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STATISTICAL CONVERGENCE OF INFINITE SERIES

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Abstract. In this paper we use the notion of statistical convergence of infinite series naturally introduced as the statistical convergence of the sequence of the partial sums of the series. We will discuss some questions related to the convergence of subseries of a given series.

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1. Introduction

The notion of statistical convergence of a sequence (of real or complex numbers) was defined in [5] (see also [6], [13], [14]).

We say that a sequence $(x_n)_{n\geqslant 1}$ (of real or complex numbers) *statistically converges* to a point ξ if for each $\varepsilon > 0$ the set $A(\varepsilon) := \{n\colon |x_n - \xi| \geqslant \varepsilon\}$ has zero asymptotic density, i.e.

$$d(A(\varepsilon)) := \lim_{n\to\infty} \frac{\operatorname{card}(A(\varepsilon)\cap\{1,\dots,n\})}{n} = 0.$$

It is obvious (see [8], pp. 70–71) that if a statistical limit of a sequence $(x_n)_{n\geqslant 1}$ exists it is unique and we will use the notation: $\operatorname{lim-stat} x_n = \xi$. Statistical limit of a sequence can be characterized as a partial limit along a thick subset of indices. Namely the following proposition is true ([13], Lemma 1.1).

1.1 Proposition. $\operatorname{lim-stat} x_n = \xi$ *if and only if there exists a set* $K = \{k_1 < k_2 < \dots\} \subset \mathbb{N}$ *with* $d(K) = 1$ *such that* $\lim\limits_{n\to\infty} x_{k_n} = \xi$.

Statistical convergence shares some good properties of the usual one. Specifically it can be proved [13], [7] that if $\text{lim-stat}\, x_n = a$ and $\text{lim-stat}\, y_n = b$ then

$$\text{lim-stat}(x_n + y_n) = a + b,$$
$$\text{lim-stat}\, x_n y_n = ab.$$

Let us recall some facts about dyadic expansions of real numbers from the half-closed unit interval $(0, 1]$. For every positive integer n we decompose the unit interval into 2^n disjoint intervals $I_n^j := (j/2^n, j + 1/2^n]$, $j = 0, 1, \ldots, 2^n - 1$; that is $(0, 1] = \bigcup_{j=0}^{2^n-1} I_n^j$. The dyadic expansion of any number $x \in I_n^j$, i.e. $x = \sum_{k=1}^{\infty} \varepsilon_k(x) 2^{-k}$ (with $\varepsilon_k(x) = 1$ for infinitely many indices k) has on the first n places the same digits $\varepsilon_k(x) =: c_k$, $(k = 1, \ldots, n)$ determined uniquely by the indices n and j. Hence, we can say that the interval I_n^j is associated with the finite sequence $c_1, \ldots, c_n$.

We are going to study subseries of a given infinite series $\sum_{n=1}^{\infty} a_n$ by using non-terminating dyadic expansions. Specifically, for each number $x \in (0, 1]$, we consider the unique infinite dyadic expansion $x = \sum_{n=1}^{\infty} \varepsilon_n(x) 2^{-n}$ and we associate with the number x the infinite series

$$(x) := \sum_{n=1}^{\infty} \varepsilon_n(x) a_n,$$

which after omitting the zero terms can be identified with an (infinite) subseries of the series $\sum_{n=1}^{\infty} a_n$ (see [10], [12]). This is a special case of series of the form $\sum_{n=1}^{\infty} X_n(x) a_n$, where $(X_n(x))_{n \geqslant 1}$ is a sequence of independent random variables and which are studied in probability theory.

We shall denote by $C\left(\sum_{n=1}^{\infty} a_n\right)$ and $D\left(\sum_{n=1}^{\infty} a_n\right)$ the set of all numbers $x \in (0, 1]$ such that the series (x) converges or diverges, respectively.

For the reader's convenience we recall the notion of homogeneous set. A measurable set $M \subset (0, 1]$ is called a *homogeneous set* if there exists $d \in [0, 1]$ such that for each interval $J \subset (0, 1]$ we have

$$\frac{\lambda(M \cap J)}{\lambda(J)} = d,$$

with λ the Lebesgue measure. It is known ([15]) that every homogeneous set is of Lebesgue measure 0 or 1.

The notion of statistical convergence of an infinite series is introduced through the associated infinite sequence of partial sums of that series.

1.2 Definition. An infinite series $\sum_{n=1}^{\infty} a_n$ (with real or complex terms) is called *statistically convergent* to a sum s ($s \in \mathbb{R}$ or $s \in \mathbb{C}$) if $\text{lim-stat}\, s_n = s$ with $s_n = \sum_{k=1}^{n} a_k$ $(n = 1, 2, \ldots)$. If $\text{lim-stat}\, s_n$ does not exist or is an infinite number, we say that the series *statistically diverges.* By analogy with the usual convergence we introduce the following notation

$$C_{\text{stat}}\Big(\sum_{n=1}^{\infty} a_n\Big) := \{x \in (0,1]\colon\ (x) \text{ is statistically convergent series}\},$$

$$D_{\text{stat}}\Big(\sum_{n=1}^{\infty} a_n\Big) := \{x \in (0,1]\colon\ (x) \text{ is statistically divergent series}\}.$$

The notion of statistically convergent series seems to be known in the theory of statistical convergence. It was probably B. C. Tripathy who introduced it explicitly for infinite series in his papers [16, 17].

In this paper we will determine the metric properties, the Lebesgue measure and the Hausdorff dimension of the set $C_{\text{stat}}\Big(\sum_{n=1}^{\infty} a_n\Big)$.

2. Some measure properties of C_{stat}

In the paper [1] (see also [2], pp. 404–405) metric properties of the set of all convergent subsequences of a given divergent sequence are studied. In [12] similar problems are studied for the set of all convergent subseries of given divergent series. It is proved in the paper [12] that for any series $\sum_{n=1}^{\infty} a_n$ the set $C\Big(\sum_{n=1}^{\infty} a_n\Big)$ is $\mathscr{F}_{\sigma\delta}$ in $(0,1]$ and therefore it is a measurable set. If $\sum_{n=1}^{\infty} a_n$ diverges, then it was shown in Theorem 1.10 of the quoted paper that the Lebesgue measure of $C\Big(\sum_{n=1}^{\infty} a_n\Big)$ is zero. These results have been generalized to series in normed linear spaces in [3].

The next theorem further generalizes those results and states the fundamental properties of the set $C_{\text{stat}}\Big(\sum_{n=1}^{\infty} a_n\Big)$.

2.1 Theorem. *For any (complex) series $\sum_{n=1}^{\infty} a_n$ the set $C_{\text{stat}}\Big(\sum_{n=1}^{\infty} a_n\Big)$ is a homogeneous Borel set of type $\mathscr{F}_{\sigma\delta\sigma\delta}$. If $\sum_{n=1}^{\infty} a_n$ statistically diverges then*

$$\lambda\bigg(C_{\text{stat}}\Big(\sum_{n=1}^{\infty} a_n\Big)\bigg) = 0.$$

P r o o f. Clearly it suffices to prove the assertion for real series only, so let us suppose that $a_n \in \mathbb{R}$. To prove that $C_{\rm stat}\Big(\sum\limits_{n=1}^{\infty} a_n\Big) \in \mathscr{F}_{\sigma\delta\sigma\delta}$ we use the Cauchy criterion for statistical convergence shown in [6]: A sequence $(x_n)_{n\geqslant 1}$ statistically converges iff it is *statistically Cauchy sequence*, i.e.,

$$\forall \eta > 0\ \exists N \in \mathbb{N}\colon\ d(\{n \in \mathbb{N}\colon\ |x_n - x_N| \geqslant \eta\}) = 0. \tag{2}$$

If we denote $X = (0,1] \setminus \mathbb{Q}$, with $\mathbb{Q}$ being the set of all rational numbers, then using (2) we get

$$X \cap C_{\rm stat}\Big(\sum_{n=1}^{\infty} a_n\Big) = \bigcap_{j=1}^{\infty} \bigcup_{N=1}^{\infty} B(N,j), \tag{3}$$

where

$$B(N,j) := \Big\{x \in X\colon\ d\Big(\Big\{k \in \mathbb{N}\colon\ |s_k(x) - s_N(x)| \geqslant \frac{1}{j}\Big\}\Big) = 0\Big\},$$

and $s_n(x) := \sum\limits_{i=1}^{n} \varepsilon_i(x) a_i$. Using the definition of asymptotic density we get

(3′)

$$x \in B(N,j) \Longleftrightarrow \forall v \geqslant 1\ \exists n_0\ \forall n \geqslant n_0\colon\ \frac{1}{n}\operatorname{card}\Big(\Big\{k \leqslant n\colon\ |s_k(x) - s_N(x)| \geqslant \frac{1}{j}\Big\}\Big) \leqslant \frac{1}{v}.$$

From (3) and (3′) we have

$$X \cap C_{\rm stat}\Big(\sum_{n=1}^{\infty} a_n\Big) = \bigcap_{j=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{v=1}^{\infty} \bigcup_{n_0=1}^{\infty} \bigcap_{n=n_0}^{\infty} A(N,n,j,v), \tag{4}$$

where

$$A(N,n,j,v) := \Big\{x \in X\colon\ \operatorname{card}\Big(\{k \leqslant n\colon\ |s_k(x) - s_N(x)| \geqslant \frac{1}{j}\Big) \leqslant \frac{n}{v}\Big\}.$$

It is easy to see that for any $k \leqslant n$ we have

$$|s_k(x) - s_n(x)| = \Big|\sum_{i=k+1}^{n} \varepsilon_i(x) a_i\Big|.$$

Denote

$$D(k,N,j) := \Big\{x \in X\colon\ |s_k(x) - s_N(x)| \geqslant \frac{1}{j}\Big\}.$$

If $k > N$ and the interval I_k^i is associated with the sequence $\varepsilon_1, \ldots, \varepsilon_k$, then

$$I_k^i \cap X \subset D(k,N,j) \Longleftrightarrow \Big|\sum_{m=N+1}^{k} \varepsilon_m a_m\Big| \geqslant \frac{1}{j}. \tag{5}$$

Similarly, if $k < N$ and the N-th order interval I_N^i is associated with the sequence $\varepsilon'_1, \dots, \varepsilon'_N$, then

$$I_N^i \cap X \subset D(k,N,j) \Longleftrightarrow \Big| \sum_{m=k+1}^{N} \varepsilon'_m a_m \Big| \geqslant \frac{1}{j}. \tag{5'}$$

Let us denote the inequalities on the right hand side of (5) and (5′) by P and P', respectively. So the way we can build the sets $A(N,n,j,v)$ is to include in them the intersection of X with an interval of k-th order or N-th order if and only if the inequalities P or P' are satisfied. This will be the case only if the total number of k which do not exceed n and satisfy P or P' is at most $\frac{n}{v}$. Hence, $A(N,n,j,v)$ is an intersection of X with a finite union of intervals of k-th or N-th order, and therefore it is a closed set in X. Consequently, $X \cap C_{\text{stat}}\Big(\sum_{n=1}^{\infty} a_n\Big)$ is of type $\mathscr{F}_{\sigma\delta\sigma\delta}$ in X. Since $\mathbb{Q}$ is countable set, we conclude that $C_{\text{stat}}\Big(\sum_{n=1}^{\infty} a_n\Big)$ is of type $\mathscr{F}_{\sigma\delta\sigma\delta}$ in $(0,1]$.

The fact that the set $C_{\text{stat}}\Big(\sum_{n=1}^{\infty} a_n\Big)$ is homogeneous follows from the observation that if $x = \sum_{n=1}^{\infty} \varepsilon_n(x) 2^{-n} \in C_{\text{stat}}\Big(\sum_{n=1}^{\infty} a_n\Big)$, then $y = \sum_{n=1}^{\infty} \varepsilon_n(y) 2^{-n} \in (0,1]$ is in $C_{\text{stat}}\Big(\sum_{n=1}^{\infty} a_n\Big)$ provided $\varepsilon_n(x) \neq \varepsilon_n(y)$ holds for finitely many n only (see [11], Lemma 1).

Therefore $C_{\text{stat}}\Big(\sum_{n=1}^{\infty} a_n\Big)$ is a homogeneous set in $(0,1]$ and its Lebesgue measure is either 0 or 1, cf. [15]. We finish the proof by showing that $\lambda\Big(C_{\text{stat}}\Big(\sum_{n=1}^{\infty} a_n\Big)\Big) = 1$ yields a contradiction. Indeed, if this was true then $\lambda\Big(C_{\text{stat}}\Big(\sum_{n=1}^{\infty} a_n\Big) \cap (0, \frac{1}{2}]\Big) = \frac{1}{2} = \lambda\Big(C_{\text{stat}}\Big(\sum_{n=1}^{\infty} a_n\Big) \cap (\frac{1}{2}, 1]\Big)$. Consider the function $g(x) := 1 - x$, $x \in (0, \frac{1}{2}]$. Obviously, $\lambda\Big(g\Big(C_{\text{stat}}\Big(\sum_{n=1}^{\infty} a_n\Big) \cap (0, \frac{1}{2})\Big)\Big) = \frac{1}{2}$. Hence, there exists an irrational number $x_0 \in C_{\text{stat}}\Big(\sum_{n=1}^{\infty} a_n\Big) \cap (0, \frac{1}{2}]$ with $1 - x_0 \in C_{\text{stat}}\Big(\sum_{n=1}^{\infty} a_n\Big) \cap (\frac{1}{2}, 1]$. For such x_0 both series (x_0) and $(1 - x_0)$ are statistically convergent and also their sum, $\sum_{n=1}^{\infty} a_n$, is statistically convergent. This is a contradiction. □

In connection with Theorem 2.1 a natural question arises. What can be said about the Lebesgue measure of the sets $C\Big(\sum_{n=1}^{\infty} a_n\Big), C_{\text{stat}}\Big(\sum_{n=1}^{\infty} a_n\Big)$ if the series $\sum_{n=1}^{\infty} a_n$ is convergent or statistically convergent? These sets can be of full measure when $\sum_{n=1}^{\infty} |a_n| = +\infty$. That this is indeed the case follows from some theorems on Rademacher series.

2.2 Theorem. *Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. The following two assertions hold:*

(i) *If $\sum_{n=1}^{\infty} |a_n|^2 < +\infty$ then $\lambda\Big(C\Big(\sum_{n=1}^{\infty} a_n\Big)\Big) = 1 = \lambda\Big(C_{\mathrm{stat}}\Big(\sum_{n=1}^{\infty} a_n\Big)\Big)$.*

(ii) *If $\sum_{n=1}^{\infty} |a_n|^2 = +\infty$ then $\lambda\Big(C\Big(\sum_{n=1}^{\infty} a_n\Big)\Big) = 0$.*

Proof. Let us define for each $k = 1, 2, \ldots$ the function $r_k(t) := 1 - 2\varepsilon_k(t)$, $t \in (0,1]$. Then $(r_k)_{k\geqslant 1}$ is the Rademacher orthonormal system and we can make a simple transformation which gives:

$$\sum_{k=1}^{n} \varepsilon_k(t)a_k(t) = \frac{1}{2}\Big(\sum_{k=1}^{n} a_k - \sum_{k=1}^{n} r_k(t)a_k\Big). \tag{6}$$

Let $M := \Big\{t \in (0,1] \colon \sum_{n=1}^{\infty} a_n r_n(t) \text{ converges}\Big\}$. If the assumption (i) holds, then by the Khinchin-Kolmogorov theorem, cf. Theorem (8.2) of [18], we have $\lambda(M) = 1$, hence by (6) we get $\lambda\Big(C\Big(\sum_{n=1}^{\infty} a_n\Big)\Big) = 1$ and the same thing for its superset. If the assumption (ii) holds, then using the result from [18] we get $\lambda(M) = 0$ and thus $\lambda\Big(C\Big(\sum_{n=1}^{\infty} a_n\Big)\Big) = 0$. □

2.3 Remark. The series $1 - 2^{-\alpha} + 3^{-\alpha} - \ldots$ is relatively convergent for $\alpha > 0$ and satisfies the assumption (i) provided $\frac{1}{2} < \alpha \leqslant 1$, and that of (ii) provided $0 < \alpha \leqslant \frac{1}{2}$.

Since $C\Big(\sum_{n=1}^{\infty} a_n\Big) \subset C_{\mathrm{stat}}\Big(\sum_{n=1}^{\infty} a_n\Big)$, clearly always

$$\lambda\Big(C\Big(\sum_{n=1}^{\infty} a_n\Big)\Big) \leqslant \lambda\Big(C_{\mathrm{stat}}\Big(\sum_{n=1}^{\infty} a_n\Big)\Big).$$

In the following example we exhibit a series $\sum_{n=1}^{\infty} a_n$ for which this inequality is strict, i.e.

$$\lambda\Big(C\Big(\sum_{n=1}^{\infty} a_n\Big)\Big) = 0 \quad \text{and} \quad \lambda\Big(C_{\mathrm{stat}}\Big(\sum_{n=1}^{\infty} a_n\Big)\Big) = 1. \tag{7}$$

2.4 Example. Consider $\sum_{n=1}^{\infty} a_n$ whose terms are defined for $k \geqslant 0$ as follows.

$$a_{4^k+j} = \begin{cases} 2^{-k}, & \text{for } j = 0, 1, \ldots, 2^k - 1, \\ -2^{-k}, & \text{for } j = 2^k, 2^k + 1, \ldots, 2^{k+1} - 1, \\ 0, & \text{for } j = 2^{k+1}, 2^{k+1} + 1, \ldots, 4^{k+1} - 4^k - 1. \end{cases} \tag{8}$$

Then (7) holds. Indeed, the first claim in (7), namely $\lambda\Big(C\Big(\sum\limits_{n=1}^{\infty} a_n\Big)\Big) = 0$ follows from the fact that the series $\sum\limits_{n=1}^{\infty} a_n$ is divergent. We want to show that $\lambda\Big(C_{\mathrm{stat}}\Big(\sum\limits_{n=1}^{\infty} a_n\Big)\Big) = 1$ is also true.

In what follows, we use the probability notation from [4]. Namely, instead of λ we use P to denote the Lebesgue measure on $(0,1]$. The reason for this change is that we do not want to move away from the usual notation used in probability. The presented proof is substantially simplified thanks to the comments from the anonymous referee of this paper.

Recall the Chebysheff inequality

$$P\left(\left|\frac{S_n}{n} - p\right| \geqslant \varepsilon\right) \leqslant \frac{1}{4n\varepsilon^2},$$

where n is the number of independent draws, p is the probability of a random event and S_n is the random variable that counts the occurrences of the event. Taking $\varepsilon = 1/k^2$ and $n = 2^k$ yields

$$\begin{aligned} &P\left(\left|\left(\sum_{n=4^k}^{4^k+2^k-1} \varepsilon_n(x) a_n\right) - \frac{1}{2}\right| \leqslant \frac{1}{k^2}\right) \geqslant 1 - \frac{k^4}{2^k}, \\ &P\left(\left|\left(\sum_{n=4^k+2^k}^{4^k+2^{k+1}-1} \varepsilon_n(x) a_n\right) + \frac{1}{2}\right| \leqslant \frac{1}{k^2}\right) \geqslant 1 - \frac{k^4}{2^k}. \end{aligned} \tag{9}$$

It follows that

$$P\left(\left|\sum_{n=4^k}^{4^k+2^{k+1}-1} \varepsilon_n(x) a_n\right| \leqslant \frac{2}{k^2}\right) \geqslant \left(1 - \frac{k^4}{2^k}\right)^2. \tag{10}$$

It follows that the probability that for all $k \geqslant K$ we have

$$\left|\sum_{n=4^k}^{4^k+2^{k+1}-1} \varepsilon_n(x) a_n\right| \leqslant \frac{2}{k^2} \tag{11}$$

is at least

$$\left(\prod_{k=K}^{\infty}\left(1 - \frac{k^4}{2^k}\right)\right)^2. \tag{12}$$

We want to argue that (12) is not zero, i.e., that this product is convergent. This follows from the convergence of the sum $\sum\limits_{k=K}^{\infty} k^4/2^k$. Hence with positive probability for all $k \geqslant K$ (11) holds. That is what we needed. Apparently, any $x \in (0,1]$ for which we have (11) (for all $k \geqslant K$) belongs to the set $\mathscr{E}$ defined as follows:

$$\mathscr{E} = \left\{ x \in (0,1] \colon \sum_{k=0}^{\infty} |s_{4^k+2^{k+1}-1}(x) - s_{4^k-1}(x)| < \infty \right\}.$$

Here $(s_n(x))_{n\geqslant 1}$ is as before the sequence of partial sums of the series $(x) = \sum\limits_{n=1}^{\infty} \varepsilon_n(x)a_n$. The set $\mathscr{E}$ is also Borel and homogeneous. It follows that its measure is one, since $P(\mathscr{E}) > 0$. Clearly for any k:

$$s_{4^k+2^{k+1}-1}(x) = s_{4^k+2^{k+1}}(x) = \ldots = s_{4^{k+1}-1}(x). \tag{13}$$

This and the definition of $\mathscr{E}$ imply that the sequence $(s_{4^k-1}(x))_{k\geqslant 1}$ is Cauchy. We conclude that the sequence $(s_n(x))_{\geqslant 1}$ converges over indices

$$n \in A = \{n \in \mathbb{N} \colon\ 4^k + 2^{k+1} - 1 \leqslant n < 4^{k+1},\ k = 0,1,2,\ldots\}.$$

The asymptotic density $d(A)$ of this set is one. Hence, we can conclude that $\mathscr{E} \subset C_{\rm stat}\Big(\sum\limits_{n=1}^{\infty} a_n\Big)$ and therefore $\lambda\Big(C_{\rm stat}\Big(\sum\limits_{n=1}^{\infty} a_n\Big)\Big) = 1$. □

Once we know that for every statistically divergent series $\sum\limits_{n=1}^{\infty} a_n$ we have $\lambda\Big(C_{\rm stat}\Big(\sum\limits_{n=1}^{\infty} a_n\Big)\Big) = 0$, an interesting problem arises, namely that of determining the Hausdorff dimension of the set $C_{\rm stat}\Big(\sum\limits_{n=1}^{\infty} a_n\Big)$. We show that the Hausdorff dimension depends on some properties of the underlying divergent series.

2.5 Theorem. *For each $t \in [0,1]$ there exists a statistically divergent series $\sum\limits_{n=1}^{\infty} a_n$ with positive terms such that $\dim C_{\rm stat}\Big(\sum\limits_{n=1}^{\infty} a_n\Big) = t$.*

Proof. It is proved in Theorem 5 of [10] that if $a_1 \geqslant a_2 \geqslant \ldots$ and $\sum\limits_{n=1}^{\infty} a_n = +\infty$, then $\dim C\Big(\sum\limits_{n=1}^{\infty} a_n\Big) = 0$. Since the statistical convergence of a series with positive terms is equivalent to the usual convergence, we obtain the proof in the case $t = 0$. Let us consider the case $0 < t < 1$ and define the set $A = \{[1/t], [2/t], \ldots, [n/t], \ldots\}$.

It can be proved easily that $d(A) = t$. Now let us define a series whose terms are as follows: $a_n = 1/n^2$ if $n \in A$ and $a_n = 1$ if $n \in \mathbb{N} \setminus A$. Then $\sum\limits_{n=1}^{\infty} a_n = +\infty$ and the series is also statistically divergent. We give lower and upper bounds for $\dim C_{\text{stat}}\Big(\sum\limits_{n=1}^{\infty} a_n\Big)$.

To get a lower bound we use the fact that the set $C_{\text{stat}}\Big(\sum\limits_{n=1}^{\infty} a_n\Big)$ contains the set Z of all $x = \sum\limits_{n=1}^{\infty} \varepsilon_n(x)2^{-n} \in (0,1]$ with $\varepsilon_n(x) = 0$ if $n \in \mathbb{N} \setminus A$ and $\varepsilon_n(x) \in \{0,1\}$ if $n \in A$. Applying Theorem 2.7 of [11] we get

$$\dim Z = \liminf_{n\to\infty} \frac{A(n)}{n} = d(A) = t.$$

Since $Z \subset C_{\text{stat}}\Big(\sum\limits_{n=1}^{\infty} a_n\Big)$ we have

$$t \leqslant \dim C_{\text{stat}}\left(\sum_{n=1}^{\infty} a_n\right). \tag{14}$$

To obtain an upper bound we realize that if $x := \sum\limits_{n=1}^{\infty} \varepsilon_n(x)2^{-n} \in C_{\text{stat}}\Big(\sum\limits_{n=1}^{\infty} a_n\Big)$ then $\varepsilon_n(x) = 1$ only for a finite number of $n \in \mathbb{N} \setminus A$, with $\varepsilon_n(x) = 0$ for the other $n \in \mathbb{N} \setminus A$, and $\varepsilon_n(x) = \{0,1\}$ if $n \in A$. Let us consider the sequence $H_1, H_2, \ldots$ of all finite subsets of $\mathbb{N} \setminus A$. Then due to what was said above we have the equality

$$C_{\text{stat}}\left(\sum_{n-1}^{\infty} a_n\right) = \bigcup_{j=1}^{\infty} C_j \tag{15}$$

where

$$C_j := \left\{ x = \sum_{n=1}^{\infty} \varepsilon_n(x)2^{-n} \in (0,1] \colon \varepsilon_n(x) = \begin{cases} 0 \text{ or } 1 & \text{for } n \in A, \\ 1 & \text{for } n \in H_j, \\ 0 & \text{for } n \in (\mathbb{N} \setminus A) \setminus H_j \end{cases} \right\}.$$

Applying Theorem 2.7 of [11] we have

$$\dim C_j = \liminf_{n\to\infty} \frac{\operatorname{card}(A \cap \{1,\ldots,n\})}{n} = d(A) = t. \tag{16}$$

Using in (16) a result proved in Lemma 4 in [9], which says that $M \subset \bigcup\limits_{n=1}^{\infty} M_n$ implies $\dim M \leqslant \sup\limits_{n\geqslant 1} \dim M_n$, and we conclude that

$$\dim C_{\text{stat}}\left(\sum_{n=1}^{\infty} a_n\right) \leqslant t. \tag{17}$$

This proves the claim of our theorem for $0 \leqslant t < 1$. The case $t = 1$ is solved in [10], as an Example to Corollary of Theorem 6. □

As we have mentioned for series with positive terms $\sum_{n=1}^{\infty} a_n$, $a_n > 0$, $n = 1, 2, \ldots$ we have

$$C_{\text{stat}}\Bigl(\sum_{n=1}^{\infty} a_n\Bigr) = C\Bigl(\sum_{n=1}^{\infty} a_n\Bigr), \tag{18}$$

and hence the following theorem is true:

2.6 Theorem. *For each $t \in [0, 1]$ there exists a divergent series with positive terms $\sum_{n=1}^{\infty} a_n$ such that $\dim C\Bigl(\sum_{n=1}^{\infty} a_n\Bigr) = t$.*

Theorem 2.6 completes the results of the paper [10].

3. Topological properties of C_{stat}

If $\sum_{n=1}^{\infty} a_n$ is a divergent series with complex terms, then $\sum_{n=1}^{\infty} |a_n| = +\infty$ and by the result from [12] mentioned above we get that the set $C\Bigl(\sum_{n=1}^{\infty} a_n\Bigr)$ is of the first Baire category in $(0, 1]$. This claim does not allow to make any conclusion about the Baire category of the superset $C_{\text{stat}}\Bigl(\sum_{n=1}^{\infty} a_n\Bigr)$. We show that the latter set has similar structure as its subset $C\Bigl(\sum_{n=1}^{\infty} a_n\Bigr)$.

3.1 Theorem. *If the series $\sum_{n=1}^{\infty} a_n$ statistically diverges then $C_{\text{stat}}\Bigl(\sum_{n=1}^{\infty} a_n\Bigr)$ is of the first Baire category in $(0, 1]$.*

P r o o f. We are going to use the notation from the proof of Theorem 2.1. First we consider real series, i.e. $a_n \in \mathbb{R}$, $n = 1, 2, \ldots$. For $X = (0, 1] \setminus \mathbb{Q}$ we have

$$X \cap C_{\text{stat}}\Bigl(\sum_{n=1}^{\infty} a_n\Bigr) = \bigcap_{j=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{v=1}^{\infty} \bigcup_{n_0=1}^{\infty} \bigcap_{n=n_0}^{\infty} A(N, n, j, v), \tag{19}$$

where $A(N, n, j, v) := \{x \in X \colon \operatorname{card}(\{k \leqslant n \colon |s_k(x) - s_N(x)| \geqslant 1/j\}) \leqslant n/v\}$. We prove that $X \cap C_{\text{stat}}\Bigl(\sum_{n=1}^{\infty} a_n\Bigr)$ is of first category in X. It follows from (19) that it suffices to prove that $\bigcap_{v=1}^{\infty} \bigcup_{n_0=1}^{\infty} \bigcap_{n=n_0}^{\infty} A(N, n, j, v)$ is of first category in X. Actually, proving that $\bigcup_{n_0=1}^{\infty} \bigcap_{n=n_0}^{\infty} A(N,n,j,2)$ is of first category in X would suffice. To check this, we establish that for any $n_0 \geqslant 1$ the set $E(N,j) = \bigcap_{n=n_0}^{\infty} A(N,n,j,2)$ is nowhere-dense. Fix $n_0 \geqslant 1$ and let $J \subset X$ be an arbitrary interval in X, i.e., an intersection of X with an interval $J^* \subset (0,1]$. If we find a nonempty interval $J' \subset J$ disjoint from the set $E(N,j)$, then $E(N,j)$ is indeed nowhere dense.

We find $m, t \in \mathbb{N}$ such that $I_m^t \subset J$, $0 \leqslant t \leqslant 2^m - 1$. We can suppose that $m \geqslant n_0$ and $m \geqslant N$. Let I_m^t be associated with the m-tuple $c_1, c_2, \ldots, c_m$ of digits 0, 1. Since the series $\sum_{n=1}^{\infty} a_n$ is not statistical ly convergent, it is not convergent in the usual sense and therefore at least one of the series $\sum_{a_n \geqslant 0} a_n$, $\sum_{a_n \leqslant 0} a_n$ has an infinite sum. Let us suppose that

$$\sum_{a_n \geqslant 0} a_n = +\infty. \tag{20}$$

If $a_{m+i} > 0$ we put $c_{m+i} = 1$, else $c_{m+i} = 0$. Due to (20) we have

$$\left| \sum_{n=1}^{m+p} c_n a_n - \sum_{n=1}^{N} c_n a_n \right| = \sum_{n=N+1}^{m+p} c_n a_n > \frac{1}{j} \tag{21}$$

for some sufficiently large p. We choose a natural number

$$q > m + p. \tag{22}$$

and the numbers $c_{m+p+1}, \ldots, c_{m+p+q}$ such that $c_i = 0$ for $a_i \leqslant 0$, $c_i = 1$ for $a_i > 0$, where $i \in \{m+p+1, \ldots, m+p+q\}$. Finally, we find the interval I_{m+p+q}^d of $(m+p+q)$-th order associated with the sequence $c_1, c_2, \ldots, c_{m+p}, c_{m+p+1}, \ldots, c_{m+p+q}$. By (21), we have

$$\sum_{n=N+1}^{m+p+i} c_n a_n > \frac{1}{j}, \tag{23}$$

for every $i = 1, 2, \ldots, q$. Consider any $x := \sum_{n=1}^{\infty} \varepsilon_n(x) 2^{-n} \in X \cap I_{m+p+q}^d$. Clearly, $\varepsilon_n(x) = c_n$, $n = 1, 2, \ldots, m+p+q$. It follows from (23) that $|s_n(x) - s_N| \geqslant 1/j$, $n = m+p+1, \ldots, m+p+q$, and consequently $\operatorname{card}(\{n \leqslant m+p+q \colon |s_n(x) - s_N(x)| \geqslant 1/j\}) \geqslant q$. The way we picked q, we must have $q > \frac{1}{2}(m+p+q)$ and so $X \cap I_{m+p+q}^d \cap E(N,j) = \emptyset$.

This proves that the set $E(N,j)$ is nowhere dense in X. Therefore the set $X \cap C_{\mathrm{stat}}\Bigl(\sum\limits_{n=1}^{\infty} a_n\Bigr)$ is of the first Baire category in X. Since $(0,1] \setminus X$ is countable, we can conclude that $C_{\mathrm{stat}}\Bigl(\sum\limits_{n=1}^{\infty} a_n\Bigr)$ is of the first Baire category in $(0,1]$.

Finally, if the series $\sum\limits_{n=1}^{\infty} a_n$ is complex, then we write $a_n = x_n + \mathrm{i} y_n$, $n = 1,2,\ldots$ At least one of the series $\sum\limits_{n=1}^{\infty} x_n$, $\sum\limits_{n=1}^{\infty} y_n$ is statistically divergent, and we have

$$C_{\mathrm{stat}}\Bigl(\sum_{n=1}^{\infty} a_n\Bigr) = C_{\mathrm{stat}}\Bigl(\sum_{n=1}^{\infty} x_n\Bigr) \cap C_{\mathrm{stat}}\Bigl(\sum_{n=1}^{\infty} y_n\Bigr). \tag{24}$$

Since at least one of the sets on the right hand side is of the first Baire category, the same must be true for the set on the left hand side. □

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