $g \in H^{1,p}(\partial\Omega)$ for some $1 < p < \infty$. Consider the solution u to the Dirichlet regularity problem:*

$$Lu = (\Delta - V)u = f \quad \text{in } \Omega, \qquad u\big|_{\partial\Omega} = g, \qquad \mathcal{M}^1 u \in L^p(\partial\Omega), \tag{3.101}$$

where $f \in \mathcal{D}^{0,q}$. (Denote the solution u by $u = L_{Dg}^{-1} f$).

There is $\varepsilon > 0$ such that for any $1 < p < 2 + \varepsilon$ the solution u to the Dirichlet problem (3.101) exists, is unique and the operator L_{Dg}^{-1}:

$$L_{Dg}^{-1} : \mathcal{D}^{0,q} \to \mathcal{D}^{1,p}, \tag{3.102}$$

is continuous and compact for any $q > \frac{(n-1)p}{n-1+p}$, provided $p \geq \frac{n-1}{n-2}$, $q \geq 1$ otherwise. Moreover, we have the following estimate on the norm of $L_{Dg}^{-1} f$:

$$\|L_{Dg}^{-1} f\|_{\mathcal{D}^{1,p}} \leq C(p, \Omega)(\|g\|_{H^{1,p}(\partial\Omega)} + \|f\|_{\mathcal{D}^{0,q}}). \tag{3.103}$$

If Ω is a domain with C^1 boundary, (3.101) is solvable for any $1 < p < \infty$. Also the estimate (3.103) remains true in this case.

4. Uniform estimates of solutions

In this section we investigate some additional properties of the solutions to the Dirichlet and Neumann boundary problems for the operator $L = \Delta - V$ on Ω. We start with the following lemma on the kernel $E(x, y)$ of the operator L^{-1} from (3.4). We keep same assumptions as in the previous section on M and Ω.

Lemma 4.1. *Let $K > 0$ be a given constant. Consider a sequence of functions V^n in $L^\infty(M)$ such that for any integer $n \geq 1$*

$$0 \leq V^n \leq K, \qquad V^n(x) = V^1(x) \text{ for } x \in M \setminus \Omega, \tag{4.1}$$

and $V^1 > 0$ on a set of positive measure on each connected component of $M \setminus \Omega$. Denote by L^n the corresponding operator $\Delta - V^n$, and by $E^n(x, y)$ the kernel of the inverse $(L^n)^{-1}$. Then there is a constant $C = C(K, \varepsilon)$ depending only on K any $\varepsilon > 0$ such that for any $n, m \in \mathbb{N}$ we have:

$$\begin{aligned} |E^n(x, y) - E^m(x, y)| &\leq C|x - y|^{-(n-2-\alpha+\varepsilon)} \\ |\nabla_x E^n(x, y) - \nabla_x E^m(x, y)| &\leq C|x - y|^{-(n-1-\alpha+\varepsilon)}, \end{aligned} \tag{4.2}$$

where α as before is the smoothness of the metric tensor on M.

Moreover, there exists a subsequence $(V^{n_k})_{k\in N}$ of the sequence $(V^n)_{n\in N}$, such that V^{n_k} converges weakly in $L^q(M)$, for any $1 \le q < \infty$. Let V be the weak limit of this sequence. It follows that $V \in L^\infty(M)$, $0 \le V \le K$. Denote by L the operator $\Delta - V$, and by $E(x,y)$ the kernel of L^{-1}. The subsequence $(V^{n_k})_{k\in N}$ can be selected such that for any $s < 1+\alpha$

$$E^{n_k}(x,y) \to E(x,y), \quad \textit{in } \; C^s_{\mathrm{loc}}(M \times M \setminus \textit{diag}). \tag{4.3}$$

Proof. We closely follow [25]. First we want to establish (4.2). The key is the kernel decomposition (3.5). Clearly, the term $e_0(x-y,y)$ does not depend on V^n and therefore

$$E^n(x,y) - E^m(x,y) = g(y)^{-1/2}\left(e_1^n(x,y) - e_1^m(x,y)\right). \tag{4.4}$$

Hence, it is sufficient to concentrate on the estimates for e_1^n. Here comes into play the boundedness of the sequence $(V^n)_{n\in N}$. It follows that in the decomposition (2.10) of [25], where we write the operator L^n as $L^n = L^{\#,n} + L^{b,n}$, the part $L^{\#,n} \in OPS^2_{1,\delta}$ does not dependent on n, and the norm of $L^{b,n}$ is uniformly bounded. This is crucial. It allows us to establish the statement of Proposition 2.3 of [MT2] uniformly in n. Here if we rephrase this proposition into our settings we get:

Let $\mathcal{O}$ be an open subset of M, which has a C^α metric tensor, $0 < \alpha < 1$. Assume

$$1 < p \le q < \infty, \qquad -1 \le \sigma < -1 + \alpha. \tag{4.5}$$

Then given any open $\widetilde{\mathcal{O}} \subset\subset \mathcal{O}$

$$u \in H^{\tau,p}(\widetilde{\mathcal{O}}), \tau > 1-\alpha, \quad L^n u \in H^{\sigma,q}(\widetilde{\mathcal{O}}) \Longrightarrow u \in H^{2+\sigma,q}(\widetilde{\mathcal{O}}). \tag{4.6}$$

Moreover, there is a constant $C = C(\tau,p,\sigma,q,\widetilde{\mathcal{O}})$ (but independent on n) such that

$$\|u\|_{H^{2+\sigma,q}(\widetilde{\mathcal{O}})} \le C(\|L^n u\|_{H^{\sigma,q}(\widetilde{\mathcal{O}})} + \|u\|_{H^{\tau,p}(\widetilde{\mathcal{O}})}). \tag{4.7}$$

It follows (see Corollary 2.4 of [25]) that given any compact set $K \subset M \times M \setminus \mathrm{diag}$:

$$E^n \in C^s(K), \quad \text{for all } s < 1+\alpha, \quad \text{and} \quad \|E^n\|_{C^s(K)} \le C(K,s). \tag{4.8}$$

We also get as uniform estimate on the part $R_3^n(x,y)$ of the decomposition (2.35) of [MT3] in the form of

$$|R_3^n(x,y)| \le C|x-y|^{-(n-2)}. \tag{4.9}$$

This finally yields an equivalent of Theorem 2.6 of [25] which gives us:

$$\begin{aligned} |e_1^n(x,y)| &\le C_\varepsilon |x-y|^{-(n-2-\alpha+\varepsilon)}, \\ |\nabla_x e_1^n(x,y)| &\le C_\varepsilon |x-y|^{-(n-1-\alpha+\varepsilon)}, \end{aligned} \tag{4.10}$$

with C_ε independent of n. From this (4.2) follows.

The second part of the lemma follows from (4.8), since for any $s < s' < 1+\alpha$ we have a compact embedding $C^{s'}(K) \subset C^s(K)$. $\square$

Using this lemma, we can improve Theorem 5.8 of [23] and Theorem 3.1 of [7]. The essence of our improvement is in establishing how the constant in the estimate (4.12) depends on V. We retain all hypothesis on M, Ω. Let us assume explicitly, that the metric tensor is of class C^α, $0 < \alpha < 1$.

Theorem 4.2. *Let Ω be a Lipschitz domain, $V \in L^\infty(\Omega)$ and $V \ge 0$. Then there is $\varepsilon > 0$ such that given $2-\varepsilon < p \le \infty$ and $f \in L^p(\partial\Omega)$ there exists a unique function $u \in C^{1+\alpha}_{\mathrm{loc}}(\Omega)$ satisfying*

$$Lu = (\Delta - V)u = 0 \text{ in } \Omega, \qquad \mathcal{M}^0 u \in L^p(\partial\Omega), \qquad u\big|_{\partial\Omega} = f \in L^p(\partial\Omega), \tag{4.11}$$

the limit on $\partial\Omega$ taken in the nontangential a.e. sense. Moreover, there is a uniform estimate

$$\|\mathcal{M}^0 u\|_{L^p(\partial\Omega)} \le C_p(\|V\|_{L^\infty})\|f\|_{L^p(\partial\Omega)}. \tag{4.12}$$

The constant C_p depends only on the $L^\infty(\Omega)$ norm of V, not on V itself. If Ω is a C^1 domain, the claim holds for any $1 < p \le \infty$.

Proof. If $p = \infty$ the estimate (4.12) follows from the weak maximum principle (Proposition 5.7 of [23]). In fact, in such case the constant $C_\infty = 1$ is independent of V.

If $p < \infty$ the solution to (4.11) is representable in the form

$$u = \mathcal{D}((\tfrac{1}{2}I + K)^{-1} f),$$

where $\mathcal{D}$ is a double layer potential:

$$\mathcal{D}f(x) = \int_{\partial\Omega} \frac{\partial E}{\partial \nu_y}(x,y) f(y)\ d\sigma(y), \qquad x \notin \partial\Omega, \tag{4.13}$$

and K is an operator on $L^p(\partial\Omega)$:

$$Kf(x) = \text{P.V.} \int_{\partial\Omega} \frac{\partial E}{\partial \nu_y}(x,y) f(y)\ d\sigma(y). \tag{4.14}$$

It follows from (4.2) that given any L^∞ uniformly bounded sequence $(V^n)_{n\in N}$ the corresponding double layer potentials $\mathcal{D}^n$ and operators K^n have norms uniformly bounded. The only remaining problem is the uniform invertibility of $\frac{1}{2}I+K^n$. Assume therefore contrary, i.e., suppose that there is a uniformly bounded sequence $(V^n)_{n\in N}$ for which

$$\lim_{n\to\infty} \inf_{\|f\|_{L^p}=1} \frac{\|(\frac{1}{2}I + K^n)f\|_{L^p}}{\|f\|_{L^p}} = 0. \tag{4.15}$$

It follows from Lemma 4.1, that there is a subsequence of $(V^n)_{n\in N}$ (for simplicity again denoted by $(V^n)_{n\in N}$), such that we have (4.2) and (4.3) for some function $V \in L^\infty(\Omega)$, $V \geq 0$. If K is the operator corresponding to this function V, we get from (4.2) and (4.3) that

$$\|K^n - K\|_{\mathcal{L}(L^p)} \to 0, \qquad \text{as } n \to \infty. \tag{4.16}$$

See (2.12)-(2.13) of [7] for estimates of similar nature. If we combine (4.15) with (4.16) we get that

$$\inf_{\|f\|_{L^p}=1} \frac{\|(\frac{1}{2}I + K)f\|_{L^p}}{\|f\|_{L^p}} = 0, \tag{4.17}$$

which would mean that $\frac{1}{2}I+K$ is not invertible. Apparently, this contradicts Theorem 5.8 of [23] and Theorem 3.1 of [7]. □

A similar theorem is true also for the Neumann problem. The main ingredients of the proof in this case are the single layer potential (c.f. [23] or [7]) and the operator K^* on $L^p(\partial\Omega)$ which is a formal dual of K defined by (4.14). Crucially, the same arguments (4.15)-(4.17) we used for K work also for K^*.

Theorem 4.3. *Let Ω be a Lipschitz domain, $q \in L^\infty(\Omega)$ a given function $q \geq 0$ and $q > 0$ on a set of positive measure in Ω. Assume that $V \in L^\infty(\Omega)$ and $V \geq q$. Then there is $\varepsilon > 0$ such that given $1 < p < 2+\varepsilon$ and $g \in L^p(\partial\Omega)$ there exists a unique function $u \in C^{1+\alpha}_{\text{loc}}(\Omega)$ satisfying*

$$Lu = (\Delta - V)u = 0 \text{ in } \Omega, \qquad \mathcal{M}^1 u \in L^p(\partial\Omega), \qquad \partial_\nu u\big|_{\partial\Omega} = g \in L^p(\partial\Omega), \tag{4.18}$$

the limit on $\partial\Omega$ taken in the nontangential a.e. sense. Moreover, there is a uniform estimate

$$\|\mathcal{M}^1 u\|_{L^p(\partial\Omega)} \leq C_p(q, \|V\|_{L^\infty})\|g\|_{L^p(\partial\Omega)}. \tag{4.19}$$

The constant C_p depends only on q and the $L^\infty(\Omega)$ norm of V, not on V itself. If Ω is a C^1 domain, the claim holds for any $1 < p < \infty$.

Finally, for the Dirichlet regularity problem we get in the same spirit:

Theorem 4.4. *Let the metric tensor on M be of class $C^{1+\alpha}$. Let Ω be a Lipschitz domain, $V \in L^\infty(\Omega)$ and $V \geq 0$. Then there is $\varepsilon > 0$ such that given $1 < p < 2 + \varepsilon$ and $f \in H^{1,p}(\partial\Omega)$ there exists a unique function $u \in C^{1+\alpha}_{\text{loc}}(\Omega)$ satisfying*

$$Lu = (\Delta - V)u = 0 \text{ in } \Omega, \qquad \mathcal{M}^1 u \in L^p(\partial\Omega), \qquad u\big|_{\partial\Omega} = f \in H^{1,p}(\partial\Omega), \tag{4.20}$$

the limit on $\partial\Omega$ taken in the nontangential a.e. sense. Moreover, there is a uniform estimate

$$\|\mathcal{M}^1 u\|_{L^p(\partial\Omega)} \leq C_p(\|V\|_{L^\infty})\|g\|_{L^p(\partial\Omega)}. \tag{4.21}$$

The constant C_p depends only the $L^\infty(\Omega)$ norm of V, not on V itself. If Ω is a C^1 domain, the claim holds for any $1 < p < \infty$.

Since the proofs are essentially identical to the proof of Theorem 4.2, we skip them.

5. The semilinear elliptic boundary problems

In the previous sections we have developed enough tools to take on the semilinear problem outlined in the introduction. We keep same assumptions as in the previous sections. Namely, M be a smooth, compact Riemannian manifold dim $M \geq 3$ with metric tensor of the Hölder class C^α, $0 < \alpha \leq 1$. The set $\Omega \subset M$ is open, connected with Lipschitz boundary.

Given a function $g \in L^p(\partial\Omega)$ we are interested in finding solution u to the equation:

$$\Delta u - a(x,u)u = 0, \quad \text{in } \Omega, \qquad u\big|_{\partial\Omega} = g, \qquad u \in \mathcal{D}^{0,p}, \tag{5.1}$$

or more generally to the equation

$$\Delta u - a(x,u)u = f, \quad \text{in } \Omega, \qquad u\big|_{\partial\Omega} = g, \qquad u \in \mathcal{D}^{0,p}, \tag{5.2}$$

Here, the function $a(x,u)$ is Caratheodory, i.e., measurable in x and continuous in u. Moreover, we assume that

$$a(x,u) \in L^\infty(\Omega \times \mathbb{R}), \qquad a(x,u) \geq 0. \tag{5.3}$$

We explain later the conditions we put on the function f. It would be desirable to relax somewhat the condition (5.3). For example the paper [14] by Isakov and Nachman treats two dimensional problems of the form (5.1) with more general function a. There is a trade-off, however. The existence theorem the authors discuss requires the boundary data g not only to be bounded, but also have some regularity, namely, $H^{1/2}(\partial\Omega)$. Our treatment will give us results for the semilinear equation comparable with those for the linear equation $Lu = (\Delta - V)u = 0$ on Lipschitz domains as they were presented for the flat $\mathbb{R}^n$ case in [3], [8], [9] or [12] and for Riemannian manifolds in [23]-[25], [22] or [7]. We also explore the Neumann and Dirichlet regularity problems.

At the end we show, that given g bounded we can somewhat relax the condition (5.3) for the Dirichlet problem. We begin with the following auxiliary lemma.

Lemma 5.1. *Let $a(x,u)$ be a Caratheodory function on $\Omega \times \mathbb{R}$, i.e, a is measurable in x and continuous in u. Let $1 \leq p \leq \infty$, and $v : \Omega \to \mathbb{R}$ be a given function in $\mathcal{D}^{0,p} \cap C_{\text{loc}}(\Omega)$. Then the function*

$$x \mapsto a(x, v(x)) \tag{5.4}$$

is measurable.

Proof. Given any open domain $\mathcal{O} \subset\subset \Omega$ the function

$$h(x) = \begin{cases} a(x, v(x)), & \text{for } x \in \mathcal{O}, \\ 0, & \text{otherwise}, \end{cases} \tag{5.5}$$

is measurable, since on $\mathcal{O}$ the function v is continuous. Now taking an increasing sequence of domains $\Omega_1 \subset \Omega_2 \subset \cdots \subset\subset \Omega$ union of which is Ω does the job, because the sequence of measurable functions h_n defined by (5.5) converges to $x \mapsto a(x, v(x))$, $x \in \Omega$. $\square$

We can formulate our first result.

Theorem 5.2. *Let $\Omega \subset M$ be a connected Lipschitz domain in a Riemannian manifold M whose metric tensor is in C^α, $0 < \alpha \leq 1$. Let a be a Caratheodory function satisfying (5.3). There exists $\varepsilon > 0$ such that for any $2 - \varepsilon < p \leq \infty$ we have the following: Given any $g \in L^p(\partial\Omega)$ there exists a function $u \in C^{1+\alpha}_{\mathrm{loc}}(\Omega)$ satisfying*

$$\Delta u - a(x, u)u = 0 \text{ in } \Omega, \qquad u\big|_{\partial\Omega} = g, \qquad u \in \mathcal{D}^{0,p}, \tag{5.6}$$

the limit on $\partial\Omega$ taken in the nontangential a.e. sense. Moreover, we have a bound

$$\|u\|_{\mathcal{D}^{0,p}} \leq C(a, p)\|g\|_{L^p(\partial\Omega)}. \tag{5.7}$$

If the boundary of Ω is C^1, the claim is true for any $1 < p \leq \infty$.

Proof. First, we establish existence of a solution to (5.6). We consider the case $p < \infty$. Fix $g \in L^p(\partial\Omega)$. We define an operator

$$T : \mathcal{D}^{0,p} \cap C_{\mathrm{loc}}(\Omega) \to \mathcal{D}^{0,p} \cap C_{\mathrm{loc}}(\Omega), \tag{5.8}$$

as follows. Given $u \in \mathcal{D}^{0,p} \cap C_{\mathrm{loc}}$, let $v = Tu$ be a solution to the linear problem

$$\Delta v - a(x, u)v = 0 \text{ in } \Omega, \qquad v\big|_{\partial\Omega} = g, \qquad v \in \mathcal{D}^{0,p}. \tag{5.9}$$

This problem is solvable for any given $u \in \mathcal{D}^{0,p} \cap C_{\mathrm{loc}}(\Omega)$, since the function $V(x) = a(x, u(x))$ is bounded by (5.3), measurable by Lemma 5.1. Hence Theorem 4.2 applies. This theorem also gives that the norm of $v = Tu$ in $\mathcal{D}^{0,p}$ is uniformly bounded (regardless of the initial u). This follows from (4.12) and interior regularity results which also guarantee that $v \in C^{1+\beta}_{\mathrm{loc}}(\Omega)$, for any $\beta < \alpha$. Hence $v \in \mathcal{D}^{0,p} \cap C_{\mathrm{loc}}(\Omega)$ and

$$\|v\|_{\mathcal{D}^{0,p}} \leq C(p, \|a\|_{L^\infty})\|g\|_{L^p(\partial\Omega)}. \tag{5.10}$$

Notice also that $\mathcal{D}^{0,p} \cap C_{\text{loc}}(\Omega)$ is a closed subspace of the Banach space $\mathcal{D}^{0,p}$. If we prove that the map T is continuous and compact, then in the light of (5.10), we may conclude from the Schauder fix point theorem that T has a fixed point $Tv = v$, for which we also have (5.10). This would establish the existence of the solution to (5.6).

Concentrate first on the continuity of T. Assume that we have a convergent sequence $(u_n)_{n\in N}$ of elements of $\mathcal{D}^{0,p} \cap C_{\text{loc}}(\Omega)$ converging to u. It follows that

$$u_n \to u, \qquad \text{in } C_{\text{loc}}(\Omega). \tag{5.11}$$

(See (2.10) for proof of this). Denote by v_n, v the solutions of the equation (5.9) corresponding to u_n, u, respectively. It would suffice to show that a certain subsequence $(v_{n_k})_{k\in N}$ of $(v_n)_{n\in N}$ converges to v.

We put $V_n(x) = a(x, u_n(x))$ on Ω, and extend the functions V_n to the whole M, so that we can use Lemma 4.1. It follows that there is a subsequence of $(V_n)_{n\in N}$ that is weakly convergent in $L^q(M)$, for any $q < \infty$, to some function $V \in L^\infty(M)$. Since we have

$$V_n(x) \to a(x, u(x)), \qquad \text{for any } x \in \Omega, \tag{5.12}$$

clearly $V(x) = a(x, u(x))$ on Ω. (For simplicity of notation we denote this subsequence of $(V_n)_{n\in N}$ again by $(V_n)_{n\in N}$). Also we have (4.2) and (4.3). These two estimates are crucial. From them exactly as in Theorem 4.2 we deduce that

$$\begin{aligned} \lim_{n\to\infty} \sup_{\|f\|_{L^p}=1} \frac{\|\mathcal{M}^0(\mathcal{D}f - \mathcal{D}_n f)\|_{L^p}}{\|f\|_{L^p}} &= 0, \\ \lim_{n\to\infty} \|K - K_n\|_{\mathcal{L}(L^p)} &= 0, \end{aligned} \tag{5.13}$$

where $\mathcal{D}_n$, $\mathcal{D}$, K_n, K are the corresponding double layer potentials (see (4.13)) or operators defined by (4.14). From the second line of (5.13) and the fact that $\frac{1}{2}I + K_n$, $\frac{1}{2}I + K$ are invertible we also get

$$\lim_{n\to\infty} \|(\tfrac{1}{2}I + K_n)^{-1} - (\tfrac{1}{2}I + K)^{-1}\|_{\mathcal{L}(L^p)} = 0. \tag{5.14}$$

From (5.13) and (5.14) together with the fact that the solutions v_n are representable in the form $v_n = \mathcal{D}_n((\frac{1}{2}I + K)^{-1} f_n)$ we finally get that

$$\lim_{n\to\infty} \|\mathcal{M}^0(v_n - v)\|_{L^p} = 0. \tag{5.15}$$

This and interior estimates give us that the map T is indeed continuous.

Now we turn to compactness. It follows from Theorem 3.6 that for $L = \Delta$ we have that the operator Δ_{Dg}^{-1} is well defined and compact on $\mathcal{D}^{0,p}$. Assume that $(u_n)_{n\in N}$ is a bounded sequence in the norm of $\mathcal{D}^{0,p} \cap C_{\mathrm{loc}}(\Omega)$. From (5.10) we get that $v_n = Tu_n$ are also uniformly bounded in the norm. If we put

$$h_n = a(x, u_n(x))v_n(x), \tag{5.16}$$

we see that each $h_n \in \mathcal{D}^{0,p}$ and there is some $C > 0$, such that $\|h_n\|_{\mathcal{D}^{0,p}} \leq C$ for all n. Finally, notice that

$$v_n = Tu_n = \Delta_{Dg}^{-1} h_n. \tag{5.17}$$

The compactness of Δ_{Dg}^{-1} gives us that a subsequence of v_n is convergent in $\mathcal{D}^{0,p}$. So T is compact and the theorem is established.

If $p = \infty$ we proceed differently. Take any $(n-1)/2 < q < \infty$ and solve the problem (5.6) with $u \in \mathcal{D}^{0,q}$. It follows that such u can be written as $u = \Delta_{Dg}^{-1} h$, where $h = a(x, u(x))u(x) \in \mathcal{D}^{0,q}$. Now, theorem 3.6 gives us that such u actually belongs to $\mathcal{D}^{0,\infty}$, i.e., u is bounded. Also the estimate (5.7) follows. □

Notice however, that in Theorem 5.2 we do not claim uniqueness of the solution u to the Dirichlet problem (5.6). This is because we do not have uniqueness for the considered problem. The next example explains the situation.

Example 5.3. Pick any connected Lipschitz domain $\Omega \subset M$. Consider the following functions u_1, u_2 solving

$$\begin{aligned} \Delta u_1 &= 0, \quad \text{in } \Omega, \qquad u_1\big|_{\partial\Omega} = 1, \\ (\Delta - c)u_2 &= 0, \quad \text{in } \Omega, \qquad u_2\big|_{\partial\Omega} = 1. \end{aligned} \tag{5.18}$$

Here the constant $c > 0$ is picked small enough such that we have $u_2 > 0$ on Ω. Such c can always be found, since for $c = 0$ we have that the corresponding function is equal to 1 everywhere. From (5.18) we get: $u_1 = 1$, $u_2 > 0$ and therefore $\Delta u_2 > 0$. This means the function u_2 is subharmonic. Also $u_1 = u_2$ on $\partial\Omega$, hence $u_2 \leq u_1 = 1$ on Ω. A variant of the maximum principle gives us that u_2 attains its maximum on $\partial\Omega$ and therefore

$$0 < u_2(x) < u_1(x) = 1, \qquad \text{for any } x \in \Omega. \tag{5.19}$$

We define for $x \in \Omega$:

$$a(x,u) = \begin{cases} c\frac{1-u}{1-u_2(x)}, & \text{if } u_2(x) < u \le 1, \\ 0, & \text{if } u > 1, \\ c, & \text{if } u < u_2(x). \end{cases} \tag{5.20}$$

Clearly the function a is Caratheodory and satisfies (5.3). However, both functions u_1, u_2 solve the problem:

$$\Delta u - a(x,u)u = 0 \text{ in } \Omega, \qquad u\big|_{\partial\Omega} = 1. \tag{5.21}$$

It follows, that uniqueness requires a stronger condition on the function a. The next theorem provides us with the answer.

Theorem 5.4. *Let all assumptions from Theorem 5.2 hold. In addition, assume that for the function $b(x,u) = a(x,u)u$ defined on $\Omega \times \mathbb{R}$ we have:*

$$\frac{\partial}{\partial u} b(x,u) \in L^\infty(\Omega \times \mathbb{R}) \quad \text{and} \quad \frac{\partial}{\partial u} b(x,u) \ge 0. \tag{5.22}$$

Then the solution u to the Dirichlet problem (5.6) is unique.

Proof. Let $u_1, u_2 \in \mathcal{D}^{0,p} \cap C_{\text{loc}}(\Omega)$ be two different solutions of (5.6) with $u_1\big|_{\partial\Omega} = u_2\big|_{\partial\Omega}$. Then, writing

$$a(x,u_1(x))u_1(x) - a(x,u_2(x))u_2(x) = V_{12}(x)(u_1(x) - u_2(x)) \tag{5.23}$$

with

$$V_{12}(x) = \int_0^1 \frac{\partial}{\partial u} b(x, u_2(x) + t(u_1(x) - u_2(x))\ dt, \tag{5.24}$$

we have $V_{12}(x) \ge 0$, $V_{12} \in L^\infty(\Omega)$. The function $w = u_1 - u_2$ satisfies

$$(\Delta - V_{12})w = 0 \quad \text{in } \Omega, \qquad w\big|_{\partial\Omega} = 0, \qquad w \in \mathcal{D}^{0,p}. \tag{5.25}$$

The uniqueness results from [23]-[25], [7] for the linear equation (5.25) guarantee that $w = 0$ on Ω. $\square$

Now we consider equation (5.2) with a term of the right hand side.

Theorem 5.5. *Let $\Omega \subset M$ be a connected Lipschitz domain in a Riemannian manifold M whose metric tensor is in C^α, $0 < \alpha \leq 1$. Assume that a is a Caratheodory function satisfying (5.3). There exists $\varepsilon > 0$ such that for any $2 - \varepsilon < p \leq \infty$ we have the following: Assume X is one of the following spaces:*

(a) $L^r(\Omega)$ for some $r > n/2$,

(b) $H^{-\beta,r}(\Omega)$ for some $1 + \alpha > \beta$, $r > 1$ and $r(2 - \beta) > n$,

(c) $\mathcal{D}^{0,r}$, where $\begin{cases} r = 1, & \text{for } p < (n-1)/(n-3), \\ r > \dfrac{(n-1)p}{n-1+2p}, & \text{otherwise.} \end{cases}$

Then given any $g \in L^p(\partial\Omega)$ and $f \in X$ there exists a function $u \in C^\delta_{\text{loc}}(\Omega)$, for some $\delta > 0$, satisfying

$$\Delta u - a(x,u)u = f \text{ in } \Omega, \qquad u\big|_{\partial\Omega} = g, \qquad u \in \mathcal{D}^{0,p}, \tag{5.26}$$

the limit on $\partial\Omega$ taken in the nontangential a.e. sense. Moreover, we have a bound

$$\|u\|_{\mathcal{D}^{0,p}} \leq C(a,p)(\|g\|_{L^p(\partial\Omega)} + \|f\|_X). \tag{5.27}$$

If the boundary of Ω is C^1, the claim is true for any $1 < p \leq \infty$. If in addition the function $b(x,u) = a(x,u)u$ satisfies (5.22) then the solution u is unique.

Proof. Let $p < \infty$. Fix $f \in X$ and consider the Dirichlet problem

$$\Delta v - a(x,u)v = f \text{ in } \Omega, \qquad v\big|_{\partial\Omega} = 0, \qquad v \in \mathcal{D}^{0,p}, \tag{5.28}$$

for $u \in \mathcal{D}^{0,p} \cap C_{\text{loc}}(\Omega)$. We put $T_1 u = v$. The claim is that T_1 is a compact continuous operator and all v are uniformly bounded in $\mathcal{D}^{0,p}$. This, together with the way we defined T in the proof of Theorem 5.3 gives us that there is $u \in \mathcal{D}^{0,p} \cap C_{\text{loc}}(\Omega)$ which is a fixed point of $T + T_1$, i.e., $(T + T_1)u = u$. Such u solves (5.26). The uniqueness follows exactly as in the proof of Theorem 5.4.

First consider the cases (a) and (b). In fact (a) is a special case of (b), but because of its importance we stated it separately. Use of embedding theorem gives us that $L^{-1}f \in C^\delta(M)$ for some $\delta > 0$. Moreover, we have a uniform bound on the $C^\delta(\Omega) \subset \mathcal{D}^{0,p}$ norm of $U = L^{-1}f$ independent on u:

$$\|U\|_{C^\delta(\Omega)} \leq C(\|a\|_{L^\infty})\|f\|_{L^p(\Omega)}. \tag{5.29}$$

Now we consider:

$$Lw = 0, \quad \text{in } \Omega, \qquad w\big|_{\partial\Omega} = -U\big|_{\partial\Omega}, \qquad w \in \mathcal{D}^{0,p}. \tag{5.30}$$

For the range of p we consider we get that:

$$\|\mathcal{M}^0 w\|_{L^p(\partial\Omega)} \leq C(\|a\|_{L^\infty})\|U\|_{C^\delta(M)}. \tag{5.31}$$

Since $T_1 u = U + w$, (5.29) and (5.31) yield:

$$\|T_1 u\|_{\mathcal{D}^{0,p}} \leq C(\|a\|_{L^\infty})\|f\|_X. \tag{5.32}$$

The proof of continuity and compactness of T_1 uses the fact that $C^\delta(\Omega)$ is compactly embedded into $\mathcal{D}^{0,p}$. Details are left to the reader.

Finally, suppose that $X = \mathcal{D}^{0,r}$ with r as in the statement of this theorem. Fix again $f \in \mathcal{D}^{0,r}$ and let $L = \Delta - a(x, u(x))$ for some $u \in \mathcal{D}^{0,p} \cap C_{\text{loc}}(\Omega)$. The second line of (3.62) gives us that

$$L^{-1} : \mathcal{D}^{0,p} \to \mathcal{D}^{0,q} \tag{5.33}$$

compactly, for any $q < \frac{n-1}{(n-1)/p-2}$, provided $p \leq (n-1)/2$, $q = \infty$ otherwise. Actually, if we look carefully at the proof of Theorem 3.5 we see that for such p and q we actually have that

$$L^{-1} : \mathcal{D}^{0,p} \to \mathcal{D}^{\delta,q}, \tag{5.34}$$

for some $\delta = \delta(p,q) > 0$. This means that given $f \in \mathcal{D}^{0,r}$ we get that $U = L^{-1}f$ belongs to $\mathcal{D}^{\varepsilon,p}$ for some $\varepsilon > 0$, r and p are as in the statement of our theorem. From here we proceed as in the previous case, i.e., we solve (5.30). Once again we establish (5.32). Then the continuity and compactness of T_1 follows from the fact that the embedding $\mathcal{D}^{\varepsilon,p}$ into $\mathcal{D}^{0,p}$ is compact.

In $p = \infty$ essentially same argument as used in Theorem 5.2 works. Once again we solve (5.26) with $u \in \mathcal{D}^{0,q}$ for some finite $q > (n-1)/2$. Such u can be written as $u = \Delta_{Dg}^{-1}h + \Delta_{D0}^{-1}f$, where h is as before the function $a(x, u(x))u(x) \in \mathcal{D}^{0,q}$ and Δ_{D0}^{-1} is a solution operator to the Dirichlet boundary problem $\Delta v = f$ in Ω, $v\big|_{\partial\Omega} = 0$. For spaces X from the statement of our theorem, v is bounded. On the other hand, $\Delta_{Dg}^{-1}h \in \mathcal{D}^{0,\infty} = L^\infty(\Omega)$ by Theorem 3.6. So the solution u we found is bounded. $\square$

A similar result will be true also for the Neumann boundary problem.

Theorem 5.6. *Let $\Omega \subset M$ be a connected Lipschitz domain in a Riemannian manifold M whose metric tensor is in C^α, $0 < \alpha \le 1$. Assume that a is a Caratheodory function satisfying (5.3) and $\inf_{u\in\mathbb{R}} a(.,u) \ge q(.)$ on Ω, for some nonnegative function q on Ω that is positive on a subset of Ω of positive measure.*

There exists $\varepsilon > 0$ such that for any $1 < p < 2+\varepsilon$ we have the following: Assume X is one of the following spaces:

(a) $L^r(\Omega)$ for some $r > n$,

(b) $H^{-\beta,r}(\Omega)$ for some $1 > \beta$, $r > 1$ and $r(1-\beta) > n$,

(c) $\mathcal{D}^{0,r}$, where $\begin{cases} r = 1, & \text{for } p < (n-1)/(n-2), \\ r > \dfrac{(n-1)p}{n-1+p}, & \text{otherwise.} \end{cases}$

Then given any $g \in L^p(\partial\Omega)$ and $f \in X$ there exists a function $u \in C^{1+\delta}_{\text{loc}}(\Omega)$, for some $\delta > 0$, satisfying

$$\Delta u - a(x,u)u = f \text{ in } \Omega, \qquad \partial_\nu u\big|_{\partial\Omega} = g, \qquad u \in \mathcal{D}^{1,p}, \tag{5.35}$$

the limit on $\partial\Omega$ taken in the nontangential a.e. sense. Moreover, we have a bound

$$\|u\|_{\mathcal{D}^{1,p}} \le C(a,q,p)(\|g\|_{L^p(\partial\Omega)} + \|f\|_X). \tag{5.36}$$

If the boundary of Ω is C^1, the claim is true for any $1 < p < \infty$. If in addition the function $b(x,u) = a(x,u)u$ satisfies

$$\frac{\partial}{\partial u} b(x,u) \in L^\infty(\Omega \times \mathbb{R}) \quad \text{and} \quad \frac{\partial}{\partial u} b(x,u) \ge q(x) \text{ for all } u \in \mathbb{R}, \tag{5.37}$$

then the solution u is unique.

Proof. The main idea of this proof is essentially same as above. Fix $g \in L^p(\partial\Omega)$ and $f \in X$. We define an operator

$$T : \mathcal{D}^{1,p} \to \mathcal{D}^{1,p}, \tag{5.38}$$

as follows. Given $u \in \mathcal{D}^{1,p}$, let $v = Tu$ be a solution to the linear problem

$$\Delta v - a(x,u)v = f \text{ in } \Omega, \qquad \partial_\nu v\big|_{\partial\Omega} = g, \qquad v \in \mathcal{D}^{1,p}. \tag{5.39}$$

This problem is solvable for any given $u \in \mathcal{D}^{1,p}$. Indeed, first we should realize that the function $V(x) = a(x,u(x))$ is bounded by (5.3) and greater than or equal to q. We write the operator T as $T_1 + T_2$ where $v_1 = T_1 u$ solves

$$\Delta v_1 - a(x,u)v_1 = 0 \text{ in } \Omega, \qquad \partial_\nu v_1\big|_{\partial\Omega} = g, \qquad v_1 \in \mathcal{D}^{1,p}, \tag{5.40}$$

and $v_2 = T_2 u$ solves

$$\Delta v_2 - a(x,u)v_2 = f \text{ in } \Omega, \qquad \partial_\nu v_2\big|_{\partial\Omega} = 0, \qquad v_2 \in \mathcal{D}^{1,p}. \tag{5.41}$$

For T_1 we apply Theorem 4.3 to get

$$\|T_1 u\|_{\mathcal{D}^{1,p}} \leq C(\|a\|_{L^\infty}, q, p)\|g\|_{L^p(\partial\Omega)}. \tag{5.42}$$

Similarly for T_2 analysis close to the one done in the previous proof shows:

$$\|T_2 u\|_{\mathcal{D}^{1,p}} \leq C(\|a\|_{L^\infty}, q, p)\|f\|_X. \tag{5.43}$$

This two estimates together with continuity and compactness of $T_1 + T_2$ again guarantee the existence of the solution to (5.35) and the estimate (5.36). (We again use Schauder fix point theorem). The proof of compactness and continuity goes basically, as in Theorem 5.3 for T_1 and Theorem 5.5 for T_2. We use that the solution to (5.40) can be written as a single layer potential

$$v_1 = \mathcal{S}((-\tfrac{1}{2}I + K^*)^{-1}g), \tag{5.44}$$

where

$$\mathcal{S}f(x) = \int_{\partial\Omega} E(x,y)f(y)\; d\sigma(y), \quad \text{for } x \in \Omega, \tag{5.45}$$

is a single layer potential (a mapping $L^p(\partial\Omega)$ into $\mathcal{D}^{1,p}$) and

$$\mathcal{K}^* f(x) = \text{P.V.} \int_{\partial\Omega} \frac{\partial E}{\partial \nu_x}(x,y)f(y)\; d\sigma(y) x \in \partial\Omega, \tag{5.46}$$

is an operator on $L^p(\partial\Omega)$ (a formal adjoint of (4.14)). The invertibility of $-\frac{1}{2}I + K^*$ for the considered range of p follows from [23]-[25] (Lipschitz domains) and [7] (C^1 domains). Similarly as before, for any bounded sequence $(u_n)_{n\in N}$ in $\mathcal{D}^{1,p}$ there is a subsequence for which $V_n = a(x, u_n(x))$ is weakly convergent to some V (Lemma 4.1). For this subsequence we get estimates corresponding to (5.13), namely

$$\lim_{n\to\infty} \sup_{\|f\|_{L^p}=1} \frac{\|\mathcal{M}^1(\mathcal{S}f - \mathcal{S}_n f)\|_{L^p}}{\|f\|_{L^p}} = 0,$$
$$\lim_{n\to\infty} \|K^* - K_n^*\|_{\mathcal{L}(L^p)} = 0. \tag{5.47}$$

Here $\mathcal{S}_n$, $\mathcal{S}$, K_n^*, K^* are the corresponding single layer potentials and operators (5.46) to V_n, V, respectively. We do not push this analysis further, we just remark that the compactness of T_1 is a consequence of compactness of Δ_{Ng}^{-1} from Theorem 3.7. Similarly the compactness of the second piece T_2 follows from Theorems 2.4 and 3.5 in case $X = \mathcal{D}^{0,r}$ and from the compactness of embedding of $C^{1+\varepsilon}$ into C^1 in the cases (a) and (b).

Finally, the proof of uniqueness is essentially same as in Theorem 5.4. The only change is that we have now for V_{12} defined by (5.24) a lower bound: $V \geq q$. Hence uniqueness results for the linear Neumann problem are applicable. □

In the same style we can establish the following regularity result for the Dirichlet problem.

Theorem 5.7. *Let $\Omega \subset M$ be a connected Lipschitz domain in a Riemannian manifold M whose metric tensor is in $C^{1+\alpha}$, $0 < \alpha < 1$. Assume that a is a Caratheodory function satisfying (5.3). There exists $\varepsilon > 0$ such that for any $1 < p < 2 + \varepsilon$ we have the following: Assume X is one of the following spaces:*

(a) $L^r(\Omega)$ for some $r > n$,

(b) $H^{-\beta,r}(\Omega)$ for some $1 > \beta$, $r > 1$ and $r(1-\beta) > n$,

(c) $\mathcal{D}^{0,r}$, where $\begin{cases} r = 1, & \text{for } p < (n-1)/(n-2), \\ r > \dfrac{(n-1)p}{n-1+p}, & \text{otherwise.} \end{cases}$

Then given any $g \in H^{1,p}(\partial\Omega)$ and $f \in X$ there exists a function $u \in C^{1+\delta}_{\mathrm{loc}}(\Omega)$, for some $\delta > 0$, satisfying

$$\Delta u - a(x,u)u = f \text{ in } \Omega, \qquad u\big|_{\partial\Omega} = g, \qquad u \in \mathcal{D}^{1,p}, \tag{5.48}$$

the limit on $\partial\Omega$ taken in the nontangential a.e. sense. Moreover, we have a bound

$$\|u\|_{\mathcal{D}^{1,p}} \leq C(a,p)(\|g\|_{H^{1,p}(\partial\Omega)} + \|f\|_X). \tag{5.49}$$

If the boundary of Ω is C^1, the claim is true for any $1 < p < \infty$. If in addition the function $b(x,u) = a(x,u)u$ satisfies (5.22) then the solution u is unique.

Proof. Again the idea is to use Schauder fix point theorem. The key fact is that for all $1 < p < 2 + \varepsilon$ in the Lipschitz case and all $1 < p < \infty$ in the C^1

case, the solution v to the problem

$$(\Delta - V)v = 0 \text{ in } \Omega, \qquad v\big|_{\partial\Omega} = g, \qquad v \in \mathcal{D}^{1,p}, \tag{5.50}$$

can be written as $v = \mathcal{S}(S^{-1}g)$, where $\mathcal{S}$ is the single layer potential (5.45) and $S : L^p(\partial\Omega) \to H^{1,p}(\partial\Omega)$, is an invertible map (the single layer operator) defined by

$$Sf(x) = \int_{\partial\Omega} E(x,y)f(y)\, d\sigma(y), \quad \text{for } x \in \partial\Omega. \tag{5.51}$$

The rest goes essentially unchanged. $\square$

Finally, we look more closely at the case when boundary data g of the equation (5.2) are bounded. In such case we can modify our assumption on the function a. We will assume that

$$\text{for any } M \in (0,\infty) \text{ we have:} \qquad \sup_{\substack{u\in[-M,M]\\ x\in\Omega}} |a(x,u)| < \infty, \quad a(x,u) \geq 0. \tag{5.52}$$

Then the following is true.

Theorem 5.8. *Let $\Omega \subset M$ be a connected Lipschitz domain in a Riemannian manifold M whose metric tensor is in C^α, $0 < \alpha < 1$. Assume that a is a Caratheodory function satisfying (5.52).*

Then given any $g \in L^\infty(\partial\Omega)$ there exists a function $u \in C^{1+\beta}_{\text{loc}}(\Omega)$, for any $\beta < \alpha$, satisfying

$$\Delta u - a(x,u)u = 0 \textit{ in } \Omega, \qquad u\big|_{\partial\Omega} = g, \qquad u \in L^\infty(\Omega), \tag{5.53}$$

the limit on $\partial\Omega$ taken in the nontangential a.e. sense. Moreover, we have a bound

$$\|u\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\partial\Omega)}. \tag{5.54}$$

If a is a Caratheodory function that satisfies (i), (ii) and (iii):

$$\begin{aligned}
&(i) \textit{ for any } M \in (0,\infty) \textit{ we have:} \quad \sup_{\substack{u\in[-M,M]\\ x\in\Omega}} |a(x,u)| < \infty, \quad a(x,u) \geq 0,\\
&(ii) \textit{ either} \lim_{u\to\infty}(\sup_{x\in\Omega} a(x,u)) < \infty \quad \textit{or} \quad \limsup_{u\to\infty}(\inf_{x\in\Omega} a(x,u)) > 0,\\
&(iii) \textit{ either} \lim_{u\to-\infty}(\sup_{x\in\Omega} a(x,u)) < \infty \quad \textit{or} \quad \limsup_{u\to-\infty}(\inf_{x\in\Omega} a(x,u)) > 0.
\end{aligned} \tag{5.55}$$

Then, given any $f \in L^\infty(\Omega)$, $g \in L^\infty(\partial\Omega)$ there exists a function $u \in C^{1+\beta}_{\mathrm{loc}}(\Omega)$, for any $\beta < \alpha$, satisfying

$$\Delta u - a(x,u)u = f \text{ in } \Omega, \qquad u\big|_{\partial\Omega} = g, \qquad u \in L^\infty(\Omega), \tag{5.56}$$

the limit on $\partial\Omega$ taken in the nontangential a.e. sense.

If in addition the function $b(x,u) = a(x,u)u$ satisfies

$$\sup_{\substack{u\in[-M,M]\\ x\in\Omega}} \left|\frac{\partial}{\partial u} b(x,u)\right| < \infty, \quad \text{for any } M > 0, \quad \text{and} \quad \frac{\partial}{\partial u} b(x,u) \geq 0, \tag{5.57}$$

then the solution u is unique.

Remark. Notice, that if the function a does not depend on x, i.e., $a(x,u) = a(u)$, then the conditions (ii), (iii) are satisfied automatically.

Proof. We start with equation (5.53). The key is to modify the function a. Let $M = \sup_{x\in\partial\Omega} |g(x)|$. Consider a function ψ_M defined as follows

$$\psi_M(x) = \begin{cases} x, & \text{for } |x| \leq 2M, \\ 2M\mathrm{sign}(x), & \text{otherwise.} \end{cases} \tag{5.58}$$

We solve a Dirichlet problem

$$\Delta u - a(x,\psi_M(u))u = 0 \text{ in } \Omega, \qquad u\big|_{\partial\Omega} = g, \qquad u \in \mathcal{D}^{0,2}. \tag{5.59}$$

The function $a(x,\psi_M(u))$ satisfies all assumptions of Theorem 5.2 and therefore there is at least one solution u to (5.59). We will show that u actually solves (5.53) as well.

Construct a sequence $(g_n)_{n\in N}$ of continuous functions on $\partial\Omega$ such that $g_n \to g$ in $L^2(\partial\Omega)$ as $n \to \infty$, and $\|g_n\|_{L^\infty(\partial\Omega)} \leq \|g\|_{L^\infty(\partial\Omega)}$. Then if u_n is a classical solution to the linear problem:

$$\Delta u_n - a(x,\psi_M(u))u_n = 0 \text{ in } \Omega, \qquad u_n\big|_{\partial\Omega} = g_n, \qquad u_n \in C(\Omega) \cap C^1_{\mathrm{loc}}(\Omega), \tag{5.60}$$

we have that $u_n \to u$ uniformly on compact subsets of Ω and the maximum principle,

$$\|u_n\|_{L^\infty(\Omega)} \leq \|g_n\|_{L^\infty(\partial\Omega)} \leq \|g\|_{L^\infty(\partial\Omega)}. \tag{5.61}$$

Hence, passing to the limit we have

$$\|u\|_{L^\infty(\Omega)} \le \|g\|_{L^\infty(\partial\Omega)} = M. \tag{5.62}$$

However, clearly for such u: $a(x, \psi_M(u)) = a(x, u)$. So we have (5.53) and (5.54).

Now we look at (5.56). We will denote the first condition in (ii) by (iia), the second one by (iib). Similarly for (iii), we denote the first condition by (iiia), the second by (iiia). There are essentially four cases to consider. Obviously if (iia), (iiia) hold, then a is bounded and therefore Theorem 5.5 applies.

For $0 < M, N \le \infty$ consider the function

$$\psi_{M,N}(u) = \begin{cases} u, & \text{for } -N < u < M, \\ M, & \text{for } u \ge M, \\ -N, & \text{for } u \le -N. \end{cases} \tag{5.63}$$

If (iib) and (iiia) hold, we take $0 < M < \infty$ and $N = \infty$. In case (iia) and (iiib) are true, we consider $M = \infty$ and $0 < N < \infty$. Finally, if (iib) and (iiib) hold, we consider $0 < M, N < \infty$.

Denote by $u_{M,N} \in L^\infty(\Omega)$ the solution to the equation

$$\Delta u_{M,N} - a(x, \psi_{M,N}(u_{M,N}))u_{M,N} = f \text{ in } \Omega, \quad u_{M,N}\big|_{\partial\Omega} = g. \tag{5.64}$$

Obviously, for considered M, N always $a(x, \psi_{M,N}(u)) \in L^\infty(\Omega \times R)$ and thus the existence of the solution $u_{M,N}$ is guaranteed by Theorem 5.5. The claim is that for some pair M, N the function $u_{M,N}$ actually solves (5.56).

To see this, consider first that (iia) holds. In such case $M = \infty$ and therefore for any $u(x) \ge 0$:

$$a(x, u(x)) = a(x, \psi_{\infty,N}(u(x))). \tag{5.65}$$

If (iib) holds then there is $\varepsilon > 0$ and an increasing sequence $M_1, M_2, \ldots$ converging to ∞ for which

$$a(x, M_i) \ge \varepsilon, \qquad \text{for all } x \in \Omega, \qquad i = 1, 2, \ldots. \tag{5.66}$$

Find the smallest $m \in \mathbb{N}$ for which $\varepsilon M_m > \|f\|_{L^\infty(\Omega)}$ and $M_m > \|g\|_{L^\infty(\partial\Omega)}$. We claim that for any $N > 0$ we have

$$u_{M_m,N} \le M_m. \tag{5.67}$$

At this stage we drop index M_m, N to keep notation simple. Assume that (5.67) does not hold, i.e., for some $x \in \Omega$, $u(x) > M_m$. Then there is $p \in \Omega$ for which

$$u(p) = \sup_{x \in \Omega} u(x). \tag{5.68}$$

At this point the function u has a global maximum, hence necessary $\Delta u(p) \leq 0$. But obviously

$$\Delta u(p) = f(p) + a(x, u(p))u(p) = f(p) + a(x, M_m)u(p) > f(p) + \varepsilon M_m > 0. \tag{5.69}$$

So, indeed (5.67) is true. Clearly, for $u(x) \geq 0$ that satisfies (5.67) we have:

$$a(x, u(x)) = a(x, \psi_{M_m,N}(u(x))). \tag{5.70}$$

So if (iia) or (iib) hold we can always find M such that for any considered N the solution $u = u_{M,N}$ to the equation (5.64) has the property:

$$a(x, u(x)) = a(x, \psi_{M,N}(u(x))), \quad \text{provided } u(x) \geq 0. \tag{5.71}$$

Using essentially the same argument one can also show that there is N such that for any considered M the solution $u = u_{M,N}$ to the equation (5.64) has the property:

$$a(x, u(x)) = a(x, \psi_{M,N}(u(x))), \quad \text{provided } u(x) \leq 0. \tag{5.72}$$

Putting (5.71) and (5.72) together we see that there is a pair M, N for which we have for $u = u_{M,N}$: $a(x, u) = a(x, \psi_{M,N}(u))$, hence u solves (5.56).

Uniqueness again uses same argument as in Theorem 5.4. We again get (5.25) for a difference w of two solutions to (5.53). Boundedness of both solutions ensures that the function V_{12} defined by (5.24) is nonnegative and bounded. So, $w = 0$. □

Remark. A good example of an equation satisfying the assumptions of Theorem 5.8 is

$$\Delta u - |u|^p u = 0 \text{ in } \Omega, \qquad u\big|_{\partial\Omega} = g \in L^\infty(\partial\Omega), \tag{5.73}$$

for any $p \geq 0$. Theorem 5.8 gives us both existence and uniqueness for this equation.

Example 5.9. There is a nice example illustrating the second part of Theorem 5.8. This example is two dimensional. We did not consider the case dim $M = 2$ in close detail, there is however no reason for all our work not to work in two dimensions. The difference is that instead of having the decomposition (3.5) with the leading term of the form (3.6) we would get logarithmic singularity of the leading term for dim $M = 2$.

Let $\Omega \subset M$ be a connected Lipschitz domain on a two dimensional compact Riemannian manifold M. We want to impose a given Gaussian curvature $K(x) < 0$ on the set Ω by conformally altering a given metric g, whose Gauss curvature is $k(x)$. As noted in [26], if g, g' are conformally related,

$$g' = e^{2u} g, \tag{5.74}$$

then K and k are related by

$$K(x) = e^{-2u}(-\Delta u + k(x)), \tag{5.75}$$

where Δ is the Laplace operator for the original metric g. So, we want to solve the PDE

$$\Delta u = k(x) - K(x)e^{2u}. \tag{5.76}$$

We might also want to impose Dirichlet boundary conditions on u, i.e, in (5.74) we exactly specify g' on $\partial\Omega$. Under certain mild conditions the equation (5.76) satisfies all assumptions of Theorem 5.8.

What we need is $k(x), K(x) \in L^\infty$, and $K(x) \le -k < 0$ for some $k > 0$. We can rewrite (5.76) as

$$\Delta u - \left(-K(x)\frac{e^{2u}-1}{u}\right) u = k(x) - K(x), \tag{5.77}$$

i.e, we have

$$a(x,u) = -K(x)\frac{e^{2u}-1}{u}, \quad \text{and} \quad f(x) = k(x) - K(x). \tag{5.78}$$

Obviously, $f \in L^\infty(\Omega)$, $a(x,u) \ge 0$,

$$\sup_{\substack{u\in[-M,M]\\ x\in\Omega}} |a(x,u)| \le \|K\|_{L^\infty} e^{2M},$$

$$\limsup_{u\to\infty}(\inf_{x\in\Omega} a(x,u)) \ge \lim_{u\to\infty} k\frac{e^{2u}-1}{u} = \infty. \tag{5.79}$$

Similarly, (iiia) also holds. The uniqueness condition is also satisfied, since $b(x,u) = -K(x)(e^{2u} - 1)$ and therefore

$$\frac{\partial}{\partial u} b(x,u) = -2K(x)e^{2u} \geq 0. \tag{5.80}$$

It follows that given any $h = u\big|_{\partial\Omega} \in L^\infty(\partial\Omega)$ we can construct on Ω a conformal metric g' with prescribed curvature $K(x) < 0$ and boundary 'values'

$$g'\big|_{\partial\Omega} = e^{2h} g\big|_{\partial\Omega}, \tag{5.81}$$

where g is the original metric tensor on M. Let us note that a different boundary value problem (with $u \to \infty$ as $x \to \partial\Omega$) is discussed in the paper by Mazzeo and Taylor [20] for $K(x) = -1$.

6. $\hbar^1(\partial\Omega)$, bmo$(\partial\Omega)$ and $C^\beta(\partial\Omega)$ boundary problems

In the last section of this paper we would like to consider the 'end point case' for Neumann and Dirichlet regularity problems, i.e., for $p = 1$. From the results for the linear problem (i.e. $a(x,u) = V(x)$) we know that given $g \in L^1(\partial\Omega)$ it is not always possible to solve the equations (5.35) and (5.48).

In this case, the natural replacement of $L^1(\partial\Omega)$ is the Hardy space $\hbar^1(\partial\Omega)$. The solvability of the linear Neumann and Dirichlet regularity problems with boundary data in $\hbar^1(\partial\Omega)$ was established in various settings in [9], [19], [24], [25] and [7]. We will establish that the same remains true even for the semilinear equation. We return to our original hypothesis (5.3) on the function a.

Theorem 6.1. *Let $\Omega \subset M$ be a connected Lipschitz domain in a Riemannian manifold M whose metric tensor is in C^α, $0 < \alpha \leq 1$. Assume that a is a Caratheodory function satisfying*

$$a(x,u) \in L^\infty(\Omega \times \mathbb{R}), \qquad \inf_{u \in \mathbb{R}} a(.,u) \geq q(.) \text{ on } \Omega,$$

for some nonnegative function q on Ω that is positive on a subset of Ω of positive measure. Let X be one of the following spaces:

(a) $L^r(\Omega)$ for some $r > n$,

(b) $H^{-\beta,r}(\Omega)$ for some $1 > \beta$, $r > 1$ and $r(1-\beta) > n$,

(c) $\mathcal{D}^{0,1}$.

Then given any $g \in \hbar^1(\partial\Omega)$ *and* $f \in X$ *there exists a function* $u \in C^{1+\delta}_{\rm loc}(\Omega)$, *for some* $\delta > 0$, *satisfying*

$$\Delta u - a(x,u)u = f \text{ in } \Omega, \qquad \partial_\nu u\big|_{\partial\Omega} = g, \qquad u \in \mathcal{D}^{1,1}, \tag{6.1}$$

the limit on $\partial\Omega$ *taken in the nontangential a.e. sense. Moreover, we have a bound*

$$\|u\|_{\mathcal{D}^{1,1}} \le C(a,q,p)(\|g\|_{\hbar^1(\partial\Omega)} + \|f\|_X). \tag{6.2}$$

If in addition the function $b(x,u) = a(x,u)u$ *satisfies*

$$\frac{\partial}{\partial u} b(x,u) \in L^\infty(\Omega \times \mathbb{R}) \quad \text{and} \quad \frac{\partial}{\partial u} b(x,u) \ge q(x) \text{ for all } u \in \mathbb{R}, \tag{6.3}$$

then the solution u *is unique.*

Proof. The main idea of the proof is essentially unchanged. We again define a map $T : \mathcal{D}^{1,1} \to \mathcal{D}^{1,1}$ by putting $Tu = v$ for $u \in \mathcal{D}^{1,1}$ and v solving the Neumann problem

$$\Delta v - a(x,u)v = f \text{ in } \Omega, \qquad \partial_\nu v\big|_{\partial\Omega} = g, \qquad v \in \mathcal{D}^{1,1}. \tag{6.4}$$

Once again, we can decompose T as a sum of two operators T_1 and T_2 defined as in (5.40)-(5.41) for $p = 1$ and $g \in \hbar^1(\partial\Omega)$. Then the result on continuity and compactness of T_2 follows directly from Theorem 5.6 since we have that T_2 actually maps $\mathcal{D}^{1,1}$ into $\mathcal{D}^{1+\delta,q}$ for some $\delta > 0$ and $q > 1$.

On the other hand the fact that T_1 is well defined is a consequence of the Proposition 5.3 of [24]. If we look closely at the proof of this proposition (by decomposing g into 'atoms') we get that

$$\|\mathcal{M}^1 T_1 u\|_{L^1(\partial\Omega)} \le C\|g\|_{\hbar^1(\partial\Omega)}, \tag{6.5}$$

with constant C independent of u. In fact C depends on the L^∞ norm of a and the function q only. The rest follows, the proof of the fact that T_1 is continuous and bounded remains same as in Theorem 5.6. To get an equivalent of the second line of (5.47), i.e. that

$$\lim_{n\to\infty} \|K^* - K^*_n\|_{\mathcal{L}(\hbar^1(\partial\Omega))} = 0, \tag{6.6}$$

we use the first part of Lemma 2.4 of [7]. □

In what follows by $H^{1,1}(\partial\Omega)$ we mean a Hardy-Sobolev space on $\partial\Omega$ defined by

$$H^{1,1}(\partial\Omega) = \{f : \partial\Omega \to \mathbb{R}; \nabla_T f \in \hbar^1(\partial\Omega)\}, \tag{6.7}$$

equipped with the norm $\|f\|_{H^{1,1}(\partial\Omega)} = \|f\|_{\hbar^1(\partial\Omega)} + \|\nabla_T f\|_{\hbar^1(\partial\Omega)}$.

Theorem 6.2. *Let $\Omega \subset M$ be a connected Lipschitz domain in a Riemannian manifold M whose metric tensor is in C^α, $0 < \alpha \le 1$. Assume that a is a Caratheodory function satisfying*

$$a(x,u) \in L^\infty(\Omega \times \mathbb{R}), \qquad a(.,u) \ge 0.$$

Let X be one of the following spaces:
(a) $L^r(\Omega)$ for some $r > n$,
(b) $H^{-\beta,r}(\Omega)$ for some $1 > \beta$, $r > 1$ and $r(1-\beta) > n$,
(c) $\mathcal{D}^{0,1}$.

Then given any $g \in H^{1,1}(\partial\Omega)$ and $f \in X$ there exists a function $u \in C^{1+\delta}_{\text{loc}}(\Omega)$, for some $\delta > 0$, satisfying

$$\Delta u - a(x,u)u = f \text{ in } \Omega, \qquad u\big|_{\partial\Omega} = g, \qquad u \in \mathcal{D}^{1,1}, \tag{6.8}$$

the limit on $\partial\Omega$ taken in the nontangential a.e. sense. Moreover, we have a bound

$$\|u\|_{\mathcal{D}^{1,1}} \le C(a,p)(\|g\|_{H^{1,1}(\partial\Omega)} + \|f\|_X). \tag{6.9}$$

If in addition the function $b(x,u) = a(x,u)u$ satisfies

$$\frac{\partial}{\partial u} b(x,u) \in L^\infty(\Omega \times \mathbb{R}) \quad \text{and} \quad \frac{\partial}{\partial u} b(x,u) \ge 0, \tag{6.10}$$

then the solution u is unique.

Proof. The map S defined by (5.51) maps $\hbar^1(\partial\Omega)$ isomorphically to $H^{1,1}(\partial\Omega)$ as has been shown in [24]. The rest goes as in the previous theorem. $\square$

Similarly, for the Dirichlet problem we can consider a bmo$(\partial\Omega)$ version of Theorem 5.5. The following theorem together with the idea of the proof has been pointed out to me by Michael Taylor.

Theorem 6.3. *Let $\Omega \subset M$ be a connected Lipschitz domain in a Riemannian manifold M whose metric tensor is in C^α, $0 < \alpha \le 1$. Assume that a is a Caratheodory function satisfying*

$$a(x,u) \in L^\infty(\Omega \times \mathbb{R}), \qquad a(.,u) \ge 0.$$

Let X be one of the following spaces:
(a) $L^r(\Omega)$ for some $r > n/2$,

(b) $H^{-\beta,r}(\Omega)$ *for some* $1+\alpha > \beta$, $r > 1$ *and* $r(2-\beta) > n$,

(c) $\mathcal{D}^{0,r}$ *for some* $r > (n-1)/2$.

Then given any $g \in bmo(\partial\Omega)$ *and* $f \in X$ *the* $\mathcal{D}^{0,2}$ *solution* u *to the equation*

$$\Delta u - a(x,u)u = f \text{ in } \Omega, \qquad u\big|_{\partial\Omega} = g, \qquad u \in \mathcal{D}^{0,2}, \tag{6.11}$$

has the following additional property:

$$\|\mathcal{M}^0 u\|_{L^p(\partial\Omega)} \leq C(\Omega)p(\|g\|_{\text{bmo}(\partial\Omega)} + \|f\|_X) \quad \text{for any } 2 \leq p < \infty. \tag{6.12}$$

It follows that there is a constant $K > 0$ *such that for any*

$$a < K(\|g\|_{\text{bmo}(\partial\Omega)} + \|f\|_X)^{-1}$$

the function $\exp(a\mathcal{M}^0 u)$ *is* $L^1(\partial\Omega)$ *integrable.*

Proof. We are not concerned with the existence of a solution, since it follows from Theorem 5.5 that there is $u \in C^{1+\beta}_{\text{loc}}(\Omega)$ that solves (5.26) for any $2 \leq p < \infty$. A crucial point is, that we can take the constant $C(a,p)$ in (5.27) independent of p, provided we consider p only in the range $[2,\infty)$. This follows from the fact that for each p the estimate (5.27) was obtained from (4.12). However, the constant C_p in (4.12) can be taken uniform for $p \in [2,\infty]$ since we can interpolate between the estimates for $p = 2$ and $p = \infty$. Hence, indeed there exists $C = C(a)$ such that for any $2 \leq p < \infty$ we have

$$\|u\|_{\mathcal{D}^{0,p}} \leq C(\|g\|_{L^p(\partial\Omega)} + \|f\|_X). \tag{6.13}$$

Now, we use the fact that given $g \in \text{bmo}(\partial\Omega)$ we have the following bound on the L^p ($p \geq 2$) norm of g:

$$\|g\|_{L^p(\partial\Omega)} \leq Cp\|g\|_{\text{bmo}(\partial\Omega)}. \tag{6.14}$$

That is, the L^p norm of g increases at most linearly, as $p \to \infty$. From this and (6.13) it follows that

$$\|\mathcal{M}^0 u\|_{L^p} \leq C(\|f\|_X + p\|g\|_{\text{bmo}(\partial\Omega)}) \leq Cp(\|f\|_X + \|g\|_{\text{bmo}(\partial\Omega)}), \tag{6.15}$$

for any $2 \leq p < \infty$. It turns out that the condition (6.15) is equivalent to the fact that for any $\alpha > 0$ the measure of the level set $\{x \in \partial\Omega; \mathcal{M}^0 u(x) > \alpha\}$

decays exponentially as $\alpha \to \infty$, i.e, there are positive constants K, b such that

$$\sigma(\{x \in \partial\Omega; \mathcal{M}^0 u(x) > \alpha\}) \leq K e^{-b\alpha}. \tag{6.16}$$

The result [15] due to John and Nirenberg shows that this property holds for any bmo function. This inequality implies integrability of the exponential of $a\mathcal{M}^0 u$ for small $a > 0$.

□

Finally, we consider a Dirichlet problem with boundary data in the Hölder class $C^\beta(\partial\Omega)$ for some β small. We have the following:

Theorem 6.4. *Let $\Omega \subset M$ be a connected Lipschitz domain in a Riemannian manifold M whose metric tensor is in $C^{1+\alpha}$, $0 < \alpha \leq 1$. Assume that a is a Caratheodory function satisfying*

$$a(x,u) \in L^\infty(\Omega \times \mathbb{R}), \qquad a(.,u) \geq 0.$$

There exists a number $\alpha_0 > 0$ such that given any $0 < \beta < \alpha_0$ the following holds. Assume X is one of the following spaces:

(a) $L^r(\Omega)$ for some r such that $\beta < 2 - \frac{n}{r}$,

(b) $H^{-\gamma,r}(\Omega)$ for some $1+\alpha > \gamma$, $r > 1$ and $\beta < 2 - \gamma - \frac{n}{r}$,

(c) $\mathcal{D}^{0,r}$ for some $\beta < 2 - \frac{n-1}{r}$.

Then given any $g \in C^\beta(\partial\Omega)$ and $f \in X$ there exists a function $u \in C^\beta(\overline{\Omega})$ satisfying

$$\Delta u - a(x,u)u = f \text{ in } \Omega, \qquad u\big|_{\partial\Omega} = g. \tag{6.17}$$

Moreover, we have a bound

$$\|u\|_{C^\beta(\overline{\Omega})} \leq C(a,\beta)(\|g\|_{C^\beta(\partial\Omega)} + \|f\|_X). \tag{6.18}$$

If in addition the function $b(x,u) = a(x,u)u$ satisfies (6.10) then the solution u is unique. If the boundary of Ω is C^1, then we can take $\alpha_0 = 1$.

If $f = 0$ we might relax our assumptions of the function a. All above on the existence and regularity of the solution remains true, provided a satisfies

$$\text{for any } M \in (0,\infty) \text{ we have: } \quad \sup_{\substack{u \in [-M,M] \\ x \in \Omega}} |a(x,u)| < \infty, \quad a(x,u) \geq 0. \tag{6.19}$$

The uniqueness requires

$$\sup_{\substack{u\in[-M,M]\\ x\in\Omega}} \left|\frac{\partial}{\partial u} b(x,u)\right| < \infty, \quad \textit{for any } M>0, \quad \textit{and} \quad \frac{\partial}{\partial u} b(x,u) \geq 0, \tag{6.20}$$

where $b(x,u) = a(x,u)u$. *Finally, if* a *satisfies (6.19) and*

$$\begin{aligned} &(i) \textit{ either } \lim_{u\to\infty} (\sup_{x\in\Omega} a(x,u)) < \infty \quad \textit{or} \quad \limsup_{u\to\infty} (\inf_{x\in\Omega} a(x,u)) > 0, \\ &(ii) \textit{ either } \lim_{u\to-\infty} (\sup_{x\in\Omega} a(x,u)) < \infty \quad \textit{or} \quad \limsup_{u\to-\infty} (\inf_{x\in\Omega} a(x,u)) > 0, \end{aligned} \tag{6.21}$$

then, we can even take $f \neq 0$, *namely we need* $f \in L^\infty(\Omega)$. *For the uniqueness we again need (6.20).*

Proof. Since given $g \in C^\beta(\partial\Omega)$ we also have that $g \in L^\infty(\partial\Omega)$, hence the existence of a function $u \in L^\infty(\Omega)$ that solves $\Delta u - a(x,u)u = f$ in Ω, $u\big|_{\partial\Omega} = g$ follows from Theorems 5.5 and 5.8.

We need to show that given the assumptions on the functions f and g we actually have $u \in C^\beta(\overline{\Omega})$. Seeing this is not difficult. We can write the solution u as a sum of two functions $u = v_1 + v_2$, where

$$v_1 = L^{-1}F, \qquad \text{for } L = \Delta - a(x,u), \qquad \text{and} \qquad F = \begin{cases} f, & \text{in } \Omega, \\ 0, & \text{otherwise.} \end{cases} \tag{6.22}$$

Given f from X, in all three cases it follows that $v_1 \in C^\beta(\Omega)$. Now we take v_2 to be a solution to the equation

$$\Delta v_2 - a(x,u)v_2 = 0, \text{ in } \Omega, \qquad v_2\big|_{\partial\Omega} = g - v_1\big|_{\partial\Omega}. \tag{6.23}$$

The fact that the linear equation (6.23) is solvable with $v_2 \in C^\beta(\overline{\Omega})$ follows from Theorem 3.4 of [7] for any $0 < \beta < 1$ (on C^1 domains) and from Corollary 7.8 of [24] for some $0 < \beta < \alpha_0 < 1$ (on Lipschitz domains). This gives $u = v_1 + v_2 \in C^\beta(\overline{\Omega})$ and also the estimate (6.18). $\square$

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References

[1] J. Bergh and J. Löfström, Interpolation spaces, Springer-Verlag, 1976.

[2] A. P. Calderón, *Cauchy integrals on Lipschitz curves and related operators*, Proc. Nat. Acad. Sci. **74** (1977), 1324–1327.

[3] A. P. Calderón, C. P. Calderón, E. Fabes, M. Jodeit and N. M. Riviere, *Applications of the Cauchy integral on Lipschitz curves*, Bull. Am. Math. Sci. **84** (1978), 287–290.

[4] A. P. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution*, Advances in Math. **16** (1975), 1–64.

[5] R. R. Coifman, A. McIntosh and Y. Meyer, *L'integrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes*, Ann. Math. **116** (1982), 361–387.

[6] Z.-Q. Chen, R. J. Williams and Z. Zhao, *On the existence of positive solution for semilinear elliptic equations with singular lower order coefficients and Dirichlet boundary conditions*, Math. Ann. **315** (1999), 735–769.

[7] M. Dindoš, *Hardy spaces and potential theory on C^1 domains in Riemannian manifolds*, Preprint (2001).

[8] B. Dahlberg, *On the Poisson integral for Lipschitz and C^1 domains*, Studia Math. LXVI. (1979), 13–26.

[9] B. Dahlberg and C. Kenig, *Hardy spaces and the Neumann problem in L^p for Laplace's equation in Lipschitz domains*, Ann. Math. **125** (1987), 437–465.

[10] B. Dahlberg, C. Kenig and G. Verchota, *Boundary value problems for*

the system of elastostatics on Lipschitz domains, Duke Math. J. **57** (1988), 795–818.

[11] E. Fabes, M. Jodeit and N. Rivère, *Potential techniques for boundary value problems on C^1-domains*, Acta Mathematatica **141** (1978), 165–186.

[12] E. Fabes, C. Kenig and G. Verchota, *The Dirichlet problem for the Stokes system on Lipschitz domains*, Duke Math. J. **57** (1988), 769–793.

[13] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*, (3rd ed.), Springer-Verlag, Berlin-New York, 1998.

[14] V. Isakov and A. I. Nachman, *Global uniqueness for a two dimensional semilinear elliptic inverse problem*, Trans. AMS **347** (1995), 3375–3390.

[15] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Trans. Comm. Pure Appl. Math. **14** (1961), 415–426.

[16] C. Kenig, Harmonic analysis: Techniques for second order elliptic boundary value problems, American Math. Society, 1994.

[17] C. Kenig, *Elliptic boundary value problems on Lipschitz domains*, Beijing lectures in Harmonic analysis (E. Stein, ed.), Princeton University Press, 1986, pp. 131–183.

[18] C. Kenig, *Weighted H^p spaces on Lipschitz domains*, Amer. J. of Math. **122** (1980), 129–163.

[19] C. Kenig and J. Pipher, *The Neumann problem for elliptic equations with nonsmooth coefficients*, Inv. Math. **113** (1983), 447–509.

[20] R. Mazzeo and M. Taylor, *Curvature and Uniformization*, Preprint (2001).

[21] D. Mitrea, M. Mitrea and J. Pipher, *Vector potential theory on nonsmooth domains in $\mathbb{R}^3$ and applications to electromagnetic scattering*, J. Fourier Anal. and Appl. **3** (1997), 131–192.

[22] D. Mitrea, M. Mitrea and M. Taylor, *Layer potentials, the Hodge Laplacian, and global boundary problems in nonsmooth Riemannian manifolds*, Memoirs AMS no. 713 (2001).

[23] M. Mitrea and M. Taylor, *Boundary layer methods for Lipschitz domains in Riemannian manifolds*, J. of Funct. Anal. no. 163 (1999), 181–251.

[24] M. Mitrea and M. Taylor, *Potential theory on Lipschitz domains in Riemannian manifolds: L^p, Hardy and Hölder space results*, Comm. in Anal. and Geom., to appear.

[25] M. Mitrea and M. Taylor, *Potential theory on Lipschitz domains in Riemannian manifolds: Hölder continuous metric tensors*, Comm. PDE **25** (2000), 1487–1536.

[26] M. Taylor, Partial differential equations, vol. 1-3, Springer-Verlag, 1996.

[27] G. Verchota, *Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains*, J. Funct. Anal. **59** (1984), 472–611.